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Consistency of Statistical Models Described by Families of Reversed Submartingales

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A large number of statistical models is described by a family of reversed submartingales converging to degenerated limits. The problem under consideration is to estimate the maximum points of the limit function. For this various maximum functions are used and consequently different concepts of consistency are introduced. In this paper we introduce and investigate a general reversed submartingale framework for these models. Our approach relies upon the i.i.d. case [6]. We show that the best known sufficient conditions for consistency in this case remain valid for conditionally S-regular families of reversed submartingales introduced in [13], which are known to include all U-processes. Moreover, by using results on uniform convergence of families of reversed submartingales [15], we deduce new conditions for consistency. These conditions are expressed by means of Hardy's regular convergence [4], and are of a total boundedness in the mean type. In this way the problem of consistency is naturally connected with the infinitely dimensional (uniform) reversed submartingale convergence theorem. Applications to a stochastic maximization of families of random processes over time sets are also given.

1. Introduction

1. Many statistical models from [1]-[3], [5]-[12], [16]-[18] can be recognized as a family of reversed submartingales $\mathcal{H} = (\{h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ defined on the probability space $(\Omega, \mathcal{F}, P)$ and indexed by a separable metric space Θ_0. From general theory of reversed submartingales we know that each $h_n(\theta)$ converges P-almost surely to a random variable $h_\infty(\theta)$ as $n \to \infty$. If the tail σ-algebra $\mathcal{S}_\infty = \cap_{n=1}^\infty \mathcal{S}_n$ is degenerated, that is $P(A) \in \{0,1\}$ for all $A \in \mathcal{S}_\infty$, then $h_\infty(\theta)$ is also degenerated, that is P-almost surely equal to some constant which depends on $\theta \in \Theta_0$. In this case *the information function* associated with $\mathcal{H}$:

$$I(\theta) = a.s. \lim_{n\to\infty} h_n(\theta) = \lim_{n\to\infty} Eh_n(\theta)$$

may be well-defined for all $\theta \in \Theta_0$. The main problem under consideration is *to determine the maximum points of* I *on* Θ_0 using only information available on $h_n(\omega,\theta)$.

2. Two concepts of maximum functions are naturally introduced in this context as follows. Let $\{ \hat{\theta}_n \mid n \geq 1 \}$ be a sequence of functions from Ω into Θ, where (Θ, d) is a compact metric

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space containing Θ_0 . Then $\{ \hat{\theta}_n \mid n \geq 1 \}$ is called *a sequence of empirical maximums* associated with $\mathcal{H}$, if there exist a function $q : \Omega \to \mathbf{N}$ and a P-null set $N \in \mathcal{F}$ satisfying:

$$\hat{\theta}_n(\omega) \in \Theta_0 \ , \quad \forall \omega \in \Omega \setminus N \ , \ \forall n \geq q(\omega) \tag{1.1}$$

$$h_n(\omega, \hat{\theta}_n(\omega)) = h_n^*(\omega, \Theta_0) \ , \quad \forall \omega \in \Omega \setminus N \ , \ \forall n \geq q(\omega) \tag{1.2}$$

where $h_n^*(\omega, B) = \sup_{\theta \in B} h_n(\omega, \theta)$ for $n \geq 1$, $\omega \in \Omega$ and $B \subset \Theta_0$. The sequence $\{ \hat{\theta}_n \mid n \geq 1 \}$ is called *a sequence of approximating maximums* associated with $\mathcal{H}$, if there exist a function $q : \Omega \to \mathbf{N}$ and a P-null set $N \in \mathcal{F}$ satisfying:

$$\hat{\theta}_n(\omega) \in \Theta_0 \ , \quad \forall \omega \in \Omega \setminus N \ , \ \forall n \geq q(\omega) \tag{1.3}$$

$$\liminf_{n \to \infty} h_n(\omega, \hat{\theta}_n(\omega)) \geq \sup_{\theta \in \Theta_0} I(\theta) \ , \quad \forall \omega \in \Omega \setminus N \ . \tag{1.4}$$

3. Despite the fact that $h_n(\omega, \cdot)$ does not need to attain its maximal value on Θ_0 , and (1.2) can fail in this case, we can always find a sequence $\{ \hat{\theta}_n \mid n \geq 1 \}$ satisfying (1.4). However, the statistical nature lying behind imposes on $\hat{\theta}_n$ to be measurable with respect to $\mathcal{S}_n$ for $n \geq 1$. This requirement makes the *existence of approximating maximums* much harder to establish and calls for the assumption of *analicity* on Θ_0 in order to ensure the existence of suitable measurable selections (see [14]). Further, the main preliminary task towards the solution of the problem is to characterize the sets of all possible accumulation and limit points of all possible sequences of approximating maximums associated with $\mathcal{H}$. It turns out that a *convergence uniformization* is important to be established in this direction (see Lemma 3.2 and the proof of Theorem 4.1 in [14]).

Both of these questions (existence and uniformization) are answered in [6] (see p.42-47). There the i.i.d. case is considered, and Θ_0 is assumed an analytic metric space. It is shown in [14] that a little stronger version of these results remains valid in the general reversed submartingale case provided that Θ_0 is a second countable Hausdorff space satisfying the Blackwell property (a second countable analytic space). Actually, a closer look into the proofs shows that the same results hold without the submartingale property as well, and the only assumption which is essentially used is the $\mathcal{S}_n \times \mathcal{B}(\Theta_0)$-measurability of $h_n(\omega, \theta)$. Finally, it is shown in [14] that for separable families of reversed submartingales (see [13]), the Blackwell property is not needed.

4. Our purpose in this paper is to use the preliminary results just described, and to present conditions for *consistency* in the reversed submartingale case. By consistency, roughly speaking, we mean that every sequence of approximating maximums associated with $\mathcal{H}$ approaches the set of all maximum points of the information function I on Θ_0 . We think that this problem appears worthy of consideration, as in the context of statistical models recalled above, as well as in the context of more general processes $\{Z_n(t)\}_{n \geq 1, \, t \in T}$ treated in Section 4 below. Classical results in this direction are established in [2], [3], [7], [8], [9], [17], [18] (see [1] and [16]). A survey of these and related results is given in [10] and [11]. The reader should note that our approach relies upon the fact that $h_n(\omega, \theta)$ approaches $I(\theta)$, so we believe that the maximum points of $h_n(\omega, \theta)$ should approach the maximum points of $I(\theta)$ on Θ_0 . This, of course, is not always the case, but it turns out to be satisfied under fairly general hypotheses as described below. Although this principle seems very natural, and useful for both theory and practice, we are unaware of a similar result in general theory of stochastic processes.

