

Quickest Real-Time Detection of a Brownian Coordinate Drift

Philip A. Ernst & Goran Peskir

Consider the motion of a Brownian particle in two or more dimensions, whose coordinate processes are standard Brownian motions with zero drift initially, and then at some random/unobservable time, one of the coordinate processes gets a (known) non-zero drift permanently. Given that the position of the Brownian particle is being observed in real time, the problem is to detect the time at which a coordinate process gets the drift as accurately as possible. We solve this problem in the most uncertain scenario when the random/unobservable time is (i) exponentially distributed and (ii) independent from the initial motion without drift. The solution is expressed in terms of a stopping time that minimises the probability of a false early detection and the expected delay of a missed late detection. To our knowledge this is the first time that such a problem has been solved exactly in the literature.

1. Introduction

Imagine the motion of a Brownian particle in two or more dimensions, whose coordinate processes are standard Brownian motions with zero drift initially, and then at some random/unobservable time θ , one of the coordinate processes gets a (known) non-zero drift μ permanently. Assuming that the position of the Brownian particle is being observed in real time, the problem is to detect the time θ at which a coordinate process gets the drift μ as accurately as possible. The purpose of the present paper is derive the solution to this problem in the most uncertain scenario when θ is assumed to be (i) exponentially distributed and (ii) independent from the initial motion without drift.

Denoting the position of the Brownian particle in two or more dimensions by X , the error to be minimised over all stopping times τ of X is expressed as the linear combination of the probability of the *false alarm* $\mathbb{P}_\pi(\tau < \theta)$ and the expected *detection delay* $\mathbb{E}_\pi(\tau - \theta)^+$ where $\pi \in [0, 1]$ denotes the probability that θ has already occurred at time 0. This problem formulation of quickest detection dates back to [18] and has been extensively studied to date (see [20] and the references therein). The linear combination represents the Lagrangian and once the optimal stopping problem has been solved in this form it will also lead to the solution of the constrained problems where an upper bound is imposed on either the probability of the false alarm or the expected detection delay respectively.

Mathematics Subject Classification 2010. Primary 60G40, 60J65, 60H30. Secondary 35J15, 45G10, 62C10.

Key words and phrases: Quickest detection, Brownian motion, optimal stopping, elliptic partial differential equation, free-boundary problem, smooth fit, nonlinear Fredholm integral equation, the change-of-variable formula with local time on surfaces.

A canonical example is the standard Brownian motion in one dimension with one *constant* drift changing to another. This problem has also been solved in finite horizon (see [6] and the references therein). Books [19, Section 4.4] and [14, Section 22] contain expositions of these results and provide further details and references. The *signal-to-noise ratio* (defined as the difference between the new drift and the old drift divided by the diffusion coefficient) in all these problems is *constant* so that the resulting optimal stopping problem for the posterior probability distribution ratio process Φ of θ given X is *one-dimensional*. A more general problem formulation for diffusion processes X in one dimension when one *non-constant* drift changes to another has been considered in [7]. A specific problem of this kind when X is a Bessel process has been solved in [9]. The signal-to-noise ratio in these problems is not constant and the resulting optimal stopping problem for Φ coupled with X (to make it Markovian) is *two-dimensional*. The infinitesimal generator of the Markov/diffusion process (Φ, X) in these problems is of *parabolic* type.

Related quickest detection problems for X in two dimensions have been studied in [1] and [2]. The change of probabilistic characteristics in these problems can affect both coordinate processes of X and not only one as in the present paper. The coordinate processes of X in [1] are Poisson processes and the resulting two-dimensional optimal stopping problem for Φ has been studied using an iteration technique. The coordinate processes in [2] are Wiener/Poisson processes and the resulting optimal stopping problem for Φ is one-dimensional.

The quickest detection setting of the observed process X in two or more dimensions may also be viewed as a *multi-channel sensor system*. Quickest detection problems of this kind in two or more dimensions have been studied in a number of papers (see [21] & [5] and the references therein). These papers usually establish ‘asymptotic optimality’ of an ‘ad-hoc’ stopping rule and no ‘exact’ (optimal) solution has been derived in the literature to date.

In contrast to the quickest detection problems solved to date, we will see below that the *multi-dimensional* Markov/diffusion process Φ in the quickest detection problem of the present paper has the infinitesimal generator of *elliptic* type. Finding the *exact* solution to the quickest detection problem for the observed process X in *two or more* dimensions is the main contribution of the present paper. To our knowledge this is the first time that such a problem has been solved exactly in the literature.

2. Formulation of the problem

In this section we formulate the quickest detection problem under consideration. The initial formulation of the problem will be reevaluated under a change of measure in the next section. To simplify the exposition we will assume throughout that the observed process is *two-dimensional*. This assumption will be extended to *three or more dimensions* in the final section below.

1. We consider a Bayesian formulation of the problem where it is assumed that one observes a sample path of the standard two-dimensional Brownian motion $X = (X^1, X^2)$, whose coordinate processes X^1 and X^2 are standard Brownian motions with zero drift initially, and then at some random/unobservable time θ taking value 0 with probability $\pi \in [0, 1]$ and being exponentially distributed with parameter $\lambda > 0$ given that $\theta > 0$, one of the coordinate processes X^1 and X^2 gets a (known) non-zero drift μ permanently. The problem is to detect the time θ at which a coordinate process gets the drift μ as accurately as possible (neither too

early nor too late). This problem belongs to the class of quickest real-time detection problems as discussed in Section 1 above.

2. The observed process $X = (X^1, X^2)$ solves the stochastic differential equations

$$(2.1) \quad dX_t^1 = \mu I(\beta=1, t \geq \theta) dt + dB_t^1$$

$$(2.2) \quad dX_t^2 = \mu I(\beta=2, t \geq \theta) dt + dB_t^2$$

driven by a standard two-dimensional Brownian motion $B = (B^1, B^2)$ under the probability measure \mathbb{P}_π specified below, where the random variable β satisfies $\mathbb{P}_\pi(\beta=1) = p_1$ and $\mathbb{P}_\pi(\beta=2) = p_2$ for some $p_1, p_2 \in [0, 1]$ with $p_1 + p_2 = 1$ given and fixed, meaning that $\beta = i$ if and only if the coordinate process X^i gets drift μ at time θ with probability p_i for $i = 1, 2$. The unobservable time θ , the unknown coordinate β , and the driving Brownian motion B are all assumed to be independent under \mathbb{P}_π for $\pi \in [0, 1]$ given and fixed.

3. Standard arguments imply that the previous setting can be realised on a probability space $(\Omega, \mathcal{F}, \mathbb{P}_\pi)$ with the probability measure \mathbb{P}_π being decomposable as follows

$$(2.3) \quad \mathbb{P}_\pi = p_1 \pi \mathbb{P}_1^0 + p_2 \pi \mathbb{P}_2^0 + p_1 (1-\pi) \int_0^\infty \lambda e^{-\lambda t} \mathbb{P}_1^t dt + p_2 (1-\pi) \int_0^\infty \lambda e^{-\lambda t} \mathbb{P}_2^t dt$$

for $\pi \in [0, 1]$ where \mathbb{P}_i^t is the probability measure under which the coordinate process X^i gets drift μ at time $t \in [0, \infty)$ for $i = 1, 2$. The decomposition (2.3) expresses the fact that the unobservable time θ is a non-negative random variable satisfying $\mathbb{P}_\pi(\theta = 0) = \pi$ and $\mathbb{P}_\pi(\theta > t | \theta > 0) = e^{-\lambda t}$ for $t > 0$. Thus $\mathbb{P}_i^t(X \in \cdot) = \mathbb{P}_\pi(X \in \cdot | \beta = i, \theta = t)$ is the probability law of the standard two-dimensional Brownian motion process $X = (X^1, X^2)$ whose coordinate process X^i gets drift μ at time $t \in [0, \infty)$ for $i = 1, 2$. To remain consistent with this notation we also denote by \mathbb{P}_i^∞ the probability measure under which the coordinate process X^i of the observed process $X = (X^1, X^2)$ gets no drift μ at a finite time for $i = 1, 2$. Thus $\mathbb{P}_i^\infty(X \in \cdot) = \mathbb{P}_\pi(X \in \cdot | \beta = i, \theta = \infty)$ is the probability law of the standard two-dimensional Brownian motion process for $i = 1, 2$. Clearly the subscript i is superfluous in this case and we will often write \mathbb{P}^∞ instead of \mathbb{P}_i^∞ for $i = 1, 2$. Moreover, by \mathbb{P}_i we denote the probability measure under which the coordinate process X^i gets drift μ at time θ for $i = 1, 2$. From (2.3) we see that

$$(2.4) \quad \mathbb{P}_\pi = p_1 \mathbb{P}_1 + p_2 \mathbb{P}_2$$

where $\mathbb{P}_i = \pi \mathbb{P}_i^0 + (1-\pi) \int_0^\infty \lambda e^{-\lambda t} \mathbb{P}_i^t dt$ for $i = 1, 2$ and $\pi \in [0, 1]$. Note that \mathbb{P}_i depends on $\pi \in [0, 1]$ as well but we will omit this dependence from its notation for $i = 1, 2$.

4. Being based upon continuous observation of $X = (X^1, X^2)$, the problem is to find a stopping time τ_* of X (i.e. a stopping time with respect to the natural filtration $\mathcal{F}_t^X = \sigma(X_s | 0 \leq s \leq t)$ of X for $t \geq 0$) that is ‘as close as possible’ to the unknown time θ . More precisely, the problem consists of computing the value function

$$(2.5) \quad V(\pi) = \inf_{\tau} \left[\mathbb{P}_\pi(\tau < \theta) + c \mathbb{E}_\pi(\tau - \theta)^+ \right]$$

and finding the optimal stopping time τ_* at which the infimum in (2.5) is attained for $\pi \in [0, 1]$ and $c > 0$ given and fixed (recalling also that $p_1, p_2 \in [0, 1]$ with $p_1 + p_2 = 1$ are given and

fixed). Note in (2.5) that $\mathbb{P}_\pi(\tau < \theta)$ is the probability of the *false alarm* and $\mathbb{E}_\pi(\tau - \theta)^+$ is the expected *detection delay* associated with a stopping time τ of X for $\pi \in [0, 1]$. Recall also that the expression on the right-hand side of (2.5) is the Lagrangian associated with the constrained problems as discussed in Section 1 above.

