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On Boundary Behaviour of One-Dimensional Diffusions: From Brown to Feller and Beyond

Goran Peskir

1 Introduction

Feller's greatest discovery in mathematics was *sticky* (or *slowly reflecting*) boundary behaviour of one-dimensional diffusion processes (characterised by the appearance of the *second derivative* at the boundary point). Before him (Dirichlet[1], Neumann[2], Robin[3]) it was not known that this was possible. A boundary condition containing the second derivative (in one dimension) should therefore be referred to as *Feller boundary condition* (in addition to *Dirichlet condition* containing the function itself, *Neumann condition* containing its first derivative, *Robin condition* containing a linear combination of the function and its first derivative) and extensions of this condition to higher dimensions should likewise contain the name of Feller.

In this brief review I will address the relevance of this discovery and Feller's work on boundary classification of one-dimensional diffusion processes within a general context of mathematics and physics. Feller's motivating aim, often stated explicitly in his papers, was to disclose the 'most general' conditions, or to describe the 'most general' situations, and this attitude happened to be the key to unlock the mystery of the still unseen boundary behaviour. Although enduring the progress was slow and it took over 10 years (1951-1965) including help and insights of other people (Itô and McKean via Lévy and Volkonskii) to complete it at the level of a sample path. The general picture in one dimension is complete at present.

After reading Feller's papers time and again I wonder whether he ever thought that the ultimate goal of the 'most general' was achieved. The line had to be always drawn somewhere but only to be stretched further in a new paper. Likewise my review will fail to draw definite lines and at the end I will briefly indicate how to go beyond the 'most general' in hope that the suggested possibility would only please Feller in his never ending quest for the 'most general' conditions and situations. I will now return to the beginning of the story leaping forward in big jumps to get to Feller's time as quickly as possible.

2 From Brown (1828) to Feller (1936)

A pillar of Newtonian mechanics is Newton's first law[4] (or Galileo's law of inertia[5]). It states that if a body is not acted upon by a force then the body stays at rest or moves at straight line with constant velocity. The observations made by Brown [2] in 1828 about the unpredictable 'zigzag' motion of spherical particles with a diameter of the order of several μm $(= 10^{-6}$m$)$

[1] Peter Gustav Lejeune Dirichlet (1805–1859) was a German mathematician.
[2] Carl Gottfried Neumann (1832–1925) was a German mathematician.
[3] Victor Gustave Robin (1855–1897) was a French mathematician.
[4] Isaac Newton (1642–1727) was an English physicist and mathematician.
[5] Galileo Galilei (1564–1642) was an Italian physicist and mathematician.

suspended in a fluid at rest (Brownian particles) were therefore curious. The numerous attempts to explain the phenomenon over the subsequent 75 years have failed until the appearance of the paper by Einstein [5] in 1905. Einstein's aim was to probe the molecular-kinetic theory of heat according to which the fluid consists of atoms/molecules in motion whose average kinetic energy is measured by the fluid temperature. The fluid atoms/molecules have a diameter of the order of several Å $(= 10^{-10}\text{m})$ and thus could not be observed by the microscope. Einstein's ingenious idea (one of his 'gedanken' experiments) was to take a single atom/molecule of the fluid, enlarge it to the size of the order of several μm that is visible by the microscope, and witness the existence of the other atoms/molecules of the fluid (around 10^{23} of them) through their bombardment and the observed motion of the enlarged atom/molecule (i.e. Brownian particle). In the first (physical) part of his paper Einstein derives a closed-form expression for the diffusion coefficient D expressed in terms of other physical quantities (the Einstein relation), and in the second (mathematical) part of his paper Einstein derives the diffusion equation

$$p_t = D p_{xx} \tag{1}$$

for the transition density $p = p(t,x)$ of the position X_t of a Brownian particle. This means that

$$\mathbb{P}(X_t \in A) = \int_A p(t,x)\,dx \tag{2}$$

for a (measurable) set A at time $t > 0$ upon assuming that $X_0 = 0$. Thus instead of inferring that the particle occupies a particular position at a particular time (Newtonian mechanics) we speak of the particle occupying a particular set at a particular time with certain probability. Since (1) can be used to estimate D through the observation of a Brownian particle, the Einstein relation then enables him to estimate the physical quantity of interest (Avogadro's number[6]). In this way Einstein succeeds in connecting the micro scale of atoms/molecules to the macro scale of the observer through the meso scale of the Brownian particle. Experiments made subsequently by Perrin[7] confirmed the predictions and established that the atoms/molecules are real.

