Stochastic Process. Appl. Vol. 81, No. 1, 1999, (25-38)
Research Report No. 389, 1997, *Dept. Theoret. Statist. Aarhus*

Designing Options Given the Risk: The Optimal Skorokhod-Embedding Problem

GORAN PESKIR

Motivated by applications in option pricing theory [9] we formulate and solve the following problem. Given a standard Brownian motion $B = (B_t)_{t \geq 0}$ and a centered probability measure μ on $I\!R$ having the distribution function F with a strictly positive density F' satisfying:

$$\int_0^\infty x \log x \;\mu(dx) < \infty$$

there exists a cost function $x \mapsto c(x)$ in the optimal stopping problem:

$$\sup_\tau E\left(\max_{0 \leq t \leq \tau} B_t - \int_0^\tau c(B_t)\, dt \right)$$

such that for the optimal stopping time τ_* we have:

$$B_{\tau_*} \sim \mu \;.$$

The cost function is explicitly given by the formula:

$$c(x) = \frac{1}{2} \frac{F'(x)}{\big(1 - F(x)\big)}$$

where one incidentally recognizes $x \mapsto F'(x)/(1-F(x))$ as *the Hazard function* of μ. There is also a simple explicit formula for the optimal stopping time τ_*, but the main emphasis of the result is on the existence of the underlying functional in the optimal stopping problem. The integrability condition on μ is natural and cannot be improved. The condition on the existence of a strictly positive density is imposed for simplicity, and more general cases could be treated similarly. The method of proof combines ideas and facts on optimal stopping of the maximum process [8] and the Azema-Yor solution of the Skorokhod-embedding problem [1]-[2]. A natural connection between these two theories is established, and new facts of interest for both are proved. The result extends in a similar form to stochastic integrals with respect to B, as well as to more general diffusions driven by B.

1. Introduction

Our main aim in this paper is to formulate and solve the following problem. Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ which starts at zero under P, and let $S = (S_t)_{t \geq 0}$ be the maximum process associated with B:

$$S_t = \max_{0 \leq r \leq t} B_r \;. \tag{1.1}$$

Let μ be a centered probability measure on $I\!R$ satisfying some minimal regularity conditions

MR 1991 Mathematics Subject Classification. Primary 60G40, 62L15, 60J65, 60J60. Secondary 90A09, 90A12, 60J25, 60G44.
Key words and phrases: Skorokhod-embedding, optimal stopping, option design, maximum process, cost function, Brownian motion, diffusion process, stochastic integral, local martingale, the maximality principle, the Stephan problem with moving (free) boundary, the principle of smooth fit, the Hazard function, the Hardy-Littlewood maximal function, infinitesimal generator, scale function, Ito formula. © goran@imf.au.dk

specified later. The problem under consideration in this paper is to answer if there exists a cost function $x \mapsto c(x) > 0$ in the optimal stopping problem:

$$\sup_{\tau} E\Big(S_\tau - \int_0^\tau c(B_t)\; dt \Big) \tag{1.2}$$

such that for the optimal stopping time τ_* we have:

$$B_{\tau_*} \sim \mu \; . \tag{1.3}$$

(The supremum in (1.2) is taken over all stopping times τ for B for which $E(\int_0^\tau c(B_t)\, dt) < \infty$.)

This problem is of interest in option pricing theory [9] where it is referred to as *the optimal Skorokhod-embedding problem*. In this context the measure μ plays the role of a risk, and the problem itself is to design an option given the risk (see [9] for more details). Therefore the main emphasis in this problem is on the existence of the underlying functional in (1.2):

$$f_t = S_t - \int_0^t c(B_r)\; dr \tag{1.4}$$

which has enough power to generate any measure μ upon optimal stopping in (1.2)-(1.3). In this context one should be aware of the fact that the maximum process is chosen and left fixed in (1.2) for two reasons. First, it is a path dependent functional which is known to produce comfort in regard to applications of option pricing theory (the Russian option of Shepp and Shiryaev [11]). Second, it is a path dependent functional which is known to offer a good solution [1]-[2] (see also [10] p.258-264) for the classic Skorokhod-embedding problem [13]. Thus the main point in the *optimal* Skorokhod-embedding problem formulated above is to show that upon choosing an appropriate cost function $x \mapsto c(x)$ in (1.2), with the maximum process being given and fixed, any measure μ can be generated by stopping B in (1.3) at an optimal stopping time τ_* for (1.2).

We note that the optimal Skorokhod-embedding problem (1.2)-(1.3) involves more difficulty than the classic Skorokhod-embedding problem, because we are not only supposed to find a stopping time for B which generates μ , but also an optimal stopping problem for which this stopping time is optimal. It is clear that by solving the optimal Skorokhod-embedding problem we also solve the classic Skorokhod-embedding problem.

The main result of the paper (Theorem 2.1) states that the answer to this problem is affirmative. Below we show that if μ has a strictly positive density F' and satisfies a natural $L \log L$-integrability condition, then (quite surprisingly) the following explicit formula is valid:

$$c(x) = \frac{1}{2}\, \frac{F'(x)}{\big(1 - F(x)\big)} \tag{1.5}$$

for all $x \in I\!R$, where F denotes the distribution function of μ . (The condition of a strictly positive density is imposed throughout for simplicity, and more general cases could be treated either similarly or by approximation.) It is interesting that in the expression (1.5) one may incidentally recognize $h(x) = F'(x)/(1-F(x))$ as *the Hazard function* of μ . Although in this paper we do not enter into explanations of its appearance in this context, we shall note that:

$$h(x) = P\big(x < X \le x + dx \mid X > x \big) \big/ dx \tag{1.6}$$

where $X \sim \mu$ is a random variable with distribution function F.