5. To tackle the optimal stopping problem (2.5) we consider the *posterior probability distribution process* $\Pi = (\Pi_t)_{t \geq 0}$ of θ given X that is defined by

$$(2.6) \quad \Pi_t = \mathbb{P}_\pi(\theta \leq t | \mathcal{F}_t^X)$$

for $t \geq 0$. Note that we have

$$(2.7) \quad \Pi_t = \Pi_t^1 + \Pi_t^2$$

where we set

$$(2.8) \quad \Pi_t^1 = \mathbb{P}_\pi(\beta = 1, \theta \leq t | \mathcal{F}_t^X) \quad \& \quad \Pi_t^2 = \mathbb{P}_\pi(\beta = 2, \theta \leq t | \mathcal{F}_t^X)$$

for $t \geq 0$. The right-hand side of (2.5) can be rewritten to read

$$(2.9) \quad V(\pi) = \inf_{\tau} \mathbb{E}_\pi \left(1 - \Pi_\tau + c \int_0^\tau \Pi_t dt \right)$$

for $\pi \in [0, 1]$.

6. To connect the process Π to the observed process X we set

$$(2.10) \quad \bar{\Pi}_t^1 = \mathbb{P}_\pi(\beta = 1, \theta > t | \mathcal{F}_t^X) \quad \& \quad \bar{\Pi}_t^2 = \mathbb{P}_\pi(\beta = 2, \theta > t | \mathcal{F}_t^X)$$

and define the *posterior probability distribution ratio process* $\Phi = (\Phi^1, \Phi^2)$ of θ given X by

$$(2.11) \quad \Phi_t^1 = \frac{\Pi_t^1}{\bar{\Pi}_t^1} \quad \& \quad \Phi_t^2 = \frac{\Pi_t^2}{\bar{\Pi}_t^2}$$

for $t \geq 0$. Using (2.3) we find that

$$(2.12) \quad \Pi_t^i = p_i \pi \frac{d\mathbb{P}_{i,t}^0}{d\mathbb{P}_{\pi,t}} + p_i (1 - \pi) \int_0^t \lambda e^{-\lambda s} \frac{d\mathbb{P}_{i,t}^s}{d\mathbb{P}_{\pi,t}} ds$$

where $\mathbb{P}_{i,t}^s$ and $\mathbb{P}_{\pi,t}$ denote the restrictions of the measures \mathbb{P}_i^s and \mathbb{P}_π to \mathcal{F}_t^X for $s \geq 0$ and $i = 1, 2$ respectively. Similarly, using (2.3) we find that

$$(2.13) \quad \bar{\Pi}_t^i = p_i (1 - \pi) e^{-\lambda t} \frac{d\mathbb{P}_t^\infty}{d\mathbb{P}_{\pi,t}}$$

where \mathbb{P}_t^∞ and $\mathbb{P}_{\pi,t}$ denote the restrictions of the measures \mathbb{P}^∞ and \mathbb{P}_π to \mathcal{F}_t^X for $t \geq 0$ and $i = 1, 2$ (notice in this derivation that $d\mathbb{P}_{i,t}^s/d\mathbb{P}_{\pi,t} = d\mathbb{P}_t^\infty/d\mathbb{P}_{\pi,t}$ for $s \geq t$). From (2.12) and (2.13) we see that taking ratios as in (2.11) removes dependence on $\mathbb{P}_{\pi,t}$ which makes

explicit calculations possible. Indeed, using the Girsanov theorem we see that the *likelihood ratio process* $L = (L^1, L^2)$ can be expressed as follows

$$(2.14) \quad L_t^i = \frac{d\mathbf{P}_{i,t}^0}{d\mathbf{P}_t^\infty} = \exp\left(\mu X_t^i - \frac{\mu^2}{2}t\right)$$

for $t \geq 0$ and $i = 1, 2$. Moreover, using (2.12) and (2.13) we find by (2.11) that

$$(2.15) \quad \Phi_t^i = e^{\lambda t} L_t^i \left(\Phi_0^i + \lambda \int_0^t \frac{ds}{e^{\lambda s} L_s^i} \right)$$

with $\Phi_0^i = \pi/(1-\pi)$ for $t \geq 0$ and $i = 1, 2$ (notice in this derivation that $d\mathbf{P}_{i,t}^s/d\mathbf{P}_{i,t}^t = L_t^i/L_s^i$ for $s \leq t$). From (2.14) and (2.15) we see that the process $\Phi = (\Phi^1, \Phi^2)$ is an explicit (path-dependent) functional of the observed process $X = (X^1, X^2)$ and hence observable (by observing a sample path of X we are also seeing a sample path of Φ both in real time).

3. Measure change

In this section we show that changing the probability measure \mathbf{P}_π for $\pi \in [0, 1]$ to \mathbf{P}^∞ in the optimal stopping problem (2.5) or (2.9) provides crucial simplifications of the setting which make the subsequent analysis possible. This will be achieved by invoking the decomposition of \mathbf{P}_π into \mathbf{P}_1 and \mathbf{P}_2 as stated in (2.4) above, changing both probability measures \mathbf{P}_1 and \mathbf{P}_2 to \mathbf{P}_1^∞ and \mathbf{P}_2^∞ respectively, and recalling that both \mathbf{P}_1^∞ and \mathbf{P}_2^∞ coincide with \mathbf{P}^∞ .

1. We show that the optimal stopping problem (2.9) admits a transparent reformulation under the probability measure \mathbf{P}^∞ in terms of the process $\Phi = (\Phi^1, \Phi^2)$ defined by (2.11) above. Recall that Φ^i starts at $\pi/(1-\pi)$ and this dependence on the initial point will be indicated by a superscript to Φ^i when needed for $i = 1, 2$.

Proposition 1. *The value function V from (2.9) satisfies the identity*

$$(3.1) \quad V(\pi) = (1-\pi) [1 + c \hat{V}(\pi)]$$

where the value function \hat{V} is given by

$$(3.2) \quad \hat{V}(\pi) = \inf_{\tau} \mathbf{E}^\infty \left[\int_0^\tau e^{-\lambda t} \left(p_1 \Phi_t^{1, \pi/(1-\pi)} + p_2 \Phi_t^{2, \pi/(1-\pi)} - \frac{\lambda}{c} \right) dt \right]$$

for $\pi \in [0, 1)$ and the infimum in (3.2) is taken over all stopping times τ of X .

Proof. Let a (bounded) stopping time τ of X be given and fixed. Set

$$(3.3) \quad \tilde{\Pi}_t^1 = \mathbf{P}_1(\theta \leq t \mid \mathcal{F}_t^X) \quad \& \quad \tilde{\Pi}_t^2 = \mathbf{P}_2(\theta \leq t \mid \mathcal{F}_t^X)$$

for $t \geq 0$. We claim that

$$(3.4) \quad \mathbf{E}_\pi \left(1 - \Pi_\tau + c \int_0^\tau \Pi_t dt \right) = p_1 \mathbf{E}_1 \left(1 - \tilde{\Pi}_\tau^1 + c \int_0^\tau \tilde{\Pi}_t^1 dt \right) + p_2 \mathbf{E}_2 \left(1 - \tilde{\Pi}_\tau^2 + c \int_0^\tau \tilde{\Pi}_t^2 dt \right)$$

for $\pi \in [0, 1]$. For this, first note that (2.7) yields

$$(3.5) \quad \mathbf{E}_\pi \left(1 - \Pi_\tau + c \int_0^\tau \Pi_t dt \right) = 1 + \mathbf{E}_\pi \left(-\Pi_\tau^1 - \Pi_\tau^2 + c \int_0^\tau \Pi_t^1 dt + c \int_0^\tau \Pi_t^2 dt \right)$$

for $\pi \in [0, 1]$. Next note that (2.4) implies that

$$(3.6) \quad \mathbf{E}_\pi(\Pi_\tau^i) = \mathbf{P}_\pi(\beta = i, \theta \leq \tau) = p_i \mathbf{P}_i(\theta \leq \tau) = p_i \mathbf{E}_i(\tilde{\Pi}_\tau^i)$$

and similarly we find that

$$(3.7) \quad \begin{aligned} \mathbf{E}_\pi \left(\int_0^\tau \Pi_t^i dt \right) &= \mathbf{E}_\pi \left(\int_0^\infty \mathbf{P}_\pi(\beta = i, \theta \leq t \leq \tau | \mathcal{F}_t^X) dt \right) = \int_0^\infty \mathbf{P}_\pi(\beta = i, \theta \leq t \leq \tau) dt \\ &= p_i \int_0^\infty \mathbf{P}_i(\theta \leq t \leq \tau) dt = p_i \mathbf{E}_i \left(\int_0^\infty \mathbf{P}_i(\theta \leq t \leq \tau | \mathcal{F}_t^X) dt \right) \\ &= p_i \mathbf{E}_i \left(\int_0^\tau \tilde{\Pi}_t^i dt \right) \end{aligned}$$

for $\pi \in [0, 1]$ and $i = 1, 2$. Finally, combining (3.5)-(3.7) we obtain (3.4) as claimed.

Focusing on each of the two expectations on the right-hand side of (3.4) separately, and noticing that the enlargement of the filtration from $\mathcal{F}_t^{X^i}$ to \mathcal{F}_t^X for $t \geq 0$ creates no difficulty for $i = 1, 2$ because X^1 and X^2 are independent under both \mathbf{P}_1 and \mathbf{P}_2 , we see that the problem of establishing (3.1) and (3.2) reduces to one dimension. Hence applying the change-of-measure identity (4.12) from [9] to each of the two expectations on the right-hand side of (3.4) separately, we obtain

$$(3.8) \quad \mathbf{E}_i \left(1 - \tilde{\Pi}_\tau^i + c \int_0^\tau \tilde{\Pi}_t^i dt \right) = (1 - \pi) \left(1 + c \mathbf{E}_i^\infty \left[\int_0^\tau e^{-\lambda t} \left(\Phi_t^i - \frac{\lambda}{c} \right) dt \right] \right)$$

for $\pi \in [0, 1]$ and $i = 1, 2$. Recalling that \mathbf{P}_1^∞ and \mathbf{P}_2^∞ coincide with \mathbf{P}^∞ and inserting (3.8) into (3.4) we see that (3.1) and (3.2) hold as claimed. \square

2. From Proposition 1 we see that the optimal stopping problem (2.5) or (2.9) is equivalent to the optimal stopping problem (3.2). Using the fact pointed out in the proof above that Φ^i and $\tilde{\Phi}^i$ coincide for $i = 1, 2$, we see from (4.7) in [9] that Φ^1 and Φ^2 solve the following stochastic differential equations

$$(3.9) \quad d\Phi_t^1 = \lambda(1 + \Phi_t^1) dt + \mu \Phi_t^1 dB_t^1$$

$$(3.10) \quad d\Phi_t^2 = \lambda(1 + \Phi_t^2) dt + \mu \Phi_t^2 dB_t^2$$

under \mathbf{P}^∞ with $\Phi_0^1 = \varphi_1$ and $\Phi_0^2 = \varphi_2$ in $[0, \infty)$ both being equal to $\pi/(1-\pi)$ for $\pi \in [0, 1]$. The system of stochastic differential equations (3.9)-(3.10) has a unique strong solution given by (2.14)+(2.15) above. Hence the process $\Phi = (\Phi^1, \Phi^2)$ is both strong Markov and strong Feller (see e.g. [17, pp 158-163 & pp 170-173]). Basic properties of the one-dimensional diffusion processes Φ^1 and Φ^2 are reviewed in [11, Section 2]. In particular, it is known that Φ^i is *recurrent* in $[0, \infty)$ if and only if $\lambda \leq \mu^2/2$ for $i = 1, 2$. If $\lambda > \mu^2/2$ then Φ^i is *transient* in $[0, \infty)$ with $\Phi_t^i \rightarrow \infty$ almost surely under \mathbf{P}^∞ as $t \rightarrow \infty$ for $i = 1, 2$.