5. The organization of the first part of the paper is as follows. First we introduce some additional information functions associated with the family of reversed submartingales $\mathcal{H}$, and present their basic properties (see Proposition 2.1). Then we show that the uniformization over compact sets outside a single null set obtained in [6] carries over to the general reversed submartingale case (see (2.17) and Corollary 2.3). Together with the fundamental existence theorem mentioned above, this uniformization is crucial for the characterization of the sets all possible accumulation and limit points of all possible sequences of approximating maximums associated with $\mathcal{H}$. It makes it possible to describe more precisely the fact that all sequences of approximating maximums approach the set of all maximum points of the information function I on Θ_0 . This is formally done by introducing a concept of *consistency* of $\mathcal{H}$, which is expressed in terms of the information functions associated with $\mathcal{H}$ just mentioned (see Proposition 2.4, Proposition 2.5, Corollary 2.6 and Remark 2.1). Finally, we complete the first part of the paper by showing that the conditions for consistency given in [6] remain valid for *conditionally S-regular* families of reversed submartingales introduced in [13] (see Theorem 3.3). It is important to realize that all U*-processes* are known to be conditionally S-regular. In this way we obtain a variety of important examples covered by the result.

6. Some facts in the first part of the paper are motivated by [6] with:

$$h_n(\omega,\theta) = \frac{1}{n}\sum_{j=1}^{n} h(X_j(\omega),\theta)$$

where $\{ X_j \mid j \geq 1 \}$ is an i.i.d. sequence of random variables and h is a given function. Since the proofs in this context are similar to the proofs in [6], their details are either omitted or briefly sketched. However, note that in this process we do not assume that the tail σ-algebra $\mathcal{S}_\infty = \cap_{n=1}^{\infty}\mathcal{S}_n$ is degenerated, which is by the Hewitt-Savage 0-1 law automatically true in the setting of [6]. Consequently, the random functions which are $\mathcal{S}_\infty$-measurable are not longer P-almost surely equal to constants. This is mainly done in order to show that the characterization of the sets of all possible accumulation and limit points of all possible sequences of approximating maximums (obtained in Remark 2.1) has nothing to do with this assumption, and without any particular application to the statistical background in mind. Yet another reason for this generality is of a technical nature. Namely, some of the desired statements concerning the functions under consideration are first proved pointwise, and then they are extended to their degenerated versions. This method can increase the clarity of relations between objects involved. As an illustration of this approach, the connection between Proposition 2.2 and Corollary 2.3, obtained by the uniformization from (2.17), may be served. Moreover, a close look into the proofs shows that the same fact is also true for the submartingale property of families of functions $\{ h_n(\,\cdot\,,\theta) \mid n \geq 1 \}$ that form $\mathcal{H}$ for $\theta \in \Theta_0$, and one can easily verify that the given characterization holds with no assumption imposed on these families, except that each $h_n(\omega,\theta)$ is $\mathcal{S}_n \times \mathcal{B}(\Theta_0)$-measurable.

7. In the second part of the paper we obtain conditions for consistency of $\mathcal{H}$ by using a different method. This approach relies upon the results on uniform convergence of families of reversed submartingales obtained in [15]. It turns out that these results can be successfully transformed into conditions for consistency. In this way we obtain Theorems 3.4-3.8. We are unaware of similar results in the general reversed submartingale context. The conditions obtained are expressed in terms of Hardy's regular convergence [4], and are of a total boundedness in the mean type. The question of comparing these conditions with those obtained earlier appears worthy

of consideration. We do not pursue this in more detail, but instead consider applications to a stochastic maximization over time sets of families of random processes (see Examples 4.1-4.2). To the best of our knowledge, this sort of maximization has not been studied previously.

8. We would like to point out that our approach in some parts of Section 2 and Section 3 is very formal. The reader who wants to see these results in a less formal setting which is more suitable for straightforward applications is referred to Section 4.

2. Characterization of accumulation and limit points

1. Let $\mathcal{H} = (\{h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ be a family of reversed submartingales defined on a probability space $(\Omega, \mathcal{F}, P)$ and indexed by an analytic metric space Θ_0 , and let $\mathcal{B}_0$ denote the Borel σ-algebra on Θ_0 . Then according to [13] the family $\mathcal{H}$ is said to be:

(2.1) *measurable*, if $(\omega,\theta) \mapsto h_n(\omega,\theta)$ is $\mathcal{S}_n \times \mathcal{B}_0$-measurable for all $n \geq 1$

(2.2) *degenerated*, if $\mathcal{S}_\infty = \cap_{n=1}^{\infty} \mathcal{S}_n$ is degenerated, that is $P(A) \in \{0,1\}$ for all $A \in \mathcal{S}_\infty$

(2.3) *separable* relative to given families $\mathcal{S} \subset 2^{\Theta_0}$ and $\mathcal{C} \subset 2^{\mathbf{R}}$, if for each $B \in \mathcal{S}$ there exists a sequence $\{\, \theta_i \mid i \geq 1 \,\}$ in Θ_0 such that for all $C \in \mathcal{C}$ we have:

$$P^*\Big(\bigcup_{n=1}^{\infty} \{\, h_n(\theta_i) \in C \mid \theta_i \in B \,\} \setminus \{\, h_n(\theta) \in C \mid \theta \in B \,\}\Big) = 0$$

(2.4) *separable*, if it is separable relative to the family $\mathcal{G}(\Theta_0)$ of all open sets in Θ_0 and the family $\mathcal{C}(\mathbf{R})$ of all closed sets in $\mathbf{R}$

(2.5) *a.s.-upper (lower) semicontinuous* on a given set $\Gamma \subset \Theta_0$, if there exists a P-null set $N \in \mathcal{F}$ such that the function $\theta \mapsto h_n(\omega,\theta)$ is upper (lower) semicontinuous on Γ for all $\omega \in \Omega \setminus N$ and all $n \geq 1$