There is another interesting feature of this solution that we want to describe in some detail. For this one should note that the optimal stopping time in (1.2) with $x \mapsto c(x)$ from (1.5) is given by:

$$\tau_* = \inf \left\{ t > 0 \mid B_t \le g_*(S_t) \right\} \tag{1.7}$$

where the optimal stopping boundary $s \mapsto g_*(s)$ is the maximal solution of the differential equation:

$$g'(s) = \frac{1}{2\, c\big(g(s)\big)\big(s - g(s)\big)} \tag{1.8}$$

which stays below and never hits the diagonal in $I\!R^2$. We shall recall that this fact holds for all one-dimensional time-homogenous diffusions and is called *the maximality principle* (see [8]). We could further note that the equation (1.8) admits the general solution in a closed form, and from this formula we may observe that the optimal stopping boundary $s \mapsto g_*(s)$ can be expressed explicitly through its inverse by the formula:

$$g_*^{-1}(x) = \frac{1}{1 - F(x)} \int_x^\infty y \; dF(y) \tag{1.9}$$

for $x \in I\!R$. This function may now be recognized as *the barycentre function* of μ which was used by Azema and Yor in their solution of the Skorokhod-imbedding problem [1]-[2], or in other words, the stopping time (1.7) is the Azema-Yor stopping time satisfying (1.3). These observations establish a fundamental connection between these two theories and offer an explanation for the choice (1.9) which is based upon general principles of optimal stopping theory; we recall that the maximality principle is equivalent to a superharmonic characterization of the payoff (see [8]). For comparison with (1.5) we note that:

$$g_*^{-1}(x) = E\big(X \mid X > x\big) \tag{1.10}$$

for all $x \in I\!R$, where $X \sim \mu$ is a random variable with distribution function F.

This connection has also an impact on the Hardy-Littlewood theory [4] which is seen as follows. Using that $F(B_{\tau_*}) \sim U(0,1)$, and substituting $F(y) = v$ in (1.9), we see (as noted in [2]) that $S_{\tau_*} = g_*^{-1}(B_{\tau_*})$ is equally distributed under P as *the Hardy-Littlewood maximal function* of μ:

$$H(u) = \frac{1}{1-u} \int_u^1 F^{-1}(v)\; dv \quad \big(0 \le u \le 1\big) \tag{1.11}$$

on the probability space $[0,1]$ with Lebesgue measure. (For a nice exposition on connections of Hardy-Littlewood theory and martingale theory we refer to [7].) From this fact and a well-known argument of Blackwell and Dubins [3], we find that the optimal stopping time (1.7) in the problem (1.2) with $x \mapsto c(x)$ from (1.5) has another good property of interest in option pricing: If σ is any stopping time such that $B_\sigma \sim \mu$ and $E(S_\sigma) < \infty$, then S_{τ_*} *stochastically* maximizes S_σ in the following sense (Proposition 2.2):

$$P\{S_\sigma \ge s\} \le P\{S_{\tau_*} \ge s\} \tag{1.12}$$

for all $s \geq 0$. Taking now the advantage of thinking in terms of optimal stopping theory for (1.2) above, this result can be refined (Corollary 2.3): If equality in (1.12) is attained, then $\tau_* \leq \sigma$ P-a.s. This shows that τ_* is *pointwise* the *smallest* possible stopping time satisfying $B_{\tau_*} \sim \mu$ with a *"largest"* possible maximum of the process B up to the time of stopping. This *minimax* property of τ_* should be favourably compared with the best known extremal property of this type that if $\sigma \leq \tau_*$ with $B_\sigma \sim \mu$ then $\tau_* = \sigma$ P-a.s. which follows from the result of Monroe [6] upon uniform integrability of $(B_{t \wedge \tau_*})_{t \geq 0}$ established by Azema and Yor in [2]

We'd like to point out that we work throughout with Brownian motion B merely for simplicity, and the reader should note that the main results can be extended in a similar form to stochastic integrals with respect to B , as well as to more general diffusions driven by B , both merely at the expense of technical complexity. In this paper our main emphasis is on the method of proof, and our main aim is to present crucial steps and arguments which make the whole construction possible. More general questions which involve other processes of interest could be treated quite similarly.

2. The results and proof

The main result of the paper is formulated in the following theorem. We emphasize that some steps in the proof are of independent interest. This result can be extended in a similar form to stochastic integrals (continuous local martingales) and more general diffusions. One of its interesting consequences is presented in Corollary 2.3 below.

Theorem 2.1 (The optimal Skorokhod-embedding problem)

1. Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ which starts at zero under P , and let $S = (S_t)_{t \geq 0}$ be the maximum process associated with B:

(2.1) $$S_t = \max_{0 \leq r \leq t} B_r \; .$$

Let μ be a centered probability measure on $I\!R$ having the distribution function F with a strictly positive density F' satisfying:

(2.2) $$\int_0^\infty x \log x \;\, \mu(dx) < \infty \; .$$

Then there exists a cost function $x \mapsto c(x) > 0$ in the optimal stopping problem:

(2.3) $$\sup_\tau E\Big(\max_{0 \leq t \leq \tau} B_t - \int_0^\tau c(B_t)\, dt \Big) := V_*(0,0)$$

such that for the optimal stopping time τ_* we have:

(2.4) $$B_{\tau_*} \sim \mu \; .$$

(The supremum in (2.3) is taken over all stopping times τ for B for which $E(\int_0^\tau c(B_t)\, dt) < \infty$.)