3. To tackle the equivalent optimal stopping problem (3.2) for the strong Markov process $\Phi = (\Phi^1, \Phi^2)$ solving (3.9)-(3.10) we will enable $\Phi = (\Phi^1, \Phi^2)$ to start at any point $\varphi = (\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty)$ under the probability measure $\mathbb{P}_\varphi^\infty$ so that the optimal stopping problem (3.2) extends as follows

$$(3.11) \quad \hat{V}(\varphi) = \inf_{\tau} \mathbb{E}_\varphi^\infty \left[\int_0^\tau e^{-\lambda t} \left(p_1 \Phi_t^1 + p_2 \Phi_t^2 - \frac{\lambda}{c} \right) dt \right]$$

for $\varphi \in [0, \infty) \times [0, \infty)$ with $\mathbb{P}_\varphi(\Phi_0 = \varphi) = 1$ where the infimum is taken over all stopping times τ of Φ and we recall that $p_1, p_2 \in [0, 1]$ with $p_1 + p_2 = 1$ are given and fixed. In this way we have reduced the initial quickest detection problem (2.5) or (2.9) to the optimal stopping problem (3.11) for the strong Markov process $\Phi = (\Phi^1, \Phi^2)$ solving (3.9)-(3.10) and being explicitly given by the Markovian flow (2.14)+(2.15) of the initial point $(\Phi_0^1, \Phi_0^2) = (\varphi_1, \varphi_2) =: \varphi$ in $[0, \infty) \times [0, \infty)$ under $\mathbb{P}_\varphi^\infty$. Note that the optimal stopping problem (3.11) is inherently/fully two-dimensional and the infinitesimal generator of $\Phi = (\Phi^1, \Phi^2)$ is of elliptic type as discussed in the next section.

4. Mayer formulation

The optimal stopping problem (3.11) is Lagrange formulated. In this section we derive its Mayer reformulation which is helpful in the subsequent analysis.

1. From (3.9)+(3.10) we read that the infinitesimal generator of the strong Markov process $\Phi = (\Phi^1, \Phi^2)$ is given by

$$(4.1) \quad \mathbb{L}_\Phi = \lambda(1+\varphi_1) \partial_{\varphi_1} + \lambda(1+\varphi_2) \partial_{\varphi_2} + \frac{\mu^2}{2} \varphi_1^2 \partial_{\varphi_1 \varphi_1} + \frac{\mu^2}{2} \varphi_2^2 \partial_{\varphi_2 \varphi_2}$$

for (φ_1, φ_2) belonging to $(0, \infty) \times (0, \infty)$. From (2.15) we see that the topological boundary $\{0\} \times [0, \infty) \cup (0, \infty) \times \{0\}$ of the state space $[0, \infty) \times [0, \infty)$ consists of *entrance* boundary points for Φ (meaning that Φ can be started at any boundary point never to return to the boundary) and clearly the differential operator \mathbb{L}_Φ is of *elliptic* type (cf. (2.12) in [13]).

For the Mayer reformulation of the problem (3.11) we need to look for a function $M : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ solving the partial differential equation

$$(4.2) \quad \mathbb{L}_\Phi M - \lambda M = L$$

on $(0, \infty) \times (0, \infty)$ where in view of (3.11) we set

$$(4.3) \quad L(\varphi_1, \varphi_2) = p_1 \varphi_1 + p_2 \varphi_2 - \lambda/c$$

for $(\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty)$. Ignoring the constant $-\lambda/c$ on the right-hand side of (4.2) for now, we see that a possible attempt to solve the resulting partial differential equation is to separate the variables φ_1 and φ_2 by considering the two ordinary differential equations

$$(4.4) \quad \lambda(1+\varphi_i) M_i' + \frac{\mu^2}{2} \varphi_i^2 M_i'' - \lambda M_i = p_i \varphi_i$$

where $M_i = M_i(\varphi_i)$ is a function/solution to be found for $\varphi_i \in (0, \infty)$ with $i = 1, 2$. Simplifying the notation we see that the equation (4.4) reads

$$(4.5) \quad x^2 y'' + \kappa(1+x)y' - \kappa y = \nu x$$

for $x \in (0, \infty)$ where $y = y(x)$ and we set $\kappa := 2\lambda/\mu^2$ and $\nu := 2p_i/\mu^2$ with $i = 1, 2$. The homogeneous part of the equation (4.5) is closely related to the Euler equation (cf. Eq. (118) in [15, Section 2.1.2]), and there exists a general transformation which reduces this part to another second-order ordinary differential equation, whose leading term is no longer quadratic but linear, and whose solutions can be expressed in terms of known special functions (see the reduction of Eq. (129) to Eq. (103) and Table 2.2 in [15, Section 2.1.2]).

Motivated by a probabilistic meaning of the posterior probability distribution ratio process in this context, and aiming to exploit the specific form of the coefficients $\kappa(1+x)$ and $-\kappa$ in (4.5) more directly, we will take a different tack and seek a solution to (4.5) by setting

$$(4.6) \quad y(x) := (1+x) z\left(\frac{x}{1+x}\right)$$

for $x \in (0, \infty)$. Setting $u = x/(1+x)$ we then find by (4.5) that $z = z(u)$ solves

$$(4.7) \quad u^2(1-u)z'' + \kappa z' = \nu \frac{u}{1-u}$$

for $u \in (0, 1)$ where the z term is no longer present. This equation can therefore be solved in closed form by reduction to a first-order ordinary differential equation. Inserting this solution back into (4.6) we find that the sought solution to (4.5) is given by

$$(4.8) \quad y(x) = \nu(1+x) \int_0^{x/(1+x)} \left(\frac{1-v}{v}\right)^\kappa e^{\kappa/v} \int_0^v \frac{u^{\kappa-1}}{(1-u)^{\kappa+2}} e^{-\kappa/u} du dv$$

for $x \in (0, \infty)$. This solution can now be used to specify the sought solutions to the equation (4.4). These solutions in turn can be used to specify the solution to the equation (4.2) above. The only matter remaining is to account for the missing constant $-\lambda/c$ on the right-hand side of (4.2) and this will be done shortly below.

2. We now consider the Mayer reformulation of the optimal stopping problem (3.11). Motivated by (4.8) and recalling that $\nu = 2p_i/\mu^2$ with $i = 1, 2$, let us define a function $M : [0, \infty) \rightarrow \mathbb{R}$ by setting

$$(4.9) \quad M(\varphi) = \frac{2}{\mu^2} (1+\varphi) \int_0^{\varphi/(1+\varphi)} \left(\frac{1-v}{v}\right)^\kappa e^{\kappa/v} \int_0^v \frac{u^{\kappa-1}}{(1-u)^{\kappa+2}} e^{-\kappa/u} du dv$$

for $\varphi \in [0, \infty)$ where we recall that $\kappa = 2\lambda/\mu^2$. In addition, let us define a function $M : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ by setting

$$(4.10) \quad M(\varphi_1, \varphi_2) = p_1 M(\varphi_1) + p_2 M(\varphi_2) + 1/c$$

for $(\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty)$. The arguments above then show that the function M from (4.10) solves the equation (4.2) above (notice that the final term $1/c$ yields the missing constant $-\lambda/c$

on the right-hand side of (4.2) as needed). Note that we use the same letter M to denote both functions in order to emphasise the ‘fractal’ nature of (4.10) expressed in terms of (4.9). The fact that the two functions have different domains can/will be used to remove any ambiguity when needed. Having defined the function M in (4.10) using (4.9) we can now describe the Mayer reformulation of the optimal stopping problem (3.11) as follows.

Proposition 2. *The value function \hat{V} from (3.11) can be expressed as*

$$(4.11) \quad \hat{V}(\varphi) = \inf_{\tau} \mathbf{E}_{\varphi}^{\infty} [e^{-\lambda\tau} M(\Phi_{\tau}^1, \Phi_{\tau}^2)] - M(\varphi)$$

for $\varphi \in [0, \infty) \times [0, \infty)$ where the infimum is taken over all (bounded) stopping times τ of $\Phi = (\Phi^1, \Phi^2)$ and the function M is given by (4.10) using (4.9) above.