(2.6) *conditionally S-regular* relative to a given family $\mathcal{M} \subset 2^{\Theta_0}$, if for each $B \in \mathcal{M}$ there exist a P-null set N in $\mathcal{F}$ and versions $\hat{E}\{h_n^*(B) \mid \mathcal{S}_{n+1}\}(\omega)$ of the conditional expectations $E\{h_n^*(B) \mid \mathcal{S}_{n+1}\}$ satisfying:

$$\hat{E}\{h_n^*(B) \mid \mathcal{S}_{n+1}\}(\omega) \geq h_{n+1}(\omega,\theta)$$

for all $\omega \in \Omega \setminus N$, all $\theta \in B$, and all $n \geq k$ for some $k \geq 1$. Here we implicitly suppose that every set B in $\mathcal{M}$ satisfies two following two conditions:

(i) The map $\omega \mapsto h_n^*(\omega, B)$ is $\mathcal{S}_n^P$-measurable

(ii) $Eh_n^*(B) < \infty$

for all $n \geq k$ with the given $k \geq 1$. For instance, condition (i) is by the projection theorem fulfilled whenever $\mathcal{H}$ is measurable and B is analytic (see [6]).

For more information on (2.1)-(2.6) we refer to [13]. We point out that all U-processes are known to be conditionally S-regular relative to all analytic (Borel) sets (see Example 4.4 in [13]).

Let (Θ, d) be a compact metric space containing Θ_0 , let $\mathcal{G}(\Theta)$ denote the family of all open sets in Θ , and let $\mathcal{B}$ denote the Borel σ-algebra on Θ . Then Θ is an analytic metric space and we will always set $f(\theta) = -\infty$ for all $\theta \in \Theta \setminus \Theta_0$, whenever $f : \Theta_0 \to \mathbf{R}$ is a function. It is easily verified that definitions (2.1)-(2.6) extend with no change under:

$$h_n(\omega, \theta) = -\infty \quad \big(\forall \theta \in \Theta \setminus \Theta_0 \ , \ \forall \omega \in \Omega \ , \ \forall n \geq 1 \big)$$

with Θ being a new index space.

2. In the sequel we shall make use of the following auxiliary functions associated with $\mathcal{H}$:

$$h_n^*(\omega, B) = \sup_{\theta \in B} h_n(\omega, \theta) \tag{2.7}$$

$$H_0^*(\omega, B) = \liminf_{n \to \infty} \ h_n^*(\omega, B) \quad , \quad H^*(\omega, B) = \limsup_{n \to \infty} \ h_n^*(\omega, B) \tag{2.8}$$

$$\bar{H}_0(\omega, B) = \inf_{G \in \mathcal{G}(\Theta), G \supset B} H_0^*(\omega, G) \quad , \quad \bar{H}(\omega, B) = \inf_{G \in \mathcal{G}(\Theta), G \supset B} H^*(\omega, G) \tag{2.9}$$

$$\bar{H}_0(\omega, \theta) = \inf_{r > 0} H_0^*(\omega, b(\theta, r)) \quad , \quad \bar{H}(\omega, \theta) = \inf_{r > 0} H^*(\omega, b(\theta, r)) \tag{2.10}$$

$$\eta(\theta) = \inf_{n \geq 1} E^* \bar{h}_n(\theta) \quad , \quad \eta^*(B) = \inf_{n \geq 1} E^* h_n^*(B) \tag{2.11}$$

where $\omega \in \Omega$, $\theta \in \Theta$, $B \subset \Theta$ and $n \geq 1$. Here $\bar{h}_n(\omega, \theta) = \lim_{\, r \downarrow 0} h_n^*(\omega, b(\theta, r))$ denotes the upper semicontinous envelope of $h_n(\omega, \cdot\,)$ for $\omega \in \Omega$, $\theta \in \Theta$ and $n \geq 1$, and E^* denotes the upper P-integral. Note that $\bar{H}_0(\omega, \theta) = \bar{H}_0(\omega, \{\theta\})$ and $\bar{H}(\omega, \theta) = \bar{H}(\omega, \{\theta\})$ whenever $\omega \in \Omega$ and $\theta \in \Theta$. According to [6], the function $h_n(\omega, \theta)$ is called *the empirical information function*, the functions $\bar{H}_0(\omega, B)$ and $\bar{H}(\omega, B)$ are called *the outer maximal functions*, the functions $\bar{H}_0(\omega, \theta)$ and $\bar{H}(\omega, \theta)$ are called *the upper information functions*, and the functions $\eta(\theta)$ and $\eta^*(B)$ are called *the mean value information functions* associated with $\mathcal{H}$.

If $\mathcal{H}$ is degenerated, then we define *the information function* associated with $\mathcal{H}$ as follows:

$$I(\theta) \ = \ a.s. \lim_{n \to \infty} h_n(\theta) = \lim_{n \to \infty} E h_n(\theta)$$

for all $\theta \in \Theta$. Note that every $\mathcal{S}_\infty$-measurable function is then P-almost surely equal to some constant, and thus if $\mathcal{H}$ is measurable, then by the projection theorem the functions $H_0^*(\,\cdot\,, B)$ and $H^*(\,\cdot\,, B)$ are degenerated for every analytic subset B of Θ . We will denote these constants by $H_0^*(B)$ and $H^*(B)$ respectively, and define the associated outer maximal functions as follows:

$$\bar{H}_0(B) = \inf_{G \in \mathcal{G}(\Theta), G \supset B} H_0^*(G) \quad , \quad \bar{H}(B) = \inf_{G \in \mathcal{G}(\Theta), G \supset B} H^*(G)$$

for all $B \subset \Theta$. If $\mathcal{H}$ is not degenerated, then respecting the statistical nature lying behind, we will define the information function associated with $\mathcal{H}$ by:

$$I(\omega, \theta) = \liminf_{n \to \infty} \ h_n(\omega, \theta)$$

whenever $\omega \in \Omega$ and $\theta \in \Theta$.

Basic properties of the objects just introduced are stated as follows.