In his physical arguments Einstein makes use of Fick's law [18] from 1855 (which is a definition of the diffusion coefficient) that in turn mimics Fourier's law [20] from 1822 (a special case of which is Newton's law of cooling). A simple integration by parts (or the divergence theorem in higher dimensions) transforms Fick's law into the *diffusion equation* and likewise Fourier's law into the *heat equation* (both being transport phenomena). A curious fact is that Einstein appears to be unfamiliar with this derivation of the diffusion equation. Instead he ingeniously postulates that the position process X has stationary and independent increments. This enables him to write down an integral equation for the transition density (later to be named after Chapman and Kolmogorov) which he then expands in powers of time (first order) and space (second order). Passing to the limit (upon violating his own claim on the breakdown of independent increments at small time scales) he then formally obtains the diffusion equation (1). I will refer to this notable derivation as Einstein's argument in the sequel (see [5, page 95 of the English translation] for details). Wiener [41] then in 1923 proves the existence of a stochastic process satisfying Einstein's postulates (stationary independent increments plus continuous sample paths which Einstein finds too obvious to state explicitly). The Einstein-Wiener process does not fit the observed data at all time scales (due to the breakdown of independent increments). The Ornstein-Uhlenbeck position process [39] based on the equation of Langevin [32] (representing Newton's second law) does. Nonetheless, due to its canonical form and tractability, the Einstein-Wiener process turns out to be the most important stochastic process that is often referred to as Brownian motion itself (in mathematical circles in particular).

[6]Lorenzo Romano Amedeo Carlo Avogadro (1776–1856) was an Italian scientist.
[7]Jean Baptiste Perrin (1870–1942) was a French physicist.

Subsequently Smoluchowski [37] in 1916 extends the forward equation (1) to the case when the Brownian particle is under influence of an external force (transforming the zero/constant drift into a function of [time and] space), and Chapman [3] in 1928 studies the case when the Brownian particle is in a fluid with non-constant temperature (transforming the constant diffusion coefficient into a function of [time and] space). Stationary and independent increments in both cases break down and get superseded by the Markov[8] property (formalised by Markov in 1906 although the idea constitutes the backbone of Newtonian mechanics) in that the future probabilistic behaviour of the process depends only on its current position when the entire past is given. Consequently the drift and the diffusion coefficient become non-constant in these more general physical situations. This more general forward equation has also been derived in some particular cases by Fokker [19] in 1914 and Planck [36] in 1917 (and it is often named after them). For further details on the facts reviewed so far see [33] and [34].

Kolmogorov [30] in 1931 undertakes a systematic study of the forward and backward equations for the transition densities starting with the integral equation (of Einstein) arising from the Markov property (named later after Chapman and Kolmogorov) as a new axiom for the stochastic motion. Kolmogorov derives the *backward equation*

$$p_s = -\mu p_x - D p_{xx} \tag{3}$$

by extending the Einstein's argument referred to above (compare incidentally the same page 95 in the English translations of both [30] and [5]) and the *forward equation*

$$p_t = -(\mu p)_y + (Dp)_{yy} \tag{4}$$

integrating by parts (twice) so that the two equations become adjoints to each other (in the sense of a scalar product). In these equations $p = p(s,x,t,y)$ denotes the transition density of the position process X in the sense that

$$\mathbb{P}(X_t \in A \,|\, X_s = x) = \int_A p(s,x,t,y)\, dy \tag{5}$$

for a (measurable) set A, while $\mu = \mu(t,x)$ is the drift of X and $\sigma = \sigma(t,x)$ is the (mathematical) diffusion coefficient of X (the two are connected by $D = \sigma^2/2$). Kolmogorov establishes some existence and uniqueness results for the backward and forward equations (in [30] when the state space is countable and in [31] when the state space is a Riemannian manifold) but largely leaves the existence and uniqueness question open.

At this point Feller enters the scene with his paper [7] that was written during his early Stockholm years (1934–1939) and published in 1936. Firstly, he establishes more definite existence and uniqueness results for the backward and forward equations (both for continuous and discontinuous Markov processes). Secondly, he also provides a more complete treatment of Markov processes with jumps (this was indicated by Kolmogorov) leading to partial integro-differential equations in place of the backward and forward differential equations (4) and (5). Feller expresses all solutions as series in his paper. His method relies upon an adaptation of the construction of fundamental solutions due to Hadamard[9] and Gevrey[10] (dating back to 1911 and 1913 respectively). Both Kolmogorov [30] and Feller [7] treat the backward and forward equations in a purely analytic fashion starting with the Chapman-Kolmogorov integral equation

[8] Andrey Andreyevich Markov (1856–1922) was a Russian mathematician.

[9] Jacques Salomon Hadamard (1865–1963) was a French mathematician.