Proof. By Itô’s formula using (3.9)+(3.10) we get

$$(4.12) \quad e^{-\lambda t} M(\Phi_t) = M(\varphi) + \int_0^t e^{-\lambda s} (\mathcal{L}_{\Phi} M - \lambda M)(\Phi_s) ds + N_t$$

for $\varphi \in [0, \infty) \times [0, \infty)$ where $N_t = \sum_{i=1}^2 \int_0^t e^{-\lambda s} M_{\varphi_i}(\Phi_s) \mu \Phi_s^i dB_s^i$ is a continuous local martingale for $t \geq 0$. Making use of a localisation sequence of stopping times for this local martingale if needed, applying the optional sampling theorem and recalling that M solves (4.2), we find by taking $\mathbf{E}_{\varphi}^{\infty}$ on both sides in (4.12) that

$$(4.13) \quad \mathbf{E}_{\varphi}^{\infty} [e^{-\lambda\tau} M(\Phi_{\tau}^1, \Phi_{\tau}^2)] = M(\varphi) + \mathbf{E}_{\varphi}^{\infty} \left[\int_0^{\tau} e^{-\lambda t} L(\Phi_t) dt \right]$$

for all $\varphi \in [0, \infty) \times [0, \infty)$ and all (bounded) stopping times τ of Φ . From (3.11) and (4.13) using (4.3) we see that (4.11) holds as claimed and the proof is complete. \square

3. From Proposition 2 we see that the optimal stopping problem (3.11) is equivalent to the optimal stopping problem defined by

$$(4.14) \quad \check{V}(\varphi) = \inf_{\tau} \mathbf{E}_{\varphi}^{\infty} [e^{-\lambda\tau} M(\Phi_{\tau}^1, \Phi_{\tau}^2)]$$

for $\varphi \in [0, \infty) \times [0, \infty)$ where the infimum is taken over all (bounded) stopping times τ of $\Phi = (\Phi^1, \Phi^2)$ and the function M is given by (4.10) using (4.9) above. The optimal stopping problem (4.14) is Mayer formulated. From (4.11) and (4.14) we see that

$$(4.15) \quad \hat{V}(\varphi) = \check{V}(\varphi) - M(\varphi)$$

for $\varphi \in [0, \infty) \times [0, \infty)$. The Mayer reformulation (4.14) has certain advantages that will be exploited in the subsequent analysis of the optimal stopping problem (3.11) below.

5. One dimension

The observed process X in the initial quickest detection problem (2.5) is two-dimensional. In this section we consider the analogue of (2.5) and the resulting optimal stopping problem

(3.11) when X is one-dimensional. The reduction of dimension from two to one corresponds to taking either p_1 or p_2 equal to 1. Then Φ standing for either Φ^1 or Φ^2 respectively is a one-dimensional Markov/diffusion process so that standard optimal stopping arguments can be used to solve the problem. The derived results for the one-dimensional optimal stopping problem (3.11) when X is one-dimensional will be used in the subsequent analysis of the two-dimensional optimal stopping problem (3.11) when X is two-dimensional.

1. Using the same arguments as in Sections 2 and 3 above, it is easily seen that the quickest detection problem (2.5) when X is one-dimensional reduces to the optimal stopping problem (3.11) with $p_1 = 1$ and $p_2 = 0$ (without loss of generality). Omitting the superscript 1 from Φ^1 for simplicity, we thus see that the optimal stopping problem (3.11) reads

$$(5.1) \quad \hat{V}(\varphi) = \inf_{\tau} \mathbf{E}_{\varphi}^{\infty} \left[\int_0^{\tau} e^{-\lambda t} \left(\Phi_t - \frac{\lambda}{c} \right) dt \right]$$

for $\varphi \in [0, \infty)$ with $\mathbf{P}_{\varphi}^{\infty}(\Phi_0 = \varphi) = 1$ where the infimum is taken over all stopping times τ of Φ . From (3.9) we see that the infinitesimal generator of Φ is given by

$$(5.2) \quad \mathbb{L}_{\Phi} = \lambda(1+\varphi) \frac{d}{d\varphi} + \frac{\mu^2}{2} \varphi^2 \frac{d^2}{d\varphi^2}$$

for φ belonging to $[0, \infty)$.

2. Noting that the optimal stopping problem (5.1) is Lagrange formulated, standard arguments imply (see e.g. [14]) that \hat{V} should solve the free-boundary problem

$$(5.3) \quad \mathbb{L}_{\varphi} \hat{V} - \lambda \hat{V} = -(\varphi - \lambda/c) \text{ for } \varphi \in [0, \varphi_*)$$

$$(5.4) \quad \hat{V}(\varphi_*) = 0 \text{ (instantaneous stopping)}$$

$$(5.5) \quad \hat{V}'(\varphi_*) = 0 \text{ smooth fit}$$

where $\varphi_* \in (\lambda/c, \infty)$ is the optimal stopping boundary/point to be found, and we set $\hat{V}(\varphi) = 0$ for $\varphi \in (\varphi_*, \infty)$ in addition to (5.4) above (note from (5.1) that considering the exit times of Φ from sufficiently small intervals shows that it is never optimal to stop at least in $[0, \lambda/c)$ as partly indicated above).

3. The general solution to the ordinary differential equation (5.3) is given by

$$(5.6) \quad \hat{V}(\varphi) = A(1+\varphi) \int_{1/2}^{\varphi/(1+\varphi)} \left(\frac{1-v}{v} \right)^{\kappa} e^{\kappa/v} dv + B(1+\varphi) + \hat{V}_p(\varphi)$$

where A and B are (unspecified) real constants and V_p is a particular solution to (5.3) obtained by modifying the solution M from (4.9) as follows

$$(5.7) \quad \hat{V}_p(\varphi) = -\frac{2}{\mu^2} (1+\varphi) \int_{\varphi_*/(1+\varphi_*)}^{\varphi/(1+\varphi)} \left(\frac{1-v}{v} \right)^{\kappa} e^{\kappa/v} \int_0^v \frac{u^{\kappa-1}}{(1-u)^{\kappa+2}} e^{-\kappa/u} du dv - 1/c$$

for $\varphi \in [0, \varphi_*]$ where $\varphi_* \in (\lambda/c, \infty)$ is to be found and we recall that $\kappa = 2\lambda/\mu^2$. This can be obtained by noticing that M from (4.10) with $p_1 = 1$ and $p_2 = 0$ solves (4.2) with \mathbb{L}_{Φ}

from (5.2) if and only if $\hat{V} := -M$ solves (5.3). Hence we see that the transformation (4.6) reduces the equation (5.3) to a solvable form. Proceeding thus as in (4.7) and (4.8) above, and applying the analogous arguments to the homogeneous part of the equation (5.3), we obtain the general solution (5.6) with (5.7) as claimed. The motivation for modifying the particular solution M from (4.9) as \hat{V}_p in (5.7) comes from the instantaneous stopping and smooth fit conditions (5.4) and (5.5) as will be clear from the calculations below.

4. A direct differentiation in (5.6) shows that $\hat{V}'(0+) = +\infty$ if $A > 0$ and $\hat{V}'(0+) = -\infty$ if $A < 0$. We thus choose $A = 0$ as a candidate value in the sequel. Using (5.4) we then find that $B = 1/c(1+\varphi_*)$. This yields the following candidate solution to the free-boundary problem (5.3)-(5.5) above

$$(5.8) \quad \hat{V}(\varphi) = -\frac{\varphi_* - \varphi}{c(1+\varphi_*)} - \frac{2}{\mu^2} (1+\varphi) \int_{\varphi_*/(1+\varphi_*)}^{\varphi/(1+\varphi)} \left(\frac{1-v}{v}\right)^\kappa e^{\kappa/v} \int_0^v \frac{u^{\kappa-1}}{(1-u)^{\kappa+2}} e^{-\kappa/u} du dv$$

for $\varphi \in [0, \varphi_*]$ and $\hat{V}(\varphi) = 0$ for (φ_*, ∞) . A direct differentiation in (5.8) then shows that (5.5) holds if and only if φ_* solves

$$(5.9) \quad \frac{e^{\kappa(1+\varphi_*)/\varphi_*}}{\varphi_*^\kappa} \int_0^{\varphi_*/(1+\varphi_*)} \frac{u^{\kappa-1}}{(1-u)^{\kappa+2}} e^{-\kappa/u} du = \frac{\mu^2}{2c}$$

where we recall that $\kappa = 2\lambda/\mu^2$.

5. To make the arguments developed above rigorous we can reverse their order and start our analysis from the end. Firstly, we claim that there exists a unique point $\varphi_* \in (\lambda/c, \infty)$ satisfying the equation (5.9). For this, define the functions F and G by setting

$$(5.10) \quad F(\varphi) = \int_0^{\varphi/(1+\varphi)} \frac{1}{u(1-u)^2} e^{\kappa(\log[u/(1-u)]-1/u)} du \quad \& \quad G(\varphi) = \frac{\mu^2}{2c} e^{\kappa(\log \varphi - (1+\varphi)/\varphi)}$$

for $\varphi \in [0, \infty)$. Note that the claim about (5.9) is equivalent to establishing that

$$(5.11) \quad F(\varphi_*) = G(\varphi_*)$$

for a unique point $\varphi_* \in (\lambda/c, \infty)$. To verify (5.11) note that $F(0) = G(0)$ and we have

$$(5.12) \quad \varphi < \frac{\lambda}{c} \iff F'(\varphi) < G'(\varphi) \quad \& \quad \varphi > \frac{\lambda}{c} \iff F'(\varphi) > G'(\varphi)$$

for $\varphi \in (0, \infty)$ as is easily verified by a direct differentiation in (5.10). This shows that $F(\varphi) < G(\varphi)$ for all $\varphi \in (0, \lambda/c]$. Moreover, applying L'Hospital's rule we find that

$$(5.13) \quad \lim_{\varphi \rightarrow \infty} \frac{F(\varphi)}{G(\varphi)} = \lim_{\varphi \rightarrow \infty} \frac{F'(\varphi)}{G'(\varphi)} = \lim_{\varphi \rightarrow \infty} \left(\frac{c}{\lambda} \varphi\right) = \infty.$$

Combining (5.12) and (5.13) we see that the graphs of F and G must intersect on $(\lambda/c, \infty)$ at a unique point φ_* establishing (5.11) as claimed. Secondly, define $\hat{V}_*(\varphi)$ by the right-hand side of (5.8) for $\varphi \in [0, \varphi_*]$ and set $\hat{V}_*(\varphi) = 0$ for (φ_*, ∞) . Then the arguments above show (or it is a matter of routine to verify) that \hat{V}_* solves the free-boundary problem (5.3)-(5.5)

above. Thirdly, applying the Itô-Tanaka formula (cf. [16, p. 223]) to \hat{V}_* composed with Φ , which reduces to Itô's formula due to smooth fit (5.5), and making use of the optional sampling theorem, it is easily verified that \hat{V}_* coincides with the value function \hat{V} from (5.1) and the optimal stopping time (at which the infimum in (5.1) is attained) is given by

$$(5.14) \quad \tau_* = \inf \{ t \geq 0 \mid \Phi_t \in [\varphi_*, \infty) \}$$

where $\varphi_* \in (\lambda/c, \infty)$ is a unique solution to (5.9) on $(0, \infty)$. These facts will be used in the subsequent analysis of the optimal stopping problem (3.11) when the observed process X is two-dimensional as assumed in Section 2 above.

6. Properties of the optimal stopping boundary

In this section we establish the existence of an optimal stopping time in the problem (3.11) and derive basic properties of the optimal stopping boundary.