2. The cost function is explicitly given by:

(2.5) $$c(x) = \frac{1}{2} \frac{F'(x)}{\big(1 - F(x)\big)} \qquad (x \in I\!R).$$

3. The optimal stopping time τ_* is explicitly given by:

$$\tau_* = \inf \left\{ \, t > 0 \mid B_t \le g_*(S_t) \, \right\} \tag{2.6}$$

where the optimal stopping boundary $s \mapsto g_*(s)$ is the maximal solution of the differential equation:

$$g'(s) = \frac{1}{2\,c\big(g(s)\big)\big(s - g(s)\big)} \tag{2.7}$$

which stays below and never hits the diagonal in $I\!R^2$ (the maximality principle). It is given explicitly through its inverse by:

$$g_*^{-1}(x) = \frac{1}{1 - F(x)} \int_x^\infty y \; dF(y) \qquad (x \in I\!R). \tag{2.8}$$

4. The payoff is explicitly given by:

$$V_*(0,0) = -\int_{-\infty}^{0} y \, \frac{F'(y)}{1 - F(y)} \; dy \tag{2.9}$$

and we have $0 < V_*(0,0) < \infty$.

5. The condition (2.2) is best possible in this context (see (2.22)-(2.23) below).

Proof. 1. Consider the following stopping time for B :

$$\tau_g = \inf \left\{ \, t > 0 \mid B_t \le g(S_t) \, \right\} \tag{2.10}$$

where $g : I\!R_+ \rightarrow I\!R$ is a strictly increasing C^1-function satisfying $g(s) < s$ for all s , as well as $g(0) = -\infty$ and $g(\infty) = \infty$.

2. *Claim.* We show that $B_{\tau_g} \sim F$ if and only if g solves the following equation:

$$g'\big(g^{-1}(x)\big) = \frac{1 - F(x)}{F'(x)\,\big(g^{-1}(x) - x\big)} \tag{2.11}$$

for $x \in I\!R$. (Note that due to $F' > 0$, we must have $g(0) = -\infty$ and $g(\infty) = \infty$.)

3. To prove (2.11) we shall recall from either [5] or [1] that:

$$P\big\{ S_{\tau_g} \ge s \big\} = \exp\left(-\int_0^s \frac{dt}{t - g(t)} \right) \tag{2.12}$$

for all $s \ge 0$. Suppose now that $B_{\tau_g} \sim F$ for some g . Then from (2.12) we get:

$$\begin{aligned} 1 - F(x) &= P\big\{ B_{\tau_g} > x \big\} = P\big\{ g(S_{\tau_g}) > x \big\} = P\big\{ S_{\tau_g} > g^{-1}(x) \big\} \\ &= \exp\left(-\int_0^{g^{-1}(x)} \frac{dt}{t - g(t)} \right) = \exp\left(-\int_{g(0)}^{x} \frac{dr}{g'\big(g^{-1}(r)\big)\big(g^{-1}(r) - r\big)} \right) \end{aligned} \tag{2.13}$$

upon substituting $g(t) = r$. Differentiating over x in (2.13), we obtain:

(2.14)
$$\frac{F'(x)}{1-F(x)} = \frac{1}{g'\big(g^{-1}(x)\big)\big(g^{-1}(x)-x\big)}$$

which is equivalent to (2.11). On the other hand, if g solves (2.11), or equivalently (2.14), then we obtain the final equality in (2.13) upon integrating in (2.14). This proves Claim 2 above.

4. Consider the optimal stopping problem with the payoff:

(2.15)
$$V_*(x,s) = \sup_\tau E_{x,s}\bigg(\max_{0\le t\le\tau} B_t - \int_0^\tau c(B_t)\,dt \bigg)$$

where under $P_{x,s}$ the Brownian motion B starts at $x \in I\!\!R$ and the maximum process S starts at $s \ge x$, and the cost function $x \mapsto c(x)$ is assumed continuous and strictly positive. The main result in [8] states that this problem has a solution if and only if the first-order nonlinear differential equation (2.7) (obtained by *the principle of smooth fit*) admits the maximal solution $s \mapsto g_*(s)$ which stays below and never hits the diagonal in $I\!\!R^2$ (the maximality principle). In this case the stopping time (2.6) is optimal for the problem (2.15) whenever:

(2.16)
$$E_{x,s}\bigg(\int_0^{\tau_*} c(B_t)\,dt \bigg) < \infty \ .$$

(If (2.16) fails to hold, then τ_* is known to be an approximate optimal stopping time.)

5. In order to choose $x \mapsto c(x)$ in (2.15) so that $B_{\tau_*} \sim F$, motivated by Claim 2 above, we shall rewrite equation (2.7) in terms of the inverse function $x \mapsto g^{-1}(x)$:

(2.17)
$$g'\big(g^{-1}(x)\big) = \frac{1}{2\,c(x)\,\big(g^{-1}(x)-x\big)}$$

and identify this equation with the equation (2.11). This shows that (2.5) defines a unique candidate for the cost function $x \mapsto c(x)$ in the problem (2.15), such that for the optimal stopping time (2.6) we have (2.4) under $P_{0,0} = P$.