[10] Maurice-Joseph Gevrey (1884–1957) was a French mathematician.

that is implied by a Markov process but also non-Markov processes as shown later by Lévy[11] using a sequence of pairwise independent random variables that are not mutually independent (Feller's paper [16] deals with that issue). While Kolmogorov retains his vision of a stochastic process as the time evolution of marginal laws, Feller will gradually soften his analytic view and start to appreciate sample paths of the process in later years. Nonetheless, the backward and forward equations will remain Feller's main pillar for stochastic processes and he will never get tired from studying them in new and different settings from different standpoints as well. In line with this intention Feller devotes the entire Section 6 in his paper [7] to purely discontinuous processes (those changing their states only by jumps). He demonstrates that this case of backward and forward equations can be treated under much weaker assumptions than the general case and continues his study in [8] using more general methods and establishing more general existence and uniqueness results. At that time Feller was already at Brown University and after passing through Cornell and settling down at Princeton in 1950 he was to enter the golden era of his research on diffusion processes. The start was again through the backward and forward equations. This time however the solutions are not to be defined on the entire state space but only on its *subspace* of interest. One of the key motivations for Feller came from genetics. If the population size is described by a diffusion process then clearly this process is *non-negative* and the question arises what happens after the process hits zero. In line with his incessant search for 'most general' Feller aims to disclose all possibilities. It turns out however that the question is not as simple as one could expect at first glance.

3 From Feller (1951) to Itô and McKean (1965)

Motivated by applications in genetics [9] where the diffusion process by its nature takes non-negative values, Feller begins his work on boundary behaviour of one-dimensional diffusions in the paper [10] by studying the forward equation

$$u_t = -((bx+c)\,u)_x + (ax\,u)_{xx} \tag{6}$$

for $x > 0$ where $u = u(t,x)$ is a function of time and space, and a,b,c are constants with $a > 0$. The key feature of this equation is its singularity at the boundary point 0 in the sense that the diffusion coefficient $D = ax$ vanishes at $x = 0$. Feller shows that this forces the equation to change its character completely depending on whether $c \le 0$ (exit), $0 < c < a$ (regular), $c \ge a$ (entrance). Roughly speaking, it means that in the first and third case one has that the desired solution (positivity preserving and norm decreasing/preserving) is uniquely determined by its initial values $u(0,x)$ for $x > 0$, while in the second case one no longer has this uniqueness and the boundary values such as $u(t,0)$ or $u_x(t,0)$ need be imposed to obtain it. In terms of sample paths of the process X to which Feller does not refer explicitly, it means that the singularity of the diffusion coefficient in the first and third case determines the sample path behaviour at the boundary uniquely, while in the second case it leaves the sample path behaviour at the boundary unspecified. Singular equations of that kind were not studied before Feller's paper [10] and the natural question that Feller takes up in the 1950s is to understand and classify all possible boundary behaviour of one-dimensional diffusion processes associated with the backward and forward equations (both singular and non-singular i.e. regular). Singularity also refers to non-existence of a finite limit of the drift or diffusion coefficient when the boundary is approached from the interior. Feller's main tool in [10] was Laplace transform but he also

[11] Paul Pierre Lévy (1886–1971) was a French mathematician.

states on page 174 that 'Our result is also of some interest for the *theory of semigroups*[12] and is a counterpart to the investigations of Yosida and Hille [21][13] who have studied the semigroups defined over $-\infty < x < \infty$ by Fokker-Planck equations with singularities at infinity'.

A year later Feller publishes his fundamental paper [11] in which he discloses a full classification of all possible boundary behaviour for one-dimensional diffusions. In his paper Feller studies the backward and forward equations on a general bounded or unbounded interval of the real line and on pages 468–469 states that 'The use of *semigroup methods* for partial differential equations was initiated by Hille [21]' and then continues 'It seems that hitherto only the question has been asked whether the right side [of the forward equation] is the (or possibly the extension of the) infinitesimal generator of a semigroup. This amounts to requiring *uniqueness* of the solution of the initial value problem. Actually, the case of uniqueness for [the forward equation] is most exceptional, and the main problems relate to the case of *non-uniqueness*'. It is exactly this non-uniqueness that is the essence of Feller' problem which enables him to depart from previous research. After addressing the method of proof with reference to semigroup theory Feller concludes on page 470 that 'Curious phenomena are encountered. The general lateral condition for [the backward equation] depends on bounded linear functionals and on differential operators of *second order*. By contrast, the lateral condition of the adjoint problem is always of a local character and depends on differential operators of at most *first order*'. This is the first reference to the *second order* conditions (see equation (19.4) on page 505 for the explicit form). Feller will rarely point this out in his papers again probably because a fuller understanding of this condition will only be obtained in the 1960s as we shall see below. When addressing the problem of uniqueness Feller reveals on pages 472-473 that 'In [7] the general case of time dependent coefficients [of the backward and forward equation] is treated. In the time independent case greater elegance can be achieved by the *semigroup method* and this approach has been used by Yosida [42] and, more specifically, by Hille [21], [22]. Both authors limit themselves to a discussion of conditions which guarantee uniqueness'. In the footnote to Hille's paper [21] Feller acknowledges that 'Its manuscript (which was available to [Feller] in the Fall of 1949) drew [Feller's] attention to the possibilities opened by the study of the resolvent'. That realisation was crucial for Feller. It opened a plethora of possibilities that he will tirelessly explore. I will now briefly explain the method that Feller applied in his papers.