1. Looking at (3.11) we may conclude that the (candidate) continuation and stopping sets in this problem need to be defined as follows

$$(6.1) \quad C = \{ \varphi \in [0, \infty) \times [0, \infty) \mid \hat{V}(\varphi) < 0 \}$$

$$(6.2) \quad D = \{ \varphi \in [0, \infty) \times [0, \infty) \mid \hat{V}(\varphi) = 0 \}$$

respectively. Recalling that (2.15) defines a Markovian functional of the initial point $\Phi_0^i := \varphi_i$ in $[0, \infty)$ of the process Φ^i for $i = 1, 2$, we see that the expectation in (4.11) defines a continuous function of the initial point $\varphi = (\varphi_1, \varphi_2)$ of the process $\Phi = (\Phi^1, \Phi^2)$ for every (bounded) stopping time τ of Φ given and fixed. Taking the infimum over all (bounded) stopping times τ of Φ we can thus conclude from (4.11) that the value function \hat{V} is upper semicontinuous on $[0, \infty) \times [0, \infty)$. From (4.10) with (4.9) we see that the loss function M in (4.11) is continuous and hence lower semicontinuous too. It follows therefore by [14, Corollary 2.9] that the first entry time of the process Φ into the closed set D defined by

$$(6.3) \quad \tau_D = \inf \{ t \geq 0 \mid \Phi_t \in D \}$$

is optimal in (4.11) and hence in (3.11) as well whenever $\mathbb{P}_\varphi^\infty(\tau_D < \infty) = 1$ for $\varphi \in [0, \infty) \times [0, \infty)$. In the sequel we will establish this and other properties of τ_D by analysing the boundary of D .

2. To derive an upper bound on the boundary of D , recall that the optimal stopping boundary/point φ_* in the one-dimensional problem (5.1) can be characterised as a unique solution to (5.9) on $(\lambda/c, \infty)$. Note from (5.9) that $\varphi_* = \varphi_*(\lambda/\mu^2, \lambda/c)$ and set

$$(6.4) \quad \varphi_1^* := \varphi_*\left(\frac{\lambda}{\mu^2}, \frac{\lambda}{p_1 c}\right) \quad \& \quad \varphi_2^* := \varphi_*\left(\frac{\lambda}{\mu^2}, \frac{\lambda}{p_2 c}\right)$$

for $\lambda > 0$, $\mu \in \mathbb{R}$, $c > 0$ and $p_1, p_2 \in (0, 1)$ with $p_1 + p_2 = 1$. Recall that $\varphi_1^* \in (\lambda/(p_1 c), \infty)$ and $\varphi_2^* \in (\lambda/(p_2 c), \infty)$. We can now expose basic properties of the the value function and the continuation/stopping set in the problem (3.11) as follows.

Proposition 3.

$$(6.5) \quad \textit{The value function } \hat{V} \textit{ is concave and continuous on } [0, \infty) \times [0, \infty).$$

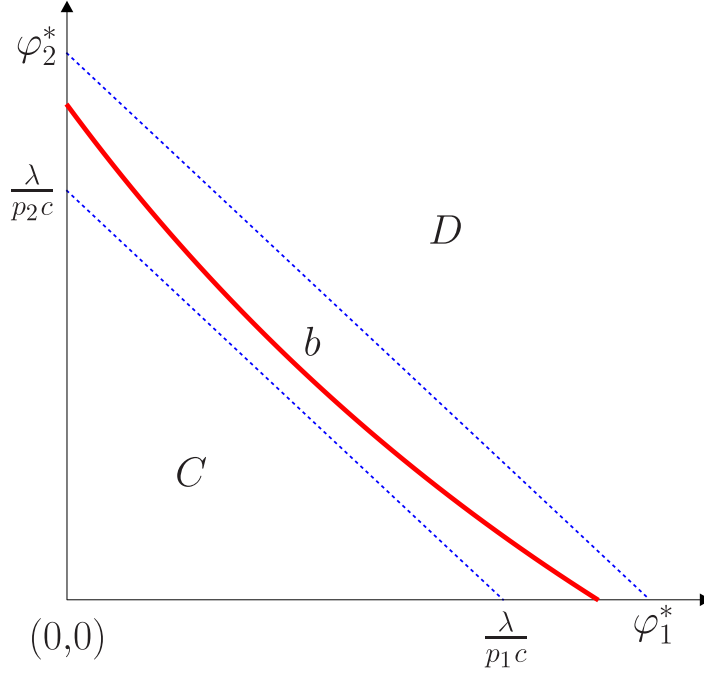


Figure 1. The optimal stopping boundary b in the problem (3.11) when $\mu = \lambda = c = 1$ and $p_1 = p_2 = 1/2$. The dotted lines are graphs of the linear functions from (6.11) below.

- (6.6) If $\varphi_1 \leq \psi_1$ & $\varphi_2 \leq \psi_2$ then $\hat{V}(\varphi_1, \varphi_2) \leq \hat{V}(\psi_1, \psi_2)$.
- (6.7) If $(\varphi_1, \varphi_2) \in D$ and $\psi_1 \geq \varphi_1$ & $\psi_2 \geq \varphi_2$ then $(\psi_1, \psi_2) \in D$.
- (6.8) The stopping set D is convex and the trigon $\{(\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty) \mid \varphi_1/\varphi_1^* + \varphi_2/\varphi_2^* - 1 \geq 0\}$ is contained in D .
- (6.9) The triangle $\{(\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty) \mid p_1\varphi_1 + p_2\varphi_2 - \lambda/c < 0\}$ is contained in the continuation set C .

Proof. (6.5): Combining the fact that the Markovian flow (2.15) is linear as a function of its initial point with the fact that the integral in (3.11) is a linear function of its argument, and using that the infimum of a convex combination is larger than the convex combination of the infima, we find that \hat{V} is concave on $[0, \infty) \times [0, \infty)$ as claimed. Hence we can also conclude that \hat{V} is continuous on the open set $(0, \infty) \times (0, \infty)$. To see that \hat{V} is continuous at the boundary points of $[0, \infty) \times [0, \infty)$ we may recall the well-known (and easily verified) fact that the concave function \hat{V} is lower semicontinuous on the closed and convex set $[0, \infty) \times [0, \infty)$. Since we also know that \hat{V} is upper semicontinuous on $[0, \infty) \times [0, \infty)$ as established following (6.2) above, we see that \hat{V} is continuous on the entire $[0, \infty) \times [0, \infty)$ as claimed.

(6.6): This is a direct consequence of the fact that the Markovian flow (2.15) is increasing as a function of its initial point being used in (3.11) above.

(6.7): By (6.6) we have $\hat{V}(\varphi_1, \varphi_2) \leq \hat{V}(\psi_1, \psi_2) \leq 0$ so that $(\varphi_1, \varphi_2) \in D$ i.e. $\hat{V}(\varphi_1, \varphi_2) = 0$ implies that $\hat{V}(\psi_1, \psi_2) = 0$ i.e. $(\psi_1, \psi_2) \in D$ as claimed.

(6.8): To see that D is convex, take any φ and ψ from D and note by (6.5) that $0 \geq \hat{V}(\alpha\varphi+(1-\alpha)\psi) \geq \alpha\hat{V}(\varphi)+(1-\alpha)\hat{V}(\psi) = 0$ so that $\hat{V}(\alpha\varphi+(1-\alpha)\psi) = 0$ i.e. $\alpha\varphi+(1-\alpha)\psi \in D$ for every $\alpha \in [0, 1]$ as claimed. To see that the trigon is contained in D , note that pulling p_1 in front of the infimum in (3.11) shows that the point $(\varphi_1^*, 0)$ belongs to D because φ_1^* as defined in (6.4) above is an optimal stopping point in the one-dimensional problem obtained by removing the (independent) positive term $(p_2/p_1)\Phi_t^2$ from the integral with respect to time in (3.11) with p_1 in front of the infimum. Similarly, we see that the point $(0, \varphi_2^*)$ belongs to D . But then the entire trigon is contained in D due to its convexity.

(6.9): Taking any point φ from the triangle and replacing τ in (3.11) by the first exit time of Φ from a sufficiently small ball around φ that is strictly contained in the triangle, we see that the integrand in (3.11) remains strictly negative so that \hat{V} takes a strictly negative value at φ itself, showing that φ belongs to the continuation set C as claimed. \square

3. From the results of Proposition 3 we see that the stopping set in the problem (3.11) can be described as follows

$$(6.10) \quad D = \{ (\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty) \mid \varphi_2 \geq b(\varphi_1) \}$$

where $b : [0, \infty) \rightarrow [0, \infty)$ is a convex, continuous, decreasing function satisfying

$$(6.11) \quad -\frac{p_1}{p_2} \varphi_1 + \frac{\lambda}{p_2 c} \leq b(\varphi_1) \leq -\frac{\varphi_2^*}{\varphi_1^*} \varphi_1 + \varphi_2^*$$

for $\varphi_1 \in [0, \lambda/(p_1 c)]$ in the first inequality and $\varphi_1 \in [0, \varphi_1^*]$ in the second inequality respectively (see Figure 1). Note that since Φ^1 and Φ^2 are independent and either recurrent or transient in $[0, \infty)$ (converging to ∞ in the latter case) as recalled following (3.9)+(3.10) above, we see from (6.11) that $\mathbf{P}_\varphi^\infty(\tau_D < \infty) = 1$ for all $\varphi \in [0, \infty) \times [0, \infty)$ as claimed following (6.3) above. We address the question of characterising/determining b in the remaining two sections.

7. Free-boundary problem

In this section we derive a free-boundary problem that stands in one-to-one correspondence with the optimal stopping problem (3.11). Using the results derived in the previous sections we show that the value function \hat{V} from (3.11) and the optimal stopping boundary b from (6.10) solve the free-boundary problem. This establishes the existence of a solution to the free-boundary problem. Its uniqueness in a natural class of functions will follow from a more general uniqueness result that will be established in Section 8 below. This will also yield an explicit integral representation of the value function \hat{V} expressed in terms of the optimal stopping boundary b .