6. Recall that the first-order linear differential equation:

(2.18)
$$y'(x) + a(x)y(x) = b(x)$$

has the general solution given by:

(2.19)
$$y(x) = Ce^{-A(x)} + e^{-A(x)} \int b(x)\,e^{A(x)}\;dx$$

with $C \in I\!\!R$ where $A'(x) = a(x)$.

It is shown above that under (2.5) the equations (2.7) and (2.11) are identical, and from (2.17) we easily see that either can be rewritten in terms of the inverse function as follows:

(2.20)
$$\big(g^{-1}\big)'(x) - \frac{F'(x)}{1-F(x)}\,g^{-1}(x) = -\frac{F'(x)}{1-F(x)}\,x \ .$$

This equation is linear, and is of type (2.19), with $A(x) = \log(1-F(x))$. Therefore the general solution of either (2.7) or (2.11) is given by:

$$g^{-1}(x) = \frac{C}{1-F(x)} + \frac{1}{1-F(x)} \int_x^\infty t \; dF(t) \tag{2.21}$$

with $C \in I\!R$, where we use the fact that μ is centered, and therefore the first moment of μ exists so that the integral over $dF(t)$ in (2.21) is well-defined and finite. It is now easily verified that the maximal solution $s \mapsto g_*(s)$ of (2.7) is obtained by taking $C = 0$, and thus (2.8) holds.

7. In order to show that $s \mapsto g_*(s)$ is an optimal stopping boundary in the problem (2.15), i.e. that τ_* from (2.6) is an optimal stopping time in (2.15), it would be enough to show that (2.16) holds with $P_{0,0} = P$. However, instead of making an attempt to show that (2.2) has the power of implying (2.16) without any reference to the problem (2.15), we shall take a more direct route to the solution of the problem (2.15) which is based upon an idea applied in [8].

The first step in this direction is contained in the following result from [2]. For the maximal solution $s \mapsto g_*(s)$ of (2.7), the following facts are equivalent:

$$E\big(S_{\tau_*}\big) \, < \, \infty \tag{2.22}$$

$$E\big(B^+_{\tau_*} \log B^+_{\tau_*}\big) \, < \, \infty \tag{2.23}$$

where τ_* is given by (2.6). (This equivalence explains the appearance of condition (2.2) above, being precisely (2.23), and shows its optimality in this context.) Thus, it is enough to show that condition (2.22) implies that the stopping time τ_* from (2.6) is optimal for the problem (2.15) with $P_{0,0} = P$, or in other words, that (2.16) with $P_{0,0} = P$ and (2.22) are equivalent.

8. To show that (2.22) implies (2.16) with $P_{0,0} = P$, we need to recall a few facts from [8]. Consider the optimal stopping problem (2.15). Then by general results of optimal stopping theory (see [12]) one is naturally led to formulate the following system for the payoff $V_*(x,s)$ in the continuation region $g_*(s) < x \leq s$:

$$(I\!\!LV)(x,s) = c(x) \quad \text{for} \;\; g(s) < x < s \;\; \text{with} \;\; s \;\; \text{fixed} \tag{2.24}$$

$$\frac{\partial V}{\partial s}(x,s)\Big|_{x=s-} = 0 \quad (\textit{normal reflection}) \tag{2.25}$$

$$V(x,s)\Big|_{x=g(s)+} = s \quad (\textit{instantaneous stopping}) \tag{2.26}$$

$$\frac{\partial V}{\partial x}(x,s)\Big|_{x=g(s)+} = 0 \quad (\textit{smooth fit}) \tag{2.27}$$

where in our case here $I\!\!L = \partial^2/2\partial x^2$. The system (2.24)-(2.27) forms a *Stephan problem with moving (free) boundary* (see [12]). It is shown in [8] that the following function solves this system:

$$V_g(x,s) = s + \int_{g(s)}^x \big(S(x)-S(y)\big)c(y) \; m(dy) \tag{2.28}$$

(with $S(x) = x$ and $m(dx) = 2dx$ in the present case) *if and only if* the C^1-function $s \mapsto g(s)$ solves the following first-order nonlinear differential equation:

(2.29) $$g'(s) = \frac{\sigma^2\big(g(s)\big)\, S'\big(g(s)\big)}{2\, c\big(g(s)\big)\Big(S(s) - S\big(g(s)\big)\Big)}$$

(with $S(x) = x$ and $\sigma(x) = 1$ in the present case). Note that this equation in the present case is precisely the equation (2.7), or equivalently, the equation (2.11) upon the choice (2.5). It was important in [8] that the following representation holds:

(2.30) $$V_g(x,s) = E_{x,s}\Big(S_{\tau_g} - \int_0^{\tau_g} c(B_t)\; dt \Big)$$

whenever both $E_{x,s}(S_{\tau_g})$ and $E_{x,s}(\int_0^{\tau_g} c(B_t)\; dt)$ are finite. (This is verified by Ito formula.)