When the Markov process X is time-homogeneous then $p(s,x,t,y) = p(0,x,t-s,y)$ for $s<t$. Setting $p = p(t,x) := p(0,x,t,y)$ this shows that the backward equation (3) reads

$$p_t = \mu p_x + D p_{xx} \tag{7}$$

for any y given and fixed. Imposing the initial condition $u(0,x) = f(x)$ and integrating over y one sees that the solution to (7) can be expressed as

$$u(t,x) = \mathbb{E}_x f(X_t) =: \mathsf{P}_t f(x) \tag{8}$$

where $X_0 = x$ under $\mathbb{P}_x$. When f is the indicator function of a (measurable) set then the solution (8) can be interpreted as the transition probability but (8) also makes sense for other functions f including continuous ones (which approximate others). The Markov property of X then reads

$$\mathsf{P}_{t+s} = \mathsf{P}_t\,\mathsf{P}_s \tag{9}$$

for $t,s \geq 0$ with $\mathsf{P}_0 = \mathsf{I}$ showing that $(\mathsf{P}_t)_{t\geq 0}$ defines a *semigroup* of operators. To describe its action one considers the *infinitesimal generator* defined by

$$\mathsf{L}f := \lim_{t\downarrow 0} \frac{\mathsf{P}_t f - f}{t} = (\mu\partial_x + D\partial_{xx})f \tag{10}$$

[12] All emphases in the quotations throughout are my own (the original emphases are not reproduced).

[13] Citations in the quotations throughout point at the end of this article (the original numbers are rewritten).

for f in its domain $D(\mathsf{L})$ where the limit exists (in the sense to be specified). Another action of interest is expressed by the *resolvent* defined by

$$\mathsf{R}_\lambda f := (\lambda \mathsf{I} - \mathsf{L})^{-1} f = \int_0^\infty \mathrm{e}^{-\lambda t}\, \mathsf{P}_t f\, dt \tag{11}$$

for $\lambda > 0$ representing the Laplace transform of the semigroup. Then under some mild regularity conditions one establishes the following identity

$$D(\mathsf{L}) = Range(\mathsf{R}_\lambda) \tag{12}$$

for any $\lambda > 0$. This is the key fact for Feller's approach: *To describe the domain of the infinitesimal generator* L *it is sufficient (and essentially the same as) to describe the range of the resolvent* R_λ *for any* $\lambda > 0$. This is what Feller does and the description is accomplished in terms of *boundary conditions* when the state space has finite boundaries in particular. It means that f satisfying these boundary conditions belongs to $D(\mathsf{L})$ and hence letting $h \downarrow 0$ in

$$\frac{u(t+h,x) - u(t,x)}{h} = \left(\frac{\mathsf{P}_{t+h} - \mathsf{P}_t}{h}\right) f(x) = \left(\frac{\mathsf{P}_h - \mathsf{I}}{h}\right) \mathsf{P}_t f(x) \tag{13}$$

$$\frac{u(t+h,x) - u(t,x)}{h} = \left(\frac{\mathsf{P}_{t+h} - \mathsf{P}_t}{h}\right) f(x) = \mathsf{P}_t \left(\frac{\mathsf{P}_h - \mathsf{I}}{h}\right) f(x) \tag{14}$$

one obtains the backward and forward equations

$$u_t = \mathsf{L}u \;(= \mathsf{L}\mathsf{P}_t f) \tag{15}$$

$$u_t = \mathsf{L}^* u \;(= \mathsf{P}_t \mathsf{L} f) \tag{16}$$

as adjoints to each other. Thus the problem reduces to determine the range of the resolvent (11) and this is what Feller does in his papers. Once this is accomplished the domain of the infinitesimal generator is determined and boundary conditions satisfied by functions belonging to the domain can be read off. The infinitesimal generator and its domain generate a unique probability measure on the space of path functions and those functions in a (minimal) set of probability one then form the sample paths of the process. This measure-theoretic aspect of the general picture is mostly implicit in Feller's papers and will only be clarified later in fuller generality. The key point in Feller's approach to boundary classification is to insist that one looks for solutions of the backward and forward equation in the form (8). This assumption is so natural from the standpoint of probability that it is hard to imagine that anything else would be possible in a more canonical form. We shall return to this issue at the end of this review.