1. Consider the optimal stopping problem (3.11) where the Markov process $\Phi = (\Phi^1, \Phi^2)$ solves the system of stochastic differential equations (3.9)-(3.10) driven by a standard Brownian motion $B = (B^1, B^2)$ under the probability measure \mathbf{P}^∞ . Recall that the infinitesimal generator of Φ is the second-order elliptic differential operator \mathcal{L}_Φ given in (4.1) above. Looking at (3.11) and relying on other properties of \hat{V} and b derived above, we are naturally led to

formulate the following free-boundary problem for finding \hat{V} and b :

$$(7.1) \quad \mathbb{L}_\Phi \hat{V} - \lambda \hat{V} = -L \quad \text{in } C$$

$$(7.2) \quad \hat{V}(\varphi) = 0 \quad \text{for } \varphi \in D \quad (\text{instantaneous stopping})$$

$$(7.3) \quad \hat{V}_{\varphi_i}(\varphi) = 0 \quad \text{for } \varphi \in \partial C \quad \text{and } i = 1, 2 \quad (\text{smooth fit})$$

where L is defined in (4.3) above, C is the (continuation) set from (6.1) above, D is the (stopping) set from (6.2)+(6.10) above, and $\partial C = \{(\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty) \mid \varphi_2 = b(\varphi_1)\}$ is the (optimal stopping) boundary between the sets C and D .

2. To formulate the existence and uniqueness result for the free-boundary problem (7.1)-(7.3), we let \mathcal{C} denote the class of functions (U, a) such that

$$(7.4) \quad U \text{ belongs to } C^1(\bar{C}_a) \cap C^2(C_a) \text{ and is continuous \& bounded on } [0, \infty) \times [0, \infty)$$

$$(7.5) \quad a \text{ is continuous \& decreasing on } [0, \infty) \text{ and satisfies } p_1 \varphi_1 + p_2 a(\varphi_1) - \lambda/c \geq 0 \text{ for } \varphi_1 \in [0, \infty)$$

where we set $C_a = \{(\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty) \mid \varphi_2 < a(\varphi_1)\}$ and $\bar{C}_a = \{(\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty) \mid \varphi_2 \leq a(\varphi_1) \text{ \& } \varphi_1 \leq \inf\{\psi_1 \in [0, \infty) \mid a(\psi_1) = 0\}\}$. Note that in the latter set we only account for the smallest zero of the function a should such zeros exist.

Theorem 4. *The free-boundary problem (7.1)-(7.3) has a unique solution (\hat{V}, b) in the class \mathcal{C} where \hat{V} is given in (3.11) and b is given in (6.10) above.*

Proof. We first show that the pair (\hat{V}, b) belongs to the class \mathcal{C} and solves the free-boundary problem (7.1)-(7.3). For this, note that the optimal stopping problem (3.11) is Lagrange formulated so that standard arguments (see e.g. the final paragraph of Section 2 in [3]) imply that \hat{V} belongs to $C^2(C)$ and satisfies (7.1). From (6.5) we know that \hat{V} is continuous on $[0, \infty) \times [0, \infty)$ and from (3.11) we readily find that

$$(7.6) \quad -\frac{1}{c} \leq \hat{V}(\varphi) \leq 0$$

for all $\varphi \in [0, \infty) \times [0, \infty)$. Moreover, recall that the process $\Phi = (\Phi^1, \Phi^2)$ is strong Feller while it is evident that each point $\varphi \in \partial C$ is probabilistically regular for the set D since b is decreasing and the coordinate processes Φ^1 & Φ^2 are independent. Finally, from (2.15) we see that the process Φ can be realised as a continuously differentiable stochastic flow of its initial point so that the integrability conditions of Theorem 8 in [3] are satisfied. Recalling that \hat{V} satisfies (7.2), and applying the result of that theorem, we can conclude that

$$(7.7) \quad \hat{V} \text{ is continuously differentiable on } [0, \infty) \times [0, \infty).$$

In particular, this shows that (7.3) holds as well as that \hat{V} belongs to $C^1(\bar{C})$ as required in (7.4) above. The fact that b satisfies (7.5) was established in the final paragraph of Section 6 above. This shows that (\hat{V}, b) belongs to \mathcal{C} and solves (7.1)-(7.3) as claimed. To derive uniqueness of the solution we will first see in the next section that any solution (U, a) to (7.1)-(7.3) from the class \mathcal{C} admits an explicit integral representation for U expressed in terms

of a , which in turn solves a nonlinear Fredholm integral equation, and we will see that this equation cannot have other solutions satisfying the required properties. From these facts we can conclude that the free-boundary problem (7.1)-(7.3) cannot have other solutions in the class \mathcal{C} as claimed. This completes the proof. \square

8. Nonlinear integral equation

In this section we show that the optimal stopping boundary b from (6.10) can be characterised as the unique solution to a nonlinear Fredholm integral equation. This also yields an explicit integral representation of the value function \hat{V} from (3.11) expressed in terms of the optimal stopping boundary b . As a consequence of the existence and uniqueness result for the nonlinear Fredholm integral equation we also obtain uniqueness of the solution to the free-boundary problem (7.1)-(7.3) as explained in the proof of Theorem 4 above. Finally, collecting the results derived throughout the paper we conclude our exposition by disclosing the solution to the initial problem.

1. Let $p = p(t; \varphi_1, \varphi_2, \psi_1, \psi_2)$ denote the transition probability density function of the Markov process $\Phi = (\Phi^1, \Phi^2)$ in the sense that

$$(8.1) \quad \mathbb{P}_{\varphi_1, \varphi_2}^\infty(\Phi_t \in A) = \iint_A p(t; \varphi_1, \varphi_2, \psi_1, \psi_2) d\psi_1 d\psi_2$$

for any measurable $A \subseteq [0, \infty) \times [0, \infty)$ with $(\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty)$ and $t \geq 0$ given and fixed. Since Φ^1 and Φ^2 are independent, we have $p(t; \varphi_1, \varphi_2, \psi_1, \psi_2) = p_1(t; \varphi_1, \psi_1) p_2(t; \varphi_2, \psi_2)$ for all (φ_1, φ_2) & (ψ_1, ψ_2) in $[0, \infty) \times [0, \infty)$ and all $t \geq 0$, where p_1 and p_2 are transition probability density functions of Φ^1 and Φ^2 respectively. Explicit expressions for p_1 and p_2 are known (see e.g. [11] and the references therein). Having p we can evaluate the expression of interest in the theorem below as follows

$$(8.2) \quad \begin{aligned} K_b(t; \varphi_1, \varphi_2) &:= \mathbb{E}_{\varphi_1, \varphi_2}^\infty [L(\Phi_t^1, \Phi_t^2) I(\Phi_t^2 < b(\Phi_t^1))] \\ &= \int_0^{\varphi_0} d\psi_1 \int_0^{b(\psi_1)} L(\psi_1, \psi_2) p(t; \varphi_1, \varphi_2, \psi_1, \psi_2) d\psi_2 \end{aligned}$$

for $t \geq 0$ and $(\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty)$ where φ_0 is the smallest zero of b on $[0, \infty)$ (recall that $\varphi_0 \in [\lambda/(p_1 c), \varphi_1^*]$ as seen in Figure 1 above) and L is defined in (4.3) above.

Theorem 5 (Existence and uniqueness). *The optimal stopping boundary b in (3.11) can be characterised as the unique solution to the nonlinear Fredholm integral equation*

$$(8.3) \quad \int_0^\infty e^{-\lambda t} K_b(t; \varphi_1, b(\varphi_1)) dt = 0$$

in the class of continuous & decreasing (convex) functions b on $[0, \infty)$ satisfying $p_1 \varphi_1 + p_2 b(\varphi_1) - \lambda/c \geq 0$ for $\varphi_1 \in [0, \varphi_0)$ where φ_0 is the smallest zero of b on $[0, \infty)$. The value function \hat{V} in (3.11) admits the following representation

$$(8.4) \quad \hat{V}(\varphi_1, \varphi_2) = \int_0^\infty e^{-\lambda t} K_b(t; \varphi_1, \varphi_2) dt$$

for $(\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty)$. The optimal stopping time in (3.11) is given by

$$(8.5) \quad \tau_b = \inf \{ t \geq 0 \mid \Phi_t^2 \geq b(\Phi_t^1) \}$$

under $\mathbb{P}_{\varphi_1, \varphi_2}^\infty$ with $(\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty)$ given and fixed.

Proof. (i) *Existence.* We first show that the optimal stopping boundary b in (3.11) solves (8.3). Recalling that b satisfies the properties stated following (6.10) above, this will establish the existence of a solution to (8.3) in the specified class of functions.

For this, to gain control over the (individual) second partial derivatives $\hat{V}_{\varphi_1\varphi_1}$ and $\hat{V}_{\varphi_2\varphi_2}$ close to the optimal stopping boundary within C (see [8] for general results of this kind), consider the sets $C_n := \{ \varphi \in [0, \infty) \times [0, \infty) \mid \hat{V}(\varphi) < -1/n \}$ and $D_n := \{ \varphi \in [0, \infty) \times [0, \infty) \mid \hat{V}(\varphi) \geq -1/n \}$ for $n \geq 1$ (large). Note that $C_n \uparrow C$ and $D_n \downarrow D$ as $n \uparrow \infty$. Moreover, using the same arguments as for the sets C and D above, we find that the set D_n is convex, and the boundary $b_n = b_n(\varphi_1)$ between C_n and D_n is a convex, continuous, decreasing function of φ_1 in $[0, \varphi_0^n]$ where φ_0^n is the smallest zero of b_n on $[0, \varphi_1^*]$ for $n \geq 1$. This also shows that $b_n \uparrow b$ uniformly on $[0, \varphi_0^n]$ with $\varphi_0^n \uparrow \varphi_0$ as $n \rightarrow \infty$ where φ_0 is the smallest zero of b on $[0, \varphi_1^*]$.