9. The key idea now is to use that fact that $s \mapsto g_*(s)$ is the maximal solution of (2.7) which stays below and never hits the diagonal in $\mathbb{R}^2$, so that there exists a decreasing sequence of solutions of (2.7) satisfying $g_n(s) \downarrow g_*(s)$ as $n \to \infty$. Each g_n must hit the diagonal in $\mathbb{R}^2$ and therefore $E_{x,s}(S_{\tau_{g_n}})$ and $E_{x,s}(\int_0^{\tau_{g_n}} c(B_t)\; dt)$ must be finite. Thus for each g_n the representation (2.30) holds, and from (2.28) we obtain:

(2.31) $$\begin{aligned} V_{g_*}(0,0) &= \lim_{n\to\infty} V_{g_n}(0,0) = \lim_{n\to\infty} E\Big(S_{\tau_{g_n}} - \int_0^{\tau_{g_n}} c(B_t)\; dt \Big) \\ &= E\Big(S_{\tau_*} - \int_0^{\tau_*} c(B_t)\; dt \Big) \end{aligned}$$

where the final equality follows by monotone convergence since (2.22) holds under (2.2). Since $V_{g_*}(0,0) = -\int_{-\infty}^0 yF'(y)/(1-F(y))\; dy$ is clearly finite (we assume that μ is centered and thus $\int |x|\, \mu(dx) < \infty$), we see that (2.16) is satisfied with $P_{0,0} = P$. Thus by the result of [8] we know that τ_* is an optimal stopping time in the problem (2.3) and the payoff is given by (2.9) above. For completeness and convenience we shall sketch how this can be formally verified.

10. Upon extending $V_*(x,s) = s$ for all $x \le g_*(s)$, by Ito formula we derive the following representation for the process (B_t, S_t) being composed with the payoff candidate $V_{g_*}(x,s)$:

(2.32) $$V_{g_*}(B_t, S_t) - \int_0^t c(B_r)\; dr = V_{g_*}(x,s) + M_t - P_t$$

where $M = (M_t)_{t\ge 0}$ is a local martingale and the process:

(2.33) $$P_t = \int_0^t c(B_r)\, 1_{\big(B_r \le g(S_r)\big)}\; dr$$

is non-negative. Thus the process on the left-hand side in (2.32) is a local supermartingale. Let τ be any stopping time for B for which $E(\int_0^\tau c(B_t)\; dt) < \infty$. Choose a localization sequence $(\sigma_n)_{n\ge 1}$ of bounded stopping times for the local martingale M. Noting that $V_{g_*}(x,s) \ge s$ for all x and all s, from (2.32) it follows:

(2.34) $$\begin{aligned} E\Big(S_{\tau\wedge\sigma_n} - \int_0^{\tau\wedge\sigma_n} c(B_t)\; dt \Big) &\le E\Big(V_{g_*}(B_{\tau\wedge\sigma_n}, S_{\tau\wedge\sigma_n}) - \int_0^{\tau\wedge\sigma_n} c(B_t)\; dt \Big) \\ &\le V_{g_*}(0,0) + E\big(M_{\tau\wedge\sigma_n}\big) = V_{g_*}(0,0)\; . \end{aligned}$$

Letting $n \to \infty$ and using Fatou's lemma, hence we get by taking supremum over all such τ:

$$\sup_{\tau} E\Big(S_\tau - \int_0^\tau c(B_t)\ dt \Big) \le V_{g_*}(0,0)\ . \tag{2.35}$$

From (2.31) we see that this supremum is attained at τ_* , and the proof of the theorem is complete. □

Remarks:

1. Note that the distribution law of B being stopped at the optimal stopping time τ_* in the problem (2.3) can be straightforwardly obtained from the cost function in (2.3) as follows:

$$F_{B_{\tau_*}}(x) = 1 - \exp\Big(-2 \int_{-\infty}^{x} c(y)\ dy \Big) \tag{2.36}$$

for $x \in I\!R$. This follows from (2.5) upon noting that $2c(x) = F'(x)/(1-F(x)) = -(\log(1-F(x)))'$ and then integrating both sides. Observe also that the payoff (2.9) depends only on the law μ restricted to the negative half-line.

2. To indicate how the result of Theorem 2.1 can be extended to more general diffusions driven by B , assume that such a diffusion $X = (X_t)_{t \ge 0}$ solves the stochastic differential equation:

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dB_t \tag{2.37}$$

where $x \mapsto \mu(x)$ and $x \mapsto \sigma(x) > 0$ are continuous. Then the scale function of X is given by:

$$S(x) = \int^x \exp\Big(-\int^y \frac{\mu(z)}{\sigma^2(z)/2}\ dz \Big) dy \tag{2.38}$$

and let us assume that $Z_t := S(X_t) \to \pm\infty$ as $t \to \infty$; it means that X is recurrent.

In parallel to the proof of Theorem 2.1, we may note that (2.12) extends as follows:

$$P\big\{ S_{\tau_g} \ge s \big\} = \exp\Big(-\int_0^s \frac{dS(t)}{S(t) - S\big(g(t)\big)} \Big) \tag{2.39}$$

and the equation (2.11) reads as follows:

$$g'\big(g^{-1}(x)\big) = \frac{\big(1-F(x)\big)\, S'\big(g^{-1}(x)\big)}{F'(x)\Big(S\big(g^{-1}(x)\big) - S(x)\Big)}\ . \tag{2.40}$$

The analogue of the equation (2.17) obtained by rewriting the equation (2.29) in terms of the inverse function $x \mapsto g^{-1}(x)$ looks like:

$$g'\big(g^{-1}(x)\big) = \frac{\sigma^2(x)\, S'(x)}{2\, c(x)\Big(S\big(g^{-1}(x)\big) - S(x)\Big)}\ . \tag{2.41}$$

Finally, the analogue of (2.8) is the following formula:

$$g_*^{-1}(x) = S^{-1}\bigg(\frac{1}{1-F(x)} \int_x^\infty S(y)\ dF(y) \bigg) \tag{2.42}$$

where it suffices to assume $\int_0^\infty S(x)\ \mu(dx) < \infty$ and $\int_0^1 S^{-1}\big((1-u)^{-1} \int_u^1 S(F^{-1}(v))\ dv\big)\, du < \infty$

(the latter condition is equivalent to $E(\max_{0 \le t \le \tau_*} X_t) < \infty$).