Proposition 2.1

Let $\mathcal{H} = (\{h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ *be a given family of reversed submartingales, then:*

(2.12) $\bar{H}_0(\omega,\cdot)$ *and* $\bar{H}(\omega,\cdot)$ *are upper semicontinuous functions on* Θ *for all* $\omega \in \Omega$

(2.13) $\sup_{\theta \in B} I(\omega,\theta) \leq H_0^*(\omega,B) \leq H^*(\omega,B)$ *, for all* $\omega \in \Omega$ *and all* $B \subset \Theta$

(2.14) $I(\omega,\theta) \leq \bar{I}(\omega,\theta) \leq \bar{H}_0(\omega,\theta) \leq \bar{H}(\omega,\theta)$ *, for all* $\omega \in \Omega$ *and all* $\theta \in \Theta$ *, where* $\bar{I}(\omega,\cdot)$ *denotes the upper semicontinuous envelope of* $I(\omega,\cdot)$ *on* Θ

(2.15) $\sup_{\theta \in B^0} \bar{H}(\omega,\theta) \leq H^*(\omega,B) \leq \bar{H}(\omega,B) \leq \bar{H}(\omega,\bar{B}) = \sup_{\theta \in \bar{B}} \bar{H}(\omega,\theta)$ *, for all* $\omega \in \Omega$ *and all* $B \subset \Theta$ *, where* $B^0 = int\ (B \cup (\Theta \setminus \Theta_0))$ *and* $\bar{B} = cl\ (B)$ *in* Θ

(2.16) $\sup_{\theta \in \Theta} \bar{I}(\omega,\theta) \leq \sup_{\theta \in \Theta} \bar{H}_0(\omega,\theta) \leq H_0^*(\omega,\Theta) \leq H^*(\omega,\Theta) = \sup_{\theta \in \Theta} \bar{H}(\omega,\theta)$ *, for all* $\omega \in \Omega$.

Moreover, if $\mathcal{H}$ *is measurable and degenerated, then:*

(2.17) *There exists a* P*-null set* $N \in \mathcal{F}$ *such that for every compact set* K *in* Θ *we have:*

$$\bar{H}_0(\omega,K) = \bar{H}_0(K) \quad \textit{and} \quad \bar{H}(\omega,K) = \bar{H}(K)$$

for all $\omega \in \Omega \setminus N$

(2.18) $\bar{H}_0$ *and* $\bar{H}$ *are upper semicontinuous functions on* Θ

(2.19) $\sup_{\theta \in B} I(\theta) \leq H_0^*(B) \leq H^*(B)$ *, for all* $B \subset \Theta$

(2.20) $I(\theta) \leq \bar{I}(\theta) \leq \bar{H}_0(\theta) \leq \bar{H}(\theta)$ *, where* $\bar{I}$ *denotes the upper semicontinuous envelope of* I *on* Θ

(2.21) $\sup_{\theta \in B^0} \bar{H}(\theta) \leq H^*(B) \leq \bar{H}(B) \leq \bar{H}(\bar{B}) = \sup_{\theta \in \bar{B}} \bar{H}(\theta)$ *, for all* $B \subset \Theta$

(2.22) $\sup_{\theta \in \Theta} \bar{I}(\theta) \leq \sup_{\theta \in \Theta} \bar{H}_0(\theta) \leq H_0^*(\Theta) \leq H^*(\Theta) = \sup_{\theta \in \Theta} \bar{H}(\theta)$.

Proof. (2.12)-(2.16): The last equality in (2.15) follows by the compactness of $\bar{B}$, and the remaining statements follow from definitions.

(2.17): Let $\mathcal{B}$ be a countable base for the topology on Θ which is closed under formations of finite unions. By our hypotheses on $\mathcal{H}$ we can find a P-null set $N \in \mathcal{F}$ such that:

$$\liminf_{n\to\infty}\ h_n^*(\omega,G) = H_0^*(G) \quad \text{and} \quad \limsup_{n\to\infty}\ h_n^*(\omega,G) = H^*(G)$$

for all $G \in \mathcal{B}$ and all $\omega \in \Omega \setminus N$. Hence by the compactness of K we find:

$$\bar{H}_0(\omega,K) = \inf_{G \in \mathcal{B}, G \supset K} H_0^*(\omega,G) = \inf_{G \in \mathcal{B}, G \supset K} H_0^*(G) = \bar{H}_0(K)$$

$$\bar{H}(\omega,K) = \inf_{G \in \mathcal{B}, G \supset K} H^*(\omega,G) = \inf_{G \in \mathcal{B}, G \supset K} H^*(G) = \bar{H}(K)$$

for all $\omega \in \Omega \setminus N$, and (2.17) is proved.

(2.18)-(2.22): Straightforward from (2.12)-(2.16) by using (2.17). □

Proposition 2.2

Let $\mathcal{H} = (\{h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ *be a given family of reversed submartingales, let* $\{\ \hat{\theta}_n \mid n \geq 1\ \}$ *be a sequence of functions from* Ω *into* Θ *, and let* B *be a subset of* Θ *. Then we have:*

$$\limsup_{n\to\infty}\ h_n^*(\omega,B) \ \leq\ \bar{H}(\omega,B) \tag{2.23}$$

$$\liminf_{n\to\infty}\ h_n^*(\omega,B) \ \leq\ \bar{H}_0(\omega,B) \tag{2.24}$$

$$\limsup_{n\to\infty}\ h_n(\omega,\hat{\theta}_n(\omega)) \ \leq\ \bar{H}(\omega,\mathcal{C}\{\hat{\theta}_n(\omega)\}) \tag{2.25}$$

$$\liminf_{n\to\infty}\ h_n(\omega,\hat{\theta}_n(\omega)) \ \leq\ \min\left\{\ \bar{H}_0(\omega,\mathcal{C}\{\hat{\theta}_n(\omega)\})\ ,\ \inf_{\theta\in\mathcal{C}\{\hat{\theta}_n(\omega)\}} \bar{H}(\omega,\theta)\ \right\} \tag{2.26}$$

for all $\omega \in \Omega$ *, where* $\mathcal{C}\{\hat{\theta}_n(\omega)\}$ *denotes the set of all accumulation points in* Θ *of the sequence* $\{\ \hat{\theta}_n(\omega) \mid n \geq 1\ \}$ *for* $\omega \in \Omega$ *.*

Proof. (2.23)-(2.24): It follows from definitions of $\bar{H}(\omega,B)$ and $\bar{H}_0(\omega,B)$.