Approximate the value function \hat{V} in (3.11) by functions \hat{V}^n defined as \hat{V} on C_n and $-1/n$ on D_n for $n \geq 1$. Note that $\hat{V}^n \uparrow \hat{V}$ uniformly on $[0, \infty) \times [0, \infty)$ as $n \rightarrow \infty$. Moreover, letting $n \geq 1$ be given and fixed in the sequel, clearly \hat{V}^n is a continuous function on $[0, \infty) \times [0, \infty)$ and \hat{V}^n restricted to C_n and D_n belongs to $C^2(\bar{C}_n)$ and $C^2(\bar{D}_n)$ respectively. Finally, since b_n is convex, we know that $b_n(\Phi^1)$ is a continuous semimartingale. This shows that the change-of-variable formula with local time on surfaces [12, Theorem 2.1] is applicable to \hat{V}^n composed with $\Phi = (\Phi^1, \Phi^2)$ and using (7.1) this gives

$$(8.6) \quad \begin{aligned} e^{-\lambda t} \hat{V}^n(\Phi_t) &= \hat{V}^n(\Phi_0) + \int_0^t e^{-\lambda s} (\mathbb{L}_\Phi \hat{V}^n - \lambda \hat{V}^n)(\Phi_s) ds + \int_0^t e^{-\lambda s} \hat{V}_{\varphi_1}^n(\Phi_s) \mu \Phi_s^1 dB_s^1 \\ &\quad + \int_0^t e^{-\lambda s} \hat{V}_{\varphi_2}^n(\Phi_s) \mu \Phi_s^2 dB_s^2 - \int_0^t e^{-\lambda s} \hat{V}_{\varphi_2}^n(\Phi_{s-}) d\ell_s^{b_n}(\Phi) \\ &= \hat{V}^n(\Phi_0) - \int_0^t e^{-\lambda s} L(\Phi_s) I(\Phi_s \in C_n) ds + \frac{\lambda}{n} \int_0^t e^{-\lambda s} I(\Phi_s \in D_n) ds \\ &\quad + M_t^n - \int_0^t e^{-\lambda s} \hat{V}_{\varphi_2}^n(\Phi_s) d\ell_s^{b_n}(\Phi) \end{aligned}$$

where $M_t^n = \int_0^t e^{-\lambda s} \hat{V}_{\varphi_1}^n(\Phi_s) \mu \Phi_s^1 I(\Phi_s \in C_n) dB_s^1 + \int_0^t e^{-\lambda s} \hat{V}_{\varphi_2}^n(\Phi_s) \mu \Phi_s^2 I(\Phi_s \in C_n) dB_s^2$ is a continuous martingale for $t \geq 0$ and $\ell^{b_n}(\Phi)$ is the local time of Φ on the curve b_n given by

$$(8.7) \quad \ell_t^{b_n}(\Phi) = \mathbf{P}\text{-}\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I(-\varepsilon < \Phi_s^2 - b_n(\Phi_s^1) < \varepsilon) d\langle \Phi^2 - b_n(\Phi^1), \Phi^2 - b_n(\Phi^1) \rangle_s$$

for $t \geq 0$. To gain control over the final term in (8.6), note that the Itô-Tanaka formula yields

$$(8.8) \quad \begin{aligned} (b_n(\Phi_t^1) - \Phi_t^2)^+ &= (b_n(\Phi_0^1) - \Phi_0^2)^+ + \int_0^t I(b_n(\Phi_s^1) - \Phi_s^2 > 0) d(b_n(\Phi^1) - \Phi^2)_s + \frac{1}{2} \ell_t^{b_n}(\Phi) \\ &= (b_n(\Phi_0^1) - \Phi_0^2)^+ + \int_0^t I(b_n(\Phi_s^1) - \Phi_s^2 > 0) (b_n'(\Phi_s^1) d\Phi_s^1 - d\Phi_s^2) \end{aligned}$$

$$+ \frac{1}{2} \int_0^t I(b_n(\Phi_s^1) - \Phi_s^2 > 0) \int_0^\infty d\ell_s^{\psi_1}(\Phi^1) db'_n(\psi_1) + \frac{1}{2} \ell_t^{b_n}(\Phi)$$

for $t \geq 0$ where b'_n denotes the first derivative of b_n whose existence follows by the implicit function theorem since smooth fit fails at b_n due to its suboptimality in the problem (3.11). Since b_n is convex we see that db'_n defines a non-negative measure on $[0, \infty)$ so that the double integral in (8.8) is non-negative. It follows therefore from (8.8) using (3.9)+(3.10) that

$$(8.9) \quad \frac{1}{2} \ell_t^{b_n}(\Phi) \leq (b_n(\Phi_t^1) - \Phi_t^2)^+ - \int_0^t I(b_n(\Phi_s^1) - \Phi_s^2 > 0) b'_n(\Phi_s^1) \lambda(1 + \Phi_s^1) ds \\ + \int_0^t I(b_n(\Phi_s^1) - \Phi_s^2 > 0) \lambda(1 + \Phi_s^2) ds + N_t^n$$

where $N_t^n = - \int_0^t I(b_n(\Phi_s^1) - \Phi_s^2 > 0) b'_n(\Phi_s^1) \mu \Phi_s^1 dB_s^1 + \int_0^t I(b_n(\Phi_s^1) - \Phi_s^2 > 0) \mu \Phi_s^2 dB_s^2$ is a continuous local martingale for $t \geq 0$. Let $(\tau_m)_{m \geq 1}$ be a localisation sequence of stopping times for N^n , define the stopping time

$$(8.10) \quad \sigma_m = \inf \{ t \geq 0 \mid \Phi_t^1 \leq \frac{1}{m} \}$$

and set $\rho_m := \tau_m \wedge \sigma_m$ for $m \geq 1$. From (8.9) we then find that

$$(8.11) \quad \frac{1}{2} \mathbf{E}_{\varphi_1, \varphi_2}^\infty [\ell_{t \wedge \rho_m}^{b_n}(\Phi)] \leq \varphi_2^* - b'_n\left(\frac{1}{m}\right) \int_0^t \lambda(1 + \mathbf{E}_{\varphi_1, \varphi_2}^\infty(\Phi_s^1)) ds \\ + \int_0^t \lambda(1 + \mathbf{E}_{\varphi_1, \varphi_2}^\infty(\Phi_s^2)) ds \leq K_m(t)$$

for $t \geq 0$ and $m \geq 1$ where the positive constant $K_m(t)$ does not depend on $n \geq 1$ because each b_n is convex and $b_n \uparrow b$ on $[0, 1/m]$ as $n \rightarrow \infty$ so that $b'_n(1/m)$ must remain bounded from below over $n \geq 1$ if b_n is to stay below b on $[0, 1/m]$ for all $n \geq 1$. In addition, by (7.7) we know that \hat{V}_{φ_2} is continuous on \bar{C} and hence uniformly continuous too because \bar{C} is a compact set. It follows therefore that $0 \leq \hat{V}_{\varphi_2}(\varphi_1, b_n(\varphi_1)) \leq \varepsilon$ for all $\varphi_1 \in [0, \varphi_0^n]$ and all $n \geq n_\varepsilon$ with $n_\varepsilon \geq 1$ large enough depending on the given and fixed $\varepsilon > 0$. Combining this fact with (8.11), upon replacing t with $t \wedge \rho_m$ in the final integral of (8.6) and taking $\mathbf{E}_\varphi^\infty$ of the resulting expression for $\varphi \in [0, \infty) \times [0, \infty)$ given and fixed, we see that

$$(8.12) \quad 0 \leq \mathbf{E}_\varphi^\infty \left[\int_0^{t \wedge \rho_m} e^{-\lambda s} \hat{V}_{\varphi_2}(\Phi_s) d\ell_s^{b_n}(\Phi) \right] \leq 2\varepsilon K_t(m)$$

for all $n \geq n_\varepsilon$ with $t \geq 0$ and $m \geq 1$ given and fixed. This shows that the expectation in (8.12) tends to zero as n tends to infinity for every $t \geq 0$ and $m \geq 1$ given and fixed. Using this fact in (8.6) upon replacing t with $t \wedge \rho_m$, taking $\mathbf{E}_\varphi^\infty$ on both sides, and letting n tend to infinity, we find by the monotone convergence theorem upon recalling (7.6) that

$$(8.13) \quad \hat{V}(\varphi) = \mathbf{E}_\varphi^\infty [e^{-\lambda(t \wedge \rho_m)} \hat{V}(\Phi_{t \wedge \rho_m})] + \mathbf{E}_\varphi^\infty \left[\int_0^{t \wedge \rho_m} e^{-\lambda s} L(\Phi_s) I(\Phi_s \in C) ds \right]$$

for all $t \geq 0$ and all $m \geq 1$. Letting $m \rightarrow \infty$ and using that $\rho_m \rightarrow \infty$ because 0 is an entrance boundary point for Φ^1 , we see from (8.13) upon recalling (7.6) and using the dominated convergence theorem that

$$(8.14) \quad \hat{V}(\varphi) = \mathbf{E}_\varphi^\infty [e^{-\lambda t} \hat{V}(\Phi_t)] + \mathbf{E}_\varphi^\infty \left[\int_0^t e^{-\lambda s} L(\Phi_s) I(\Phi_s \in C) ds \right]$$

for all $t \geq 0$. Finally, letting $t \rightarrow \infty$ in (8.14) and using the dominated and monotone convergence theorems upon recalling (7.6), we find that

$$(8.15) \quad \hat{V}(\varphi) = \mathbb{E}_{\varphi}^{\infty} \left[\int_0^{\infty} e^{-\lambda s} L(\Phi_s) I(\Phi_s \in C) ds \right]$$

for all $\varphi \in [0, \infty) \times [0, \infty)$. Recalling (6.10) and (8.2) above we see that this establishes the representation (8.4) as claimed. Moreover, the fact that τ_b from (8.5) is optimal in (3.11) follows by (6.10) above. Finally, inserting $\varphi_2 = b(\varphi_1)$ in (8.4) and using that $\hat{V}(\varphi_1, b(\varphi_1)) = 0$, we see that b solves (8.3) as claimed.

(ii) *Uniqueness.* To show that b is a unique solution to the equation (8.3) in the specified class of functions, one can adopt the four-step procedure from the proof of uniqueness given in [4, Theorem 4.1] extending and further refining the original uniqueness arguments from [10, Theorem 3.1]. Given that the present setting creates no additional difficulties we will omit further details of this verification and this completes the proof. \square

2. The nonlinear Fredholm integral equation (8.3) can be used to find the optimal stopping boundary b numerically (using Picard iteration). Inserting this b into (8.4) via (8.2) we also obtain a closed form expression for the value function \hat{V} . Collecting the results derived throughout the paper we now disclose the solution to the initial problem.

Corollary 6. *The value function in the initial problem (2.5) is given by*

$$(8.16) \quad V(\pi) = (1-\pi) \left[1 + c \hat{V} \left(\frac{\pi}{1-\pi}, \frac{\pi}{1-\pi} \right) \right]$$

for $\pi \in [0, 1]$ where the function \hat{V} is given by (8.4) above. The optimal stopping time in the initial problem (2.5) is given by

$$(8.17) \quad \tau_* = \inf \left\{ t \geq 0 \mid e^{\mu X_t^2 + (\lambda - \frac{\mu^2}{2})t} \left(\frac{\pi}{1-\pi} + \lambda \int_0^t e^{-\mu X_s^2 - (\lambda - \frac{\mu^2}{2})s} ds \right) \geq b \left(e^{\mu X_t^1 + (\lambda - \frac{\mu^2}{2})t} \left(\frac{\pi}{1-\pi} + \lambda \int_0^t e^{-\mu X_s^1 - (\lambda - \frac{\mu^2}{2})s} ds \right) \right) \right\}$$

where b is a unique solution to (8.3) above (see Figure 1).