Identifying (2.40) and (2.41) and using (2.42), we obtain the following formula for the cost function which extends (2.5):

$$c(x) = \frac{\sigma^2(x)\, F'(x)\, S'(x)}{2\,\big(1-F(x)\big)\, S'\bigg(S^{-1}\Big(\frac{1}{1-F(x)} \int_x^\infty S(y)\; dF(y)\Big)\bigg)} \; . \tag{2.43}$$

It is now a matter of routine to reformulate and extend the result of Theorem 2.1 with Brownian motion B being replaced by diffusion X . In this context one should recall that the diffusion $Z_t = S(X_t)$ solves the following equation:

$$dZ_t = S'\big(S^{-1}(Z_t)\big)\, \sigma\big(S^{-1}(Z_t)\big)\, dB_t \tag{2.44}$$

and thus is in natural scale (the scale function equals identity). We shall omit explicit statements and further applications of this result for conciseness. Note also that continuous local martingales can also be reduced to Theorem 2.1 by means of standard time change techniques.

3. A good way to experience and learn the full power of the argument used in Part 9 in the proof of Theorem 2.1 above is to try to derive (2.16) directly from (2.2) or (2.22) by means of any standard technique.

We proceed by examining some extremal properties of the stopping time τ_* from (2.6) satisfying $B_{\tau_*} \sim \mu$. Our main result is presented in Corollary 2.3 below. The following result combines the observation of Azema and Yor in [2] that S_{τ_*} is equally distributed as the Hardy-Littlewood maximal function of μ , and the well-known argument of Blackwell and Dubins [3] in a somewhat clearer form.

Proposition 2.2

In the context of Theorem 2.1 assume that σ *is any stopping time for* B *such that:*

$$B_\sigma \sim \mu \tag{2.45}$$

$$E(S_\sigma) < \infty \; . \tag{2.46}$$

Then the following inequality holds:

$$P\{S_\sigma \ge s\} \le P\{S_{\tau_*} \ge s\} \tag{2.47}$$

for all $s \ge 0$.

Proof. By Doob's maximal inequality (see [10]) we have:

$$sP\{S_{\sigma\wedge t} \ge s\} \le \int_{\{S_{\sigma\wedge t} \ge s\}} B_{\sigma\wedge t}\; dP \tag{2.48}$$

for all $t > 0$ with $s > 0$ given and fixed. Since (2.46) is satisfied, we may use Fatou's lemma, and by letting $t \to \infty$ in (2.48), we get:

(2.49) $$sP\{S_\sigma \ge s\} \le \int_{\{S_\sigma \ge s\}} B_\sigma \; dP \; .$$

Denote $A = \{S_\sigma \ge s\}$, then we have:

(2.50) $$\begin{aligned} \int_A B_\sigma \; dP &\le \int_A \big(B_\sigma \vee x\big) \; dP = \int_A \big(x + (B_\sigma - x)^+\big) \; dP \\ &\le x\,P(A) + \int (B_\sigma - x)^+ \, dP = x\,P(A) + \int_{\{B_\sigma > x\}} B_\sigma \; dP - x\,P\{B_\sigma > x\} \\ &= x\Big(P(A) - P\{B_\sigma > x\}\Big) + \int_{\{B_\sigma > x\}} B_\sigma \; dP \end{aligned}$$

for all $x \in I\!R$. Thus, if we choose x such that $P(A) - P\{B_\sigma > x\} = 0$, or in other words, $F(x) = 1 - P(A)$, then by (2.49) and (2.50) we get:

(2.51) $$sP\{S_\sigma \ge s\} \le \int_{\{B_\sigma > x\}} B_\sigma \; dP \; .$$

Since $F(B_\sigma) \sim U(0,1)$ by (2.45), we have:

(2.52) $$\begin{aligned} \int_{\{B_\sigma > x\}} B_\sigma \; dP &= \int_{\{F(B_\sigma) > F(x)\}} F^{-1}\big(F(B_\sigma)\big) \; dP \\ &= \int_{F(x)}^1 F^{-1}(v) \; dv = \int_{1-P(A)}^1 F^{-1}(v) \; dv \; . \end{aligned}$$

On the other hand, since $F(B_{\tau_*}) \sim U(0,1)$, we have:

(2.53) $$\begin{aligned} P\{S_{\tau_*} \ge s\} &= P\{g_*^{-1}(B_{\tau_*}) \ge s\} = P\bigg(\frac{1}{1 - F(B_{\tau_*})} \int_{F(B_{\tau_*})}^1 F^{-1}(v) \; dv \ge s\bigg) \\ &= \lambda\bigg(u \in [0,1] \;\Big|\; \frac{1}{1-u} \int_u^1 F^{-1}(v) \; dv \ge s \bigg) = 1 - u_* \end{aligned}$$

where u_* is the *smallest* u in $[0,1]$ for which:

(2.54) $$\frac{1}{1-u} \int_u^1 F^{-1}(v) \; dv \ge s$$

since the function on the left-hand side in (2.54) is increasing in u .