Equipped with this machinery Feller classifies all boundary behaviour in his Section 11 of [11] into (i) *regular* (non-singular), (ii) *exit*; (iii) *entrance*; and (iv) *natural*. The conditions are expressed in terms of the infinitesimal characteristics of the process (drift and diffusion coefficient). In the footnote of the same page 487 where this classification is stated Feller acknowledges that 'In the case of $r_1 = -\infty$, $r_2 = +\infty$ [i.e. when the state space equals the entire real line] E. Hille [22] has, by different methods, discovered that the (exit) boundary condition (ii) is necessary and sufficient for the uniqueness for the forward equation and the (entrance) boundary condition (iii) is necessary and sufficient for the uniqueness for the backward equation'. Feller discusses the implications of the boundary classification on the backward and forward equations in his Section 23 of [11]. To express the boundary conditions of Section 11 he introduces at the bottom of page 515 ('following Hille') what is later to be termed as the (derivative of the) *scale function* s. Feller's discussion is mainly analytic in this section/paper and he makes no attempt to interpret the results in terms of a sample path of the process. Regular boundaries are briefly addressed at the end of the paper. On page 518 Feller writes ‘If r_1 is a natural but r_2 a regular boundary, then there exist *infinitely many* solutions for the initial value problem both for [the backward and forward equation]’. He then continues ‘It is curious that the lateral condition for [the backward equation] involves a differential operator of *second order* and terms of global character, whereas the lateral conditions determining the adjoint problem are always of the simple form [specified by two conditions in the paper]’. Clearly these ‘second order’ conditions have been puzzling Feller quite a bit, however he was ahead of his time as early as in 1952. A curious fact in this regard is that Feller allows his diffusion process to jump into the interior of the state space after hitting the boundary. It might be that this was partly motivated by what Feller states on page 474 of his paper [11] that ‘Various probability considerations and a longstanding problem in genetics made it increasingly clear that, in addition to the *classical* types of boundary conditions there exist some of a *new* type. Physically speaking, there exist diffusion processes where a particle can be absorbed, and stay absorbed for a finite time, after which it penetrates *slowly* back into the interior. Attempts at a mathematical formulation of such phenomena remained *unsuccessful* for several years. It is now *possible* to identify one of our new boundary conditions as the one describing this phenomenon’. This might be viewed as another hint of sticky (or slowly reflecting) boundary behaviour, however, it is not entirely clear what Feller meant since a strong Markov process must leave a holding point by a jump (as it is well known now). If the jump density is highly concentrated in the interior close to the boundary then this could create the impression that the particle virtually ‘penetrates *slowly* back into the interior’ although strictly speaking jumps off the boundary are still present.

Feller addresses probabilistic interpretations of the analytic results from his 1952 paper [11] in his 1954 paper [12]. In that paper he also attempts to give a rigorous definition of a *diffusion process* and on pages 1-2 states ‘Probably it soon will be possible to define diffusion processes as Markov processes with continuous path functions’. The adjective ‘strong’ is not present in front of ‘Markov’ as it will only be recognised as a necessity through the work of Dynkin [4] and Hunt [23] a couple of years later. In his paper Feller refers to X_t as ‘the position of the particle at time t’ and states that the sample paths $t \mapsto X_t$ are continuous (except perhaps at the boundary). Feller discusses the existence of a stochastic processes associated with the backward and forward equations. These elements demonstrate Feller’s shift from the almost purely analytic view to a more complete probabilistic understanding. His analysis includes absorption at the boundary, reinsertion in the interior, termination (killing), instantaneous reflection, and elastic boundary behaviour. In relation to the latter Feller states on page 4 that his ‘limiting procedure sheds some new light on the *elastic barrier* processes, but does not give a measure-theoretic description of the path functions near the boundary’. Later he will form a conjecture on the latter that will be proved by Itô and McKean (see below). No mention of sticky (or slowly reflecting) boundary behaviour is made in Feller’s paper. I am not aware of any conjecture on that either. It seems that Feller had no more definite idea on how to construct sticky processes (we will return to this point in the final section below). Feller concludes on page 4 that ‘the diffusion processes constructed in Section 12 [of his paper] represent the most general type of diffusion process in the open interval (r_1, r_2). For the closed interval we have *no* such complete result’.

In his 1957 paper [15] Feller establishes the canonical infinitesimal generator form

$$\mathsf{L} = \frac{d}{dm}\frac{d}{ds} \tag{17}$$

where s is the *scale function* (due to Hille) and m is the *speed measure* (original to Feller) with probabilistic interpretations given in [12, p. 25] and [13, p. 95]+[12, p. 12] respectively. Feller writes on page 459 of his paper that 'It will be observed that (17) is much more general than the final expression in (10)'. He then states that 'This generalization introduces a remarkable simplification and unification of the theory'. It is interesting that Feller relates the speed measure to a vibrating string. On page 460 he writes 'The latter [speed measure] plays an important role in the theory of the vibrating string, m representing the distribution of the actual mass'. He then states that 'As a matter of curiosity it is possible to treat the motion of a continuous string with the entire mass concentrated at rational points' referring to his paper [14] with McKean. In terms of a diffusion process X it is known that large m corresponds to slow motion of X. This finally provides some clue to the sticky (slowly reflecting) boundary behaviour, however no explicit claim appears in the paper and no further probabilistic discussion is given either. Feller also writes a number of other papers in the 1950s related to his work in [11], [12], [15] however these three papers remain central. Among others in his 1959 paper [17] Feller extends the canonical infinitesimal generator form (17) to include a *killing measure* (corresponding to disappearance of Brownian particles due to chemical reactions for instance).