Proof. The identity (8.16) was established in (3.1) above. The explicit form of the optimal stopping time (8.17) follows from (8.5) in Theorem 5 combined with (2.14)+(2.15) above. The final claim on b was derived in Theorem 5 above. This completes the proof. \square

9. Higher dimensions

The quickest detection problem formulated in Section 2 and the results derived in Sections 3-4 and 6-8 extend in a straightforward way from dimension *two* to dimension *three or higher*. This is readily obtained by replacing the coordinate number two of the observed process X by the coordinate number three or higher throughout and only the notation gets more complicated. In this section we briefly highlight this extension for future reference.

In the more general case, we consider a Bayesian formulation of the problem (2.5) where it is assumed that one observes a sample path of the standard n -dimensional Brownian motion $X = (X^1, \dots, X^n)$, whose coordinate processes X^1, \dots, X^n are standard Brownian motions with zero drift initially, and then at some random/unobservable time θ taking value 0 with probability $\pi \in [0, 1]$ and being exponentially distributed with parameter $\lambda > 0$ given that $\theta > 0$, one of the coordinate processes X^1, \dots, X^n gets a (known) non-zero drift μ permanently. The problem is to detect the time θ at which a coordinate process gets the drift μ as accurately as possible (neither too early nor too late).

Remark 7 (Higher dimensions). All the results and arguments in Sections 2-4 and 6-8 extend in an obvious way and remain valid when the coordinate number n is three or higher. The optimal stopping boundary b is no longer a curve but a surface in $[0, \infty)^n$ which is obtained by replacing $b(\varphi_1)$ by $b(\varphi_1, \dots, \varphi_{n-1})$ above. In particular, the existence and uniqueness results of Theorems 4 and 5 remain valid when n is three or higher and so does the solution to the initial problem (2.5) as discussed in Corollary 6 above.

Remark 8 (Signal-to-noise ratio). An interesting question is what we gain, if anything, by observing *all* coordinate processes of $X = (X^1, \dots, X^n)$ simultaneously in real time instead of a particular/individual coordinate process only when $n \geq 2$. It appears to be evident that observing a single coordinate process of one's choice is suboptimal, if for nothing else, then because the drift μ may not appear in the chosen coordinate process at all. Moreover, even if this deficiency is removed by adding all coordinate processes and forming $Y_t := X_t^1 + \dots + X_t^n$ as the observed one-dimensional process for $t \geq 0$, we see from (2.1)+(2.2) that Y solves

$$(9.1) \quad dY_t = \mu I(t \geq \theta) dt + \sqrt{n} dW_t$$

where $W_t := (\sum_{i=1}^n B_t^i)/\sqrt{n}$ is a standard Brownian motion for $t \geq 0$. Comparing (9.1) with either (2.1) or (2.2) we see that the *signal-to-noise ratio* (defined as the difference between the new drift and the old drift divided by the diffusion coefficient) has *decreased* in (9.1) because $\mu/\sqrt{n} < \mu$ when μ is positive and $n \geq 2$. Similarly, setting $Z_t := (\sum_{i=1}^n X_t^i)/\sqrt{n}$ for $t \geq 0$ we see from (9.1) that Z solves

$$(9.2) \quad dZ_t = \frac{\mu}{\sqrt{n}} I(t \geq \theta) dt + dW_t.$$

Thus, assuming that Z_t is being observed for $t \geq 0$, we see that the quickest detection problem for Z reduces to the problem in one dimension considered in Section 5 above. From (9.2) we see however that the drift μ/\sqrt{n} in the former problem is *strictly smaller* than the drift μ in the latter problem when $n \geq 2$ so that quickest detection for the observed process Z is *harder*. The final result of Corollary 6 above (combined with Remark 7) shows that quickest detection of a coordinate drift requires full knowledge of all coordinate processes X^1, \dots, X^n , so that observing one of them only, or even their sum, is insufficient to reach full optimality.

Remark 9 (Ad-hoc stopping rules). Recall from Section 1 above that the quickest detection setting of the observed process $X = (X^1, \dots, X^n)$ may also be viewed as a *multi-channel sensor system*. Quickest detection problems of this kind, including different performance measures and more complicated settings as well, have been studied in a number of papers (see [21]

& [5] and the references therein). These papers, as stated in Section 1 above, usually establish ‘asymptotic optimality’ of an ‘ad-hoc’ stopping rule and no ‘exact’ (optimal) solution has been derived in the literature to date. These ‘ad-hoc’ stopping rules can be broadly described by two classes of stopping times. In the first class one considers the sufficient statistics Φ^1, \dots, Φ^n in one dimension (Section 5) for the coordinate processes X^1, \dots, X^n respectively and postulates the ‘ad-hoc’ stopping rule to be the first hitting time of any among Φ^1, \dots, Φ^n to an optimal threshold in dimension one. In the setting of Figure 1 when $n = 2$ this ‘ad-hoc’ rule coincides with the first exit time of $\Phi = (\Phi^1, \dots, \Phi^n)$ from a rectangle (or an orthotope in higher dimensions) with two of its sides lying on the φ_1 and φ_2 axes and meeting at the origin (note that Φ never exits the triangle at either of these two sides). In the second class one considers the sufficient statistics Φ^1, \dots, Φ^n in one dimension (Section 5) for the coordinate processes X^1, \dots, X^n respectively and postulates the ‘ad-hoc’ stopping rule to be the first hitting time of the sum (or a linear combination) of Φ^1, \dots, Φ^n to a suitably determined threshold in dimension one. In the setting of Figure 1 when $n = 2$ this ‘ad-hoc’ rule coincides with the first exit time of $\Phi = (\Phi^1, \dots, \Phi^n)$ from a triangle (or a simplex in higher dimensions) with two of its sides lying on the φ_1 and φ_2 axes and meeting at the origin (note that Φ never exits the triangle at either of these two sides). The distinct curvature of the optimal stopping boundary b displayed in Figure 1 shows that no rectangle/orthotope or triangle/simplex will ever be able to achieve full/true optimality in this sense. Determining and characterising the *exact* curvature of the optimal stopping boundary for Φ in the canonical quickest detection problem of the observed process X is the main contribution of the present paper. It is envisaged that the exact solution found can be used as a basic building block for finding exact solutions to more general quickest detection problems of this kind. This development raises a number of questions which are left for future research.

Acknowledgements. The authors gratefully acknowledge support from the United States Army Research Office Grant ARO-YIP-71636-MA.

References

- [1] BAYRAKTAR, E. *and* POOR, H. V. (2007). Quickest detection of a minimum of two Poisson disorder times. *SIAM J. Control Optim.* 46 (308–331).
- [2] DAYANIK, S. POOR, H. V. *and* SEZER, S. O. (2008). Multisource Bayesian sequential change detection. *Ann. Appl. Probab.* 18 (552–590).
- [3] DE ANGELIS, T. *and* PESKIR, G. (2020). Global C^1 regularity of the value function in optimal stopping problems. *Ann. Appl. Probab.* 30 (1007–1031).
- [4] DU TOIT, J. *and* PESKIR, G. (2009). Selling a stock at the ultimate maximum. *Ann. Appl. Probab.* 19 (983–1014).
- [5] FELLOURIS, G. *and* SOKOLOV, G. (2016). Second-order asymptotic optimality in multi-sensor sequential change detection. *IEEE Trans. Inform. Theory* 62 (3662–3675).
- [6] GAPEEV, P. V. *and* PESKIR, G. (2006). The Wiener disorder problem with finite horizon. *Stochastic Process. Appl.* 116 (1770–1791).

- [7] GAPEEV, P. V. and SHIRYAEV, A. N. (2013). Bayesian quickest detection problems for some diffusion processes. *Adv. in Appl. Probab.* 45 (164–185).
- [8] GILBARG, D. and TRUDINGER, N. S. (2001). *Elliptic Partial Differential Equations of the Second Order*. Springer.
- [9] JOHNSON, P. and PESKIR, G. (2017). Quickest detection problems for Bessel processes. *Ann. Appl. Probab.* 27 (1003–1056).
- [10] PESKIR, G. (2005). On the American option problem. *Math. Finance* 15 (169–181).
- [11] PESKIR, G. (2006). On the fundamental solution of the Kolmogorov-Shiryayev equation. *The Shiryayev Festschrift* (Metabief, 2005), Springer (535–546).
- [12] PESKIR, G. (2007). A change-of-variable formula with local time on surfaces. *Sém. de Probab.* XL, *Lecture Notes in Math.* 1899, Springer (69–96).
- [13] PESKIR, G. (2019). Continuity of the optimal stopping boundary for two-dimensional diffusions. *Ann. Appl. Probab.* 29 (505–530).
- [14] PESKIR, G. and SHIRYAEV, A. N. (2006). *Optimal Stopping and Free-Boundary Problems*. Lectures in Mathematics, ETH Zürich, Birkhäuser.
- [15] POLYANIN, A. D. and ZAITSEV, V. F. (1995). *Handbook of Exact Solutions for Ordinary Differential Equations*. CRC Press.
- [16] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*. Springer-Verlag.
- [17] ROGERS, L. C. G. and WILLIAMS, D. (2000). *Diffusions, Markov Processes and Martingales: Itô Calculus* (Vol 2). Cambridge University Press.
- [18] SHIRYAEV, A. N. (1961). The problem of the most rapid detection of a disturbance in a stationary process. *Soviet Math. Dokl.* 2 (795–799).
- [19] SHIRYAEV, A. N. (1978). *Optimal Stopping Rules*. Springer-Verlag.
- [20] SHIRYAEV, A. N. (2010). Quickest detection problems: Fifty years later. *Sequential Anal.* 29 (345–385).
- [21] ZHANG, H. RODOSTHENOUS, N. and HADJILIADIS, O. (2015). Robustness of the N -CUSUM stopping rule in a Wiener disorder problem. *Ann. Appl. Probab.* 25 (3405–3433).

Philip A. Ernst
 Department of Statistics
 Rice University
 6100 Main Street
 Houston TX 77005
 United States
 philip.ernst@rice.edu

Goran Peskir
 Department of Mathematics
 The University of Manchester
 Oxford Road
 Manchester M13 9PL
 United Kingdom
 goran@maths.man.ac.uk