From (2.51) and (2.52) we now see that:

(2.55) $$sP(A) \le \int_{1-P(A)}^1 F^{-1}(v) \; dv$$

and therefore $u_* \le 1 - P(A)$. This by (2.53) shows that:

(2.56) $$P\{S_\sigma \ge s\} = P(A) \le 1 - u_* = P\{S_{\tau_*} \ge s\}$$

and the proof is complete. □

Taking the advantage of our approach in Theorem 2.1, we may now think of τ_* in terms of optimal stopping theory, and this enables us to exhibit yet another extremal property of τ_* when the equality in (2.47) is attained.

Corollary 2.3 (The minimax property)
In the context of Theorem 2.1 assume that σ *is any stopping time for* B *such that:*

$$B_\sigma \sim \mu \tag{2.57}$$

$$E(S_\sigma) = E(S_{\tau_*}) . \tag{2.58}$$

Then the following inequality holds:

$$\tau_* \le \sigma \quad P\text{-a.s.} \tag{2.59}$$

Proof. We show that under (2.57) and (2.58) the stopping time σ is optimal for the problem (2.3). For this, since (2.58) holds, it is enough to show that:

$$E\Big(\int_0^\sigma c(B_t)\; dt \Big) = E\Big(\int_0^{\tau_*} c(B_t)\; dt \Big) . \tag{2.60}$$

Motivated by (2.57) we shall prove (2.60) by Ito formula.

Due to some convergence requirements appearing below, we shall approximate function $x \mapsto c(x) > 0$ by continuous functions $x \mapsto c_n(x)$ satisfying:

$$c_n = c \quad \text{on} \quad \big[-(n-1)\, ,\; n-1 \big] \tag{2.61}$$

$$c_n = 0 \quad \text{on} \quad \big] -\infty\, ,\; -n \big] \cup \big[\, n\, ,\; +\infty \big[\tag{2.62}$$

$$0 < c_n \le c \quad \text{on} \quad \big] -n\, ,\; -(n-1) \big[\;\cup\; \big]\, (n-1)\, ,\; n \big[\tag{2.63}$$

for $n \ge 1$. Then clearly $0 \le c_n \uparrow c$ as $n \to \infty$. Denote $G_n(x) = \int_{-\infty}^x c_n(y)\; dy$ and define $H_n(x) = \int_{-\infty}^x G_n(y)\; dy$ for $x \in I\!R$. Then $H_n \in C^2$ and $H_n'' = c_n$ for all $n \ge 1$. Moreover, it is easily seen that:

$$0 \le H_n(x) \le A_n\, (x-n)^+ + B_n \tag{2.64}$$

for some constants $A_n\, ,\, B_n > 0$ with $n \ge 1$.

By Ito formula we get:

$$\begin{aligned} H_n(B_t) &= H_n(0) + \int_0^t H_n'(B_r)\; dB_r + \frac{1}{2} \int_0^t H_n''(B_r)\; dr \\ &:= H_n(0) + M_t + \frac{1}{2} \int_0^t c_n(B_r)\; dr \end{aligned} \tag{2.65}$$

where $M = (M_t)_{t \ge 0}$ is a continuous local martingale. Choose a localization sequence $(\tau_k)_{k \ge 1}$ of bounded stopping times for M . Then by the optional sampling theorem we find:

$$E\Big(H_n\big(B_{\sigma\wedge\tau_k}\big)\Big) = H_n(0) + \frac{1}{2}\,E\bigg(\int_0^{\sigma\wedge\tau_k} c_n(B_r)\;dr\bigg)\;. \tag{2.66}$$

Letting $k\rightarrow\infty$ and using that by (2.64) we have $0 \le H_n(B_{\sigma\wedge\tau_k}) \le A_n\,(B_{\sigma\wedge\tau_k}-n)^+ + B_n \le A_n\,B^+_{\sigma\wedge\tau_k} + B_n \le A_n\,S_\sigma + B_n$ with $E(S_\sigma) < \infty$, we get:

$$E\Big(H_n\big(B_\sigma\big)\Big) = H_n(0) + \frac{1}{2}\,E\bigg(\int_0^{\sigma} c_n(B_r)\;dr\bigg)\;. \tag{2.67}$$

Applying exactly the same argument to τ_* , we obtain:

$$E\Big(H_n\big(B_{\tau_*}\big)\Big) = H_n(0) + \frac{1}{2}\,E\bigg(\int_0^{\tau_*} c_n(B_r)\;dr\bigg)\;. \tag{2.68}$$

Since by (2.57) above we have $B_\sigma \sim B_{\tau_*} \sim \mu$, we see that $E\big(H_n(B_\sigma)\big) = E\big(H_n(B_{\tau_*})\big)$, and therefore from (2.67) and (2.68) we may conclude:

$$E\bigg(\int_0^{\sigma} c_n(B_t)\;dt\bigg) = E\bigg(\int_0^{\tau_*} c_n(B_t)\;dt\bigg) \tag{2.69}$$

for all $n \ge 1$. Letting $n\rightarrow\infty$ we finish up with (2.60) as claimed.