(2.25)-(2.26): If $G \in \mathcal{G}(\Theta)$ with $G \supset \mathcal{C}\{\hat{\theta}_n(\omega)\}$, then there exists $n_0 \geq 1$ such that $\hat{\theta}_n(\omega) \in G$ for all $n \geq n_0$. Hence we get:

$$\limsup_{n\to\infty}\ h_n(\omega,\hat{\theta}_n(\omega)) \ \leq\ \limsup_{n\to\infty}\ h_n^*(\omega,G) = H^*(\omega,G)$$

$$\liminf_{n\to\infty}\ h_n(\omega,\hat{\theta}_n(\omega)) \ \leq\ \liminf_{n\to\infty}\ h_n^*(\omega,G) = H_0^*(\omega,G)\ .$$

Taking the infimum over all $G \in \mathcal{G}(\Theta)$ with $G \supset \mathcal{C}\{\hat{\theta}_n(\omega)\}$ we find that (2.25) and the first part of (2.26) are satisfied. For the second part of (2.26) let $\theta \in \mathcal{C}\{\hat{\theta}_n(\omega)\}$ be a given point. Then there exist integers $\sigma(1) < \sigma(2) < \dots$ such that $\hat{\theta}_{\sigma(j)}(\omega) \to \theta$ for $j \to \infty$. Put $\sigma(0) = 0$ and define:

$$\check{\theta}_k(\omega) = \hat{\theta}_{\sigma(j)}(\omega) \quad ,\ \text{for all}\ \ \sigma(j-1) < k \leq \sigma(j)\ \text{ and all }\ j \geq 1\ .$$

Let $A_p = \{\check{\theta}_k(\omega) \mid k \geq p\}$, then $\check{\theta}_k(\omega) \to \theta$ for $k \to \infty$, and hence $\bar{A}_p = A_p \cup \{\theta\}$ for all $p \geq 1$. By (2.23) and the last equality from (2.15) we obtain:

$$\begin{aligned}\liminf_{n\to\infty}\ h_n(\omega,\hat{\theta}_n(\omega)) \ &\leq\ \liminf_{j\to\infty}\ h_{\sigma(j)}(\omega,\hat{\theta}_{\sigma(j)}(\omega)) \leq \limsup_{k\to\infty}\ h_k(\omega,\check{\theta}_k(\omega)) \leq \\ &\leq \limsup_{n\to\infty}\ h_n^*(\omega,A_p) \leq \bar{H}(\omega,A_p) \leq \bar{H}(\omega,\bar{A}_p) = \\ &= \max\left\{\ \bar{H}(\omega,\theta)\ ,\ \sup_{k\geq p}\bar{H}(\omega,\check{\theta}_k(\omega))\ \right\}\ .\end{aligned}$$

By (2.12) we know that the function $\theta \mapsto \bar{H}(\omega,\theta)$ is upper semicontinuous on Θ . Thus letting $p \to \infty$, and taking the infimum over all $\theta \in \mathcal{C}\{\hat{\theta}_n(\omega)\}$, we get:

$$\liminf_{n\to\infty}\ h_n(\omega,\hat{\theta}_n(\omega)) \ \leq\ \inf_{\theta\in\mathcal{C}\{\hat{\theta}_n(\omega)\}} \bar{H}(\omega,\theta)\ .$$

This fact proves (2.26) and completes the proof. $\square$

Corollary 2.3

Let $\mathcal{H} = (\{h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ *be a given family of reversed submartingales. If* $\mathcal{H}$ *is measurable and degenerated, then there exists a* P*-null set* $N \in \mathcal{F}$ *such that for any sequence* $\{\ \hat{\theta}_n \mid n \geq 1\ \}$ *of functions from* Ω *into* Θ *and any* B *subset of* Θ *we have:*

(2.27) $$\limsup_{n\to\infty}\ h_n^*(\omega,B)\ \leq\ \bar{H}(\bar{B})$$

(2.28) $$\liminf_{n\to\infty}\ h_n^*(\omega,B)\ \leq\ \bar{H}_0(\bar{B})$$

(2.29) $$\limsup_{n\to\infty} h_n(\omega,\hat{\theta}_n(\omega))\ \leq\ \bar{H}(\mathcal{C}\{\hat{\theta}_n(\omega)\})$$

(2.30) $$\liminf_{n\to\infty} h_n(\omega,\hat{\theta}_n(\omega))\ \leq\ \min\big\{\bar{H}_0(\mathcal{C}\{\hat{\theta}_n(\omega)\})\ ,\ \inf_{\theta\in\mathcal{C}\{\hat{\theta}_n(\omega)\}} \bar{H}(\theta)\big\}\ .$$

Proof. Let N be the P-null set constructed in the proof of (2.17). Then (2.27)-(2.28) follows from (2.23)-(2.24) and (2.17). Moreover, (2.29)-(2.30) follows from (2.25)-(2.26) in the same way. These facts complete the proof. $\square$

Proposition 2.4

Let $\mathcal{H} = (\{h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ *be a given family of reversed submartingales, let* $\{\ \hat{\theta}_n \mid n \geq 1\ \}$ *be a sequence of functions from* Ω *into* Θ *, let* F *be a function from* Ω *into* $\bar{\mathbf{R}}$ *, and let us define:*

$$\Omega_F = \Big\{\ \omega\in\Omega \ \Big|\ \liminf_{n\to\infty}\ h_n(\omega,\hat{\theta}_n(\omega)) \geq F(\omega)\ \Big\}$$

$$\Omega^F = \Big\{\ \omega\in\Omega \ \Big|\ \limsup_{n\to\infty}\ h_n(\omega,\hat{\theta}_n(\omega)) \geq F(\omega)\ \Big\}.$$

Then we have:

(2.31) $\mathcal{C}\{\hat{\theta}_n(\omega)\} \subset \{\ \theta\in\Theta \mid \bar{H}(\omega,\theta) \geq F(\omega)\ \}$ *, for all* $\omega \in \Omega_F$