The state of the art on the boundary classification established in Feller's papers (from the standpoint of the semigroup theory) by the end of 1950s was as follows. All possible boundary conditions were known in their analytic form. Some of these were classical (Dirichlet, Neumann, Robin) and some were new (like those involving the second derivative). A natural question considered already by Feller in his 1954 paper [12] was to construct the underlying diffusion process corresponding to the given boundary condition and in this way obtain a fuller understanding of its meaning. In fact the entire boundary classification problem considered by Feller could be recast in these purely probabilistic terms. To illustrate the idea consider the simplest case of a standard Brownian motion and the state space $[0,\infty)$. The problem then is to describe/construct all strong Markov continuous processes X with values in $[0,\infty)$ that behave like a standard Brownian motion when off zero in $(0,\infty)$. In this case 0 is a regular boundary point and Feller's result states that the functions belonging to the domain of the infinitesimal generator of X can be characterised by the condition

$$p_1 f(0) - p_2 f'(0) + p_3 f''(0) = 0 \tag{18}$$

for some $p_1, p_2, p_3 \geq 0$ satisfying $p_1 + p_2 + p_3 = 1$. The case $f(0) = 0$ (Dirichlet condition) corresponds to X being killed upon reaching zero (i.e. moved to a coffin state); the case $f'(0) = 0$ (Neumann condition) corresponds to X being instantaneously reflected upon reaching zero; the case $f(0) = (p_2/p_1) f'(0)$ (Robin condition) corresponds to an elastic boundary behaviour of X at zero; the case $f''(0) = 0$ (Feller condition) corresponds to X being absorbed upon reaching zero (infinite stickiness); the case $f'(0) = (p_3/p_2) f''(0)$ (Feller condition) corresponds to X being stickily (slowly) reflected upon reaching zero; the case $f(0) = -(p_3/p_2) f''(0)$ (Feller condition) corresponds to X being absorbed upon reaching zero and then killed after some independent exponentially distributed time has elapsed; the general case (18) with $p_1, p_2, p_3 \neq 0$ corresponds to X being stickily (slowly) reflected upon reaching zero, then absorbed at zero after some independent exponentially distributed time has elapsed, and then finally killed after another independent exponentially distributed time has elapsed. All these cases can be expressed more systematically in terms of the speed measure and the killing measure (see e.g. [1, Chapter II] for a modern exposition). Elastic and sticky (slowly reflecting) constructions were not known when Feller had finished his work on the paper [15].

Shortly afterwards Feller came up with a conjecture on how to construct a diffusion process exhibiting *elastic* boundary behaviour. This has been recalled by Itô in his memoirs [26, pp 1-2] who spent the years 1954-56 at Princeton: 'Feller had just finished his works on the most general one-dimensional diffusion process, especially representing its local generator as (17) by means of a canonical scale function s and a speed measure m. I learned about these from

Henry McKean, a graduate student of Feller, while I explained my previous work to McKean. There was once an occasion when McKean tried to explain to Feller my work on the stochastic differential equations along with the above mentioned idea of tangent. It seemed to me that Feller did not fully understand its significance, but when I explained Lévy's *local time* to Feller, he immediately appreciated its relevance to the study of the one-dimensional diffusion. Indeed, Feller later gave us a *conjecture* that the Brownian motion on $[0,\infty)$ with an *elastic* boundary condition could be constructed from the reflecting barrier Brownian motion by killing its local time ℓ^0 at the origin by an independent exponentially distributed random time, which was eventually substantiated in my joint paper with McKean [27] published in 1963'. Section 10 (pp 200–201) of that paper contains the elastic Brownian motion construction described above. Itô and McKean state on page 184 of their paper that 'M. Kac [29] cited the problem of describing the sample paths of the elastic Brownian motion'. In that paper of Kac I could not find a direct reference to the elastic boundary behaviour; however, when discussing one drawback of his approach on page 208 in relation to the Neumann condition on Γ he states that 'one can again appeal to the theory of Brownian motion except that the *absorbing barrier* Γ must now be replaced by the *reflecting barrier*. Unfortunately, no rigorous treatment of reflecting barriers seems to be available'. Feller published his paper [9] in the same proceedings as Kac and most likely was familiar with these claims.