To prove now that (2.59) holds, we shall repeat the argument from [8]. Suppose for this that $P\{\sigma<\tau_*\}>0$, and note that τ_* can be written as follows:

$$\tau_* = \inf\;\big\{\,t>0 \;|\; V_*(B_t,S_t)=S_t\,\big\} \tag{2.70}$$

so that $S_\sigma < V_*(B_\sigma,S_\sigma)$ if $\sigma<\tau_*$. Hence we easily get:

$$\begin{aligned} E\bigg(S_\sigma - \int_0^\sigma c(B_t)\;dt\bigg) &< E\bigg(V_*(B_\sigma,S_\sigma) - \int_0^\sigma c(B_t)\;dt\bigg) \\ &\le V_*(0,0) = \sup_\tau E\bigg(S_\tau - \int_0^\tau c(B_t)\;dt\bigg) \end{aligned} \tag{2.71}$$

where the final inequality follows easily from the fact that the process $V_*(B_t,S_t) - \int_0^t c(B_r)\;dr$ is a local supermartingale. However, the strict inequality in (2.71) contradicts the fact that σ is an optimal stopping time, and thus we must have $P\{\sigma<\tau_*\}=0$. The proof is complete. □

Remarks:

1. The preceding result refines the well-known extremal property of τ_* which goes back to Monroe [6] and follows by uniform integrability of $(B_{\tau_*\wedge t})_{t\ge0}$ as shown by Azema and Yor in [2]: If $\sigma\le\tau_*$ and $B_\sigma\sim\mu$, then $\tau_*=\sigma$ P-a.s. Note, however, if $\sigma\le\tau_*$ then $B^+_{\sigma\wedge t} \le S_\sigma \le S_{\tau_*}$, and since $E(S_{\tau_*})<\infty$, we see that $(B^+_{\sigma\wedge t})_{t\ge0}$ is uniformly integrable. A closer look into the proof above shows that this is sufficient to derive (2.60), and since $x\mapsto c(x)$ is strictly positive, this implies that $\sigma=\tau_*$ P-a.s. The result above demonstrates the advantage of considering τ_* as an optimal stopping time, and thinking about it within optimal stopping theory.

2. Note that the preceding result states that τ_* is *pointwise* the smallest possible stopping time satisfying $B_{\tau_*}\sim\mu$ with a “largest” possible maximum of the process B up to the time

of stopping. Observe that this *minimax* property characterises τ_* uniquely, and that equality in (2.59) actually holds (by extending Monroe's argument quoted above). Observe also by (2.47) and integration by parts that the assumption (2.58) is equivalent to $S_\sigma \sim S_{\tau_*}$.

Acknowledgment. I am grateful to Jørgen Hoffmann-Jørgensen for showing me his elegant argument (2.50) and for telling me that the cost function appearing in the solution above is called the Hazard function.

REFERENCES

[1] AZEMA, J. *and* YOR, M. (1979). Une solution simple au probleme de Skorokhod. *Sem. Probab.* XIII, *Lecture Notes in Math.* 721, Springer (90-115).

[2] AZEMA, J. *and* YOR, M. (1979). Le probleme de Skorokhod: Complements a l'expose precedent. *Sem. Probab.* XIII, *Lecture Notes in Math.* 721, Springer (625-633).

[3] BLACKWELL, D. *and* DUBINS, L. (1963). A converse to the dominated convergence theorem. *Illinois J. Math.* 7 (508-514).

[4] HARDY, G. H. *and* LITTLEWOOD, J. E. (1930). A maximal theorem with function theoretic applications. *Acta Math.* 54 (81-116).

[5] LEHOCZKY, J. P. (1977). Formulas for stopped diffusion processes with stopping times based on the maximum. *Ann. Probab.* 5 (601-607).

[6] MONROE, I. (1972). On embedding right continuous martingales in Brownian motion. *Ann. Math. Statist.* 43 (1293-1311).

[7] GILAT, D. (1986). The best bound in the $L \log L$ inequality of Hardy and Littlewood and its martingale counterpart. *Proc. Amer. Math. Soc.* 97 (429-436).

[8] PESKIR, G. (1997). Optimal stopping of the maximum process: The maximality principle. *Research Report* No. 377, *Dept. Theoret. Statist. Aarhus* (30 pp). *Ann. Probab.* 26, 1998 (1614-1640).

[9] PESKIR, G. (1997). A true buyer's risk and classification of options. *Research Report* No. 386, *Dept. Theoret. Statist. Aarhus* (19 pp). *Inform. Technol. Econom. Management* No. 1, 2001 (1-20).

[10] REVUZ, D. *and* YOR, M. (1994). *Continuous Martingales and Brownian Motion.* (Second Edition) Springer-Verlag.

[11] SHEPP, L. A. *and* SHIRYAEV, A. N. (1993). The Russian option: Reduced regret. *Ann. Appl. Probab.* 3 (631-640).

[12] SHIRYAEV, A. N. (1978). *Optimal stopping rules.* Springer-Verlag, Berlin-New York.

[13] SKOROKHOD, A. (1965). *Studies in the theory of random processes.* Addison-Wesley, Reading.

Goran Peskir
Department of Mathematical Sciences
University of Aarhus, Denmark
Ny Munkegade, DK-8000 Aarhus
home.imf.au.dk/goran
goran@imf.au.dk