(2.32) $\mathcal{C}\{\hat{\theta}_n(\omega)\} \cap \{\ \theta\in\Theta \mid \bar{H}(\omega,\theta) \geq F(\omega)\ \} \neq \emptyset$ *, for all* $\omega \in \Omega^F$

(2.33) $\mathcal{L}\{\hat{\theta}_n(\omega)\} \subset \{\ \theta\in\Theta \mid \bar{H}_0(\omega,\theta) \geq F(\omega)\ \}$ *, for all* $\omega \in \Omega_F$

where $\mathcal{C}\{\hat{\theta}_n(\omega)\}$ *and* $\mathcal{L}\{\hat{\theta}_n(\omega)\}$ *denote the sets of all accumulation and limit points in* Θ *of the sequence* $\{\ \hat{\theta}_n(\omega) \mid n \geq 1\ \}$ *for* $\omega \in \Omega$ *respectively. In particular, if* $\mathcal{H}$ *is measurable and degenerated and* F *is a constant in* $\bar{\mathbf{R}}$*, then there exists a* P*-null set* $N \in \mathcal{F}$ *such that:*

(2.34) $\mathcal{C}\{\hat{\theta}_n(\omega)\} \subset \{\ \theta\in\Theta \mid \bar{H}(\theta) \geq F\ \}$ *, for all* $\omega \in \Omega_F \setminus N$

(2.35) $\mathcal{C}\{\hat{\theta}_n(\omega)\} \cap \{\ \theta\in\Theta \mid \bar{H}(\theta) \geq F\ \} \neq \emptyset$ *, for all* $\omega \in \Omega^F \setminus N$

(2.36) $\mathcal{L}\{\hat{\theta}_n(\omega)\} \subset \{\ \theta\in\Theta \mid \bar{H}_0(\theta) \geq F\ \}$ *, for all* $\omega \in \Omega_F \setminus N$ *.*

Proof. (2.31): It follows from (2.26).

(2.32): Since the upper semicontinuous function $\bar{H}(\omega,\ \cdot\ )$ attains its maximal value on the compact set $\mathcal{C}\{\hat{\theta}_n(\omega)\}$, we see that (2.32) follows from (2.25) and the last equality in (2.15).

(2.33): If $\theta \in \mathcal{L}\{\hat{\theta}_n(\omega)\}$, then $\mathcal{L}\{\hat{\theta}_n(\omega)\} = \mathcal{C}\{\hat{\theta}_n(\omega)\} = \{\theta\}$, and (2.33) follows from (2.26).

(2.34)-(2.36): It follows from (2.31)-(2.33) respectively, by using (2.17). □

3. Let $\mathcal{H} = (\{h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ be a given family of reversed submartingales. A sequence of functions $\{ \hat{\theta}_n \mid n \geq 1 \}$ from Ω into Θ is called:

(2.37) a *sequence of empirical maximums* associated with $\mathcal{H}$, if there exist a function $q : \Omega \to \mathbf{N}$ and a P-null set $N \in \mathcal{F}$ satisfying:

(i) $\hat{\theta}_n(\omega) \in \Theta_0 \;\; , \;\; \forall n \geq q(\omega) \; , \; \forall \omega \in \Omega \setminus N$

(ii) $h_n(\omega, \hat{\theta}_n(\omega)) = h_n^*(\omega, \Theta_0) \;\; , \;\; \forall n \geq q(\omega) \; , \; \forall \omega \in \Omega \setminus N$

(2.38) a *sequence of asymptotic maximums* associated with $\mathcal{H}$, if there exist a function $q : \Omega \to \mathbf{N}$ and a P-null set $N \in \mathcal{F}$ satisfying (i) in (2.37) and:

$$\liminf_{n\to\infty} \; h_n(\omega, \hat{\theta}_n(\omega)) \;\geq\; H_0^*(\omega, \Theta_0) \;\; , \;\; \forall \omega \in \Omega \setminus N$$

(2.39) a *sequence of approximating maximums* associated with $\mathcal{H}$, if there exist a function $q : \Omega \to \mathbf{N}$ and a P-null set $N \in \mathcal{F}$ satisfying (i) in (2.37) and:

$$\liminf_{n\to\infty} \; h_n(\omega, \hat{\theta}_n(\omega)) \;\geq\; \beta(\omega) \;\; , \;\; \forall \omega \in \Omega \setminus N$$

where $\beta(\omega) := \beta = \sup_{\theta\in\Theta_0} I(\theta)$ if $\mathcal{H}$ is degenerated, and $\beta(\omega) := \sup_{\theta\in\Theta_0} I(\omega,\theta)$ otherwise, for all $\omega \in \Omega$.

It is easily verified that every sequence of empirical maximums is a sequence of asymptotic maximums, and that every sequence of asymptotic maximums is a sequence of approximating maximums. Although $h_n(\omega, \cdot\,)$ does not need to attain its maximal value on Θ_0 , and (ii) in (2.37) may fail in this case, we can always find a sequence of functions $\{ \hat{\theta}_n \mid n \geq 1 \}$ satisfying:

$$h_n(\omega, \hat{\theta}_n(\omega)) \geq h_n^*(\omega, \Theta_0) - \varepsilon_n(\omega) \text{ , if } \; h_n^*(\omega, \Theta_0) < +\infty$$

$$h_n(\omega, \hat{\theta}_n(\omega)) \geq n \text{ , if } \; h_n^*(\omega, \Theta_0) = +\infty$$

for all $\omega \in \Omega$ and all $n \geq 1$, where $\varepsilon_n \to 0$ as $n \to \infty$. Passing to the limit inferior above we see that *sequences of approximating and asymptotic maximums always exist*. We emphasize that this fact is by itself of theoretical and practical interest.