It seems that Feller had no conjecture on how to construct a diffusion process exhibiting *sticky* boundary behaviour. Equipped with the concept of the local time mentioned above (which is an additive functional), and the result by Volkonskii [40] that a Markov process run by the clock obtained as the inverse of an additive functional preserves the Markov property, it was also accomplished by Itô and McKean. In his memoirs [26, p. 3] Itô recalls the background and relevance of this construction: 'My joint paper [27] with McKean in 1963 gave a probabilistic construction of the Brownian motion on $[0,\infty)$ subjected to the most general boundary condition whose analytic study had been established by Feller under some restrictions. Our methods involved the probabilistic idea originated in Lévy about the local time and excursions away from 0'. He then continues by explaining how the idea was extended in his fundamental paper [25] on excursion theory.

The sample-path approach to one-dimensional diffusions (return to Einstein) has been completed in the book by Itô and McKean [28]. In his memoirs [26, p. 2] Itô explains their key insight in relation to Feller's vision (with the mathematical notation slightly adjusted to the present style): 'A popular saying by Feller goes as follows: A one-dimensional diffusion traveler X makes a trip in accordance with the road map indicated by the scale function s and with the speed indicated by the measure m appearing in the generator L of X. This was substantiated in my joint book with McKean in the following fashion. Given a one-dimensional standard Brownian motion X which corresponds to $ds = dx$ and $dm = 2dx$, consider its local time ℓ^x at $x \in I\!R$ and the additive functional A defined by $A_t = \int_{I\!R} \ell_t^x \, m(dx)$ for $t \geq 0$. Then the time-changed process X_T by means of the inverse T of A turns out to be the diffusion governed by the generator $d^2/dm\,dx$'. In other words, recalling that the scale function s composed with the diffusion process X produces a new diffusion process (local martingale) on natural scale (for which the scale function is identity), this argument has established that each one-dimensional diffusion process can be obtained as a scale-&-time-changed standard Brownian motion [i.e. a standard Brownian motion run by a new clock (controlled by the speed measure) in a new path space (shaped by the scale function)]. Section 5.7 in Itô-McKean's book [28] entitled 'Feller's Brownian motions' provides a review of Feller's boundary classification results for Brownian motion in $(0,\infty)$ including the sample path constructions of elastic and sticky (slowly reflecting) processes. It remains the most complete book on the topic to date.

4 Further developments

In this section I will briefly address two subsequent developments among many others present in the literature. The first one deals with stochastic differential equations (in the sense of Itô's integral [24]) for *sticky* reflecting Brownian motion. The second one deals with the initial boundary value of Stroock and Williams [38] to which Feller's *semigroup approach* is *not* applicable in some cases of the parameters involved.

1. Recall that *sticky* boundary behaviour of diffusion processes was discovered by Feller in his 1952 paper [11]. The problem considered by Feller was to describe domains of the infinitesimal generators associated with strong Markov processes X in $[0,\infty)$ that behave like a standard Brownian motion B while in $(0,\infty)$. Using a semigroup approach Feller showed that a possible boundary behaviour of X at 0 is described by the following condition

$$f'(0+) = \tfrac{1}{2\mu} f''(0+) \tag{19}$$

where $\mu \in (0,\infty)$ is a given and fixed constant. In addition to the classic boundary conditions of (i) Dirichlet ($f(0+)=0$), (ii) Neumann ($f'(0+)=0$) and (iii) Robin ($f'(0+)=(1/2\mu)f(0+)$) corresponding to (i) killing, (ii) instantaneous reflection and (iii) elastic boundary behaviour of X at 0 respectively, the key novelty of the condition (19) is the appearance of the second derivative $f''(0+)$ measuring the 'stickiness' of X at 0.

The problem of how to construct the sample paths of X in a canonical manner was solved by Itô and McKean [27, Section 10]). They showed that X can be obtained from the reflecting Brownian motion $|B|$ by the time change $t \mapsto T_t := A_t^{-1}$ where $A_t = t+(1/\mu)\ell_t^0(B)$ for $t \geq 0$ and $\ell^0(B)$ is the local time of B at 0. When $X = |B_T|$ is away from 0, then both the additive functional A and the 'new clock' T run ordinarily (like the 'old clock') and hence the motion of X is the same as the motion of $|B|$. When X is at 0 however, then A gets an extra increase from the local time $\ell^0(B)$ causing the new clock T to slow down and forcing X to stay longer at 0 (in comparison with the reflecting Brownian motion $|B|$ under the old clock). Thus the two processes X and $|B|$ have the same sample paths and the two motions differ only by their speeds (the former being slower).

The problem whether the sticky reflecting Brownian motion arising from (19) above can be obtained from a stochastic differential equation (SDE) driven by B has been considered in the literature. Skorokhod[14] conjectured in the 1980s that the stochastic differential equation for the sticky reflecting Brownian motion X has no strong solution. Building on the Itô–McKean construction one finds that the SDE system for X is given as follows:

$$dX_t = \tfrac{1}{2} d\ell_t^0(X) + I(X_t > 0)\, dB_t \tag{20}$$

$$I(X_t = 0)\, dt = \tfrac{1}{2\mu} d\ell_t^0(X) \tag{21}$$

where $X_0 = x$ in $[0,\infty)$ and $\ell^0(X)$ is the local time of X at 0. Moreover one can verify using the Itô-Tanaka formula that the system (20)+(21) is equivalent to the single equation

$$dX_t = \mu I(X_t = 0)\, dt + I(X_t > 0)\, dB_t \tag{22}$$

obtained by incorporating (21) into (20). Then it can be shown that (i) the system (20)+(21) or equivalently the equation (22) has a jointly unique weak solution, and (ii) the system (20)+(21) or equivalently the equation (22) has no strong solution thus verifying Skorokhod's conjecture. For details see [6] and the references therein.