4. In order to describe the sets of accumulation and limit points of the sequences of maximum functions just introduced we shall introduce the following sets:

$$\hat{M} = \hat{M}(\mathcal{H}) = \{ \; \theta \in \bar{\Theta}_0 \mid \bar{H}(\omega,\theta) \geq \beta(\omega) \;\; P\text{-}a.s. \; \}$$

$$\hat{L} = \hat{L}(\mathcal{H}) = \{ \; \theta \in \bar{\Theta}_0 \mid \bar{H}_0(\omega,\theta) \geq \beta(\omega) \;\; P\text{-}a.s. \; \}$$

$$M^* = M^*(\mathcal{H}) = \{ \; \theta \in \bar{\Theta}_0 \mid \bar{H}(\omega,\theta) \geq H^*(\omega,\Theta) \;\; P\text{-}a.s. \; \}$$

$$M_0^* = M_0^*(\mathcal{H}) = \{\ \theta \in \bar{\Theta}_0 \ \mid \ \bar{H}(\omega,\theta) \ge H_0^*(\omega,\Theta) \ \ P\text{-}a.s.\ \}$$

$$L_0^* = L_0^*(\mathcal{H}) = \{\ \theta \in \bar{\Theta}_0 \ \mid \ \bar{H}_0(\omega,\theta) \ge H_0^*(\omega,\Theta) \ \ P\text{-}a.s.\ \}\ .$$

If $\mathcal{H}$ is measurable and degenerated, then from (2.17) and definition of β we find:

$$\hat{M} = \hat{M}(\mathcal{H}) = \{\ \theta \in \bar{\Theta}_0 \ \mid \ \bar{H}(\theta) \ge \beta\ \}$$

$$\hat{L} = \hat{L}(\mathcal{H}) = \{\ \theta \in \bar{\Theta}_0 \ \mid \ \bar{H}_0(\theta) \ge \beta\ \}$$

$$M^* = M^*(\mathcal{H}) = \{\ \theta \in \bar{\Theta}_0 \ \mid \ \bar{H}(\theta) \ge H^*(\Theta)\ \}$$

$$M_0^* = M_0^*(\mathcal{H}) = \{\ \theta \in \bar{\Theta}_0 \ \mid \ \bar{H}(\theta) \ge H_0^*(\Theta)\ \}$$

$$L_0^* = L_0^*(\mathcal{H}) = \{\ \theta \in \bar{\Theta}_0 \ \mid \ \bar{H}_0(\theta) \ge H_0^*(\Theta)\ \}\ .$$

The next proposition and the existence Theorem 4.1 in [14] provide a complete description of the sets of all accumulation and all limit points of the sequences of maximum functions introduced in (2.37)-(2.39) above (see Remark 2.1 below).

Proposition 2.5

Let $\mathcal{H} = (\{h_n(\omega,\theta), \mathcal{S}_n \mid n \ge 1\} \mid \theta \in \Theta_0)$ *be a given family of reversed submartingales.*

(2.40) *If* $\{\ \hat{\theta}_n \mid n \ge 1\ \}$ *is a sequence of empirical maximums associated with* $\mathcal{H}$ *, then there exists a* P*-null set* $N \in \mathcal{F}$ *such that for all* $\omega \in \Omega \setminus N$ *we have:*

(i) $\mathcal{C}\{\hat{\theta}_n(\omega)\} \subset M_0^*(\mathcal{H})$

(ii) $\mathcal{C}\{\hat{\theta}_n(\omega)\} \cap M^*(\mathcal{H}) \ne \emptyset$

(iii) $\lim\limits_{n\to\infty} d(\hat{\theta}_n(\omega), M_0^*(\mathcal{H})) = \liminf\limits_{n\to\infty} d(\hat{\theta}_n(\omega), M^*(\mathcal{H})) = 0$

(iv) $\mathcal{L}\{\hat{\theta}_n(\omega)\} \subset L_0^*(\mathcal{H})$

(2.41) *If* $\{\ \hat{\theta}_n \mid n \ge 1\ \}$ *is a sequence of asymptotic maximums associated with* $\mathcal{H}$ *, then there exists a* P*-null set* $N \in \mathcal{F}$ *such that for all* $\omega \in \Omega \setminus N$ *we have:*

(i) $\mathcal{C}\{\hat{\theta}_n(\omega)\} \subset M_0^*(\mathcal{H})$

(ii) $\lim\limits_{n\to\infty} d(\hat{\theta}_n(\omega), M_0^*(\mathcal{H})) = 0$

(iii) $\mathcal{L}\{\hat{\theta}_n(\omega)\} \subset L_0^*(\mathcal{H})$

(2.42) *If* $\{\ \hat{\theta}_n \mid n \ge 1\ \}$ *is a sequence of approximating maximums associated with* $\mathcal{H}$ *, then there exists a* P*-null set* $N \in \mathcal{F}$ *such that for all* $\omega \in \Omega \setminus N$ *we have:*

(i) $\mathcal{C}\{\hat{\theta}_n(\omega)\} \subset \hat{M}(\mathcal{H})$

(ii) $\lim\limits_{n\to\infty} d(\hat{\theta}_n(\omega), \hat{M}(\mathcal{H})) = 0$

(iii) $\mathcal{L}\{\hat{\theta}_n(\omega)\} \subset \hat{L}(\mathcal{H})$.

Proof. It follows by definitions from Proposition 2.4. $\square$

Corollary 2.6
Let $\mathcal{H} = (\{h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ *be a given family of reversed submartingales.*

(2.43) *For every* $\theta \in \hat{M}(\mathcal{H})$ $\big(M_0^*(\mathcal{H})\big)$ *there exist a sequence of approximating (asymptotic) maximums* $\{ \hat{\theta}_n \mid n \geq 1 \}$ *associated with* $\mathcal{H}$ *and a* P*-null set* $N \in \mathcal{F}$ *satisfying:*

(i) $\hat{\theta}_n$ *is* $\mathcal{S}_n$*-measurable for all* $n \geq 1$

(ii) $\theta \in \mathcal{C}\{\hat{\theta}_n(\omega)\}$, $\forall \omega \in \Omega \setminus N$

(2.44) *For every* $\theta \in \hat{L}(\mathcal{H})$ $\big(L_0^*(\mathcal{H})\big)$ *there exist a sequence of approximating (asymptotic) maximums* $\{ \hat{\theta}_n \mid n \geq 1 \}$ *associated with* $\mathcal{H}$ *and a* P*-null set* $N \in \mathcal{F}$ *satisfying:*

(i) $\hat{\theta}_n$ *is* $\mathcal{S}_n$*-measurable for all* $n \geq 1$

(ii) $\hat{\theta}_n \rightrightarrows \{\theta\}$ *on* Ω

(iii) $\bar{H}_0(\omega,\theta) = \liminf_{n\to\infty} \; h_n(\omega,\hat{\theta}_n(\omega)) \leq \limsup_{n\to\infty} \; h_n(\omega,\hat{\theta}_n(\omega)) = \bar{H}(\omega,\theta)$, $\forall \omega \in \Omega \backslash N$.