[14] Anatoliy Volodymyrovych Skorokhod (1930–2011) was a Soviet and Ukrainian mathematician.

2. Consider the initial boundary value problem

$$u_t = \mu u_x + \tfrac{1}{2} u_{xx} \quad (t > 0, x \geq 0) \tag{23}$$

$$u(0,x) = f(x) \quad (x \geq 0) \tag{24}$$

$$u_t(t,0) = \nu\, u_x(t,0) \quad (t > 0) \tag{25}$$

of Stroock and Williams [38] where $\mu, \nu \in I\!R$ and the boundary condition is not of Feller's type ([11], [12], [15]) when $\nu < 0$. If $\nu > 0$ then it is known that the solution to (23)-(25) with $f \in C_b([0,\infty))$ can be represented as

$$u(t,x) = \mathbb{E}_x f(\tilde{X}_t) \tag{26}$$

where $\tilde{X}$ starts at x under $\mathbb{P}_x$, behaves like Brownian motion with drift μ when in $(0,\infty)$, and exhibits a sticky boundary behaviour at 0. The process $\tilde{X}$ can be constructed by a time change of the reflecting Brownian motion X with drift μ (the inverse of the running time plus the local time of X at 0 divided by ν) forcing it to spend more time at 0 (cf. [27, p. 186]). If $\nu = 0$ then (26) remains valid with $\tilde{X}$ being absorbed at 0 (corresponding to the limiting case of infinite stickiness). If $\nu < 0$ then Feller's semigroup approach ([11], [12], [15]) is no longer applicable since the speed measure of $\tilde{X}$ cannot be negative. Stroock and Williams [38] show that the minimum principle breaks down in this case (non-negative f can produce negative u) so that the solution to (23)-(25) cannot be represented by (26) where $\tilde{X}$ is a strong Markov process which behaves like Brownian motion with drift μ when in $(0,\infty)$.

Inspired by their insights in the recent paper [35] we develop an entirely different approach to solving (23)-(25) probabilistically that applies to smooth initial data f vanishing at ∞. Firstly, exploiting higher degrees of smoothness of the solution u in the interior of the domain (which is a well-known fact from the theory of parabolic PDEs) we reduce the *sticky* boundary behaviour at 0 to (i) a *reflecting* boundary behaviour when $\nu = \mu$ and (ii) an *elastic* boundary behaviour when $\nu \neq \mu$. Secondly, writing down the probabilistic representations of the solutions to the resulting initial boundary value problems expressed in terms of the reflecting Brownian motion with drift μ and its local time at 0, choosing joint realisations of these processes where the initial point is given explicitly so that the needed algebraic manipulations are possible (making use of the extended Lévy's distributional theorem), we find that the following probabilistic representation of the solution is valid

$$u(t,x) = \mathbb{E}_x F(X_t, \ell_t^0(X)) \tag{27}$$

where X is a reflecting Brownian motion with drift μ starting at x under $\mathbb{P}_x$, and $\ell^0(X)$ is the local time of X at 0. The function F is explicitly given by

$$F(x,\ell) = f(x) - f'(x) \int_0^\ell e^{-2(\nu-\mu)s}\, ds \tag{28}$$

for $x \geq 0$ and $\ell \geq 0$. The derivation applies simultaneously to all μ and ν with no restriction on the sign of ν, and the process X (with its local time) plays the role of a fundamental solution in this context (a building block for all other solutions).

Since $(X, \ell^0(X))$ is a Markov process we see that the solution u is generated by the semigroup of transition operators $(\mathsf{P}_t)_{t\geq 0}$ acting on f by means of (27) and (28) (in the reverse order). Moreover, it is clear from (27) and (28) that the solution can be interpreted in terms of X and its *creation* in 0 at rate proportional to $\ell^0(X)$. Note that this also holds when $\nu < 0$ in which case Feller's semigroup approach based on the probabilistic representation (26) is not applicable. Finally, invoking the law of $(X_t, \ell_t^0(X))$ we derive a closed integral formula for u expressed in terms of μ, ν and f. Integrating further by parts yields a closed formula for u where smoothness of f is no longer needed (see [35] for details).

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Goran Peskir
School of Mathematics
The University of Manchester
Oxford Road
Manchester M13 9PL
United Kingdom
goran@maths.man.ac.uk