Proof. The proof of the first part of (2.43) and (2.44) for non-degenerated families of reversed submartingales $\mathcal{H}$ is given in [14] (see Corollary 4.2). It is easily verified that the same proof works for the second part of (2.43) and (2.44) as well. If $\mathcal{H}$ is degenerated, then the proof may be carried out in exactly the same way by using (2.17) above. $\square$

Remark 2.1
(1) Combining (i) in (2.42) with the first part of (2.43) we see that $\hat{M}(\mathcal{H})$ *is exactly the set of all possible accumulation points of all possible sequences of approximating maximums* associated with $\mathcal{H}$. Similarly, combining (iii) in (2.42) with the first part of (2.44) we see that $\hat{L}(\mathcal{H})$ *is exactly the set of all possible limit points of all possible sequences of approximating maximums* associated with $\mathcal{H}$.

(2) Combining (i) in (2.41) with the second part of (2.43) we see that $M_0^*(\mathcal{H})$ *is exactly the set of all possible accumulation points of all possible sequences of asymptotic maximums* associated with $\mathcal{H}$. Similarly, combining (iii) in (2.41) with the second part of (2.44) we see that $L_0^*(\mathcal{H})$ *is exactly the set of all possible limit points of all possible sequences of asymptotic maximums.*

3. Consistency theorems

1. Let $\mathcal{H} = (\{h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ be a family of reversed submartingales defined on a probability space $(\Omega, \mathcal{F}, P)$ and indexed by an analytic metric space Θ_0 . Suppose that $\mathcal{H}$ is degenerated and define the set:

$$M = M(\mathcal{H}) = \{ \, \theta \in \Theta_0 \mid I(\theta) = \beta \, \}$$

where $\beta = \sup_{\theta \in \Theta_0} I(\theta)$. Let $\Gamma \subset \Theta$, then $\mathcal{H}$ is said to be S*-consistent* on Γ , if for every sequence of approximating maximums $\{ \hat{\theta}_n \mid n \geq 1 \}$ associated with $\mathcal{H}$ we have that

$\mathcal{C}\{\hat{\theta}_n(\omega)\} \cap \Gamma \subset M$ for all $\omega \in \Omega \setminus N$, where N is a P-null set in $\mathcal{F}$. In particular $\mathcal{H}$ is said to be S-*consistent*, if it is S-consistent on Θ. Note that $\mathcal{H}$ is S-consistent on Γ if and only if, every accumulation point of any sequence of approximating maximums $\{ \hat{\theta}_n \mid n \geq 1 \}$ associated with $\mathcal{H}$ which belongs to Γ is a maximum point of the information function I on Θ_0.

2. By (1) in Remark 2.1 we know that $\hat{M}(\mathcal{H})$ is exactly the set of all possible accumulation points of all possible sequences of approximating maximums associated with $\mathcal{H}$. Therefore the following statements are equivalent:

(3.1) $\qquad \mathcal{H}$ is S-consistent on Γ

(3.2) $\qquad \mathcal{H}$ is S-consistent on $\Gamma \cap (\hat{M}(\mathcal{H}) \setminus M(\mathcal{H}))$

(3.3) $\qquad \Gamma \cap \hat{M}(\mathcal{H}) \subset \mathcal{M}(\mathcal{H})$

(3.4) $\qquad \bar{H}(\theta) < \beta \ , \forall \theta \in \Gamma \setminus M(\mathcal{H})$.

Suppose that $\{ \hat{\theta}_n \mid n \geq 1 \}$ is a Γ-*tight* sequence of approximating maximums associated with $\mathcal{H}$. This means that $\mathcal{C}\{\hat{\theta}_n(\omega)\} \subset \Gamma$ for all $\omega \in \Omega \setminus N$, where N is a P-null set in $\mathcal{F}$. If $\mathcal{H}$ is S-consistent on Γ, then we have:

(3.5) $\qquad \mathcal{C}\{\hat{\theta}_n(\omega)\} \subset M$

(3.6) $\qquad \lim_{n\to\infty} d(\hat{\theta}_n(\omega), M) = 0$

for all $\omega \in \Omega \setminus N$, where N is a P-null set in $\mathcal{F}$.

3. Our next aim is to show that the main conditions for consistency of $\mathcal{H}$ given in [6] remain valid for conditionally S-regular families of reversed submartingales introduced in [13]. In the next two propositions we collect some information of independent interest, which is motivated by [6], and offers more than really needed to complete our main aim. The main result on consistency is presented in Theorem 2.3 below. Although its proof in part follows by results of the next two propositions, we independently present a complete self-contained proof.

In the next we will use $\mathcal{A}(\Theta)$ to denote the family of all analytic sets in Θ. We further set:

$$B(\theta, r_0) = \{ \ b(\theta, r) \ \mid \ r \in \mathbf{Q}_+ \ , \ r \leq r_0 \ \}$$

for all $\theta \in \Theta$ and all $r_0 > 0$. We finally recall that $B^0 = int\ (B \cup (\Theta \setminus \Theta_0))$ for any $B \subset \Theta$.

Proposition 3.1

Let $\mathcal{H} = (\{h_n(\omega, \theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ *be a given family of reversed submartingales, and let us suppose that for some* $\theta \in \Theta$ *and* $B \in \mathcal{A}(\Theta)$ *we have:*

$$\eta(\theta) < \infty \quad \textit{and} \quad \eta^*(B) < \infty \ .$$

If $\mathcal{H}$ *is measurable, then we have:*

(3.7) $\quad H^*(\omega, B) = H_0^*(\omega, B)$ P-*a.s., if* $\mathcal{H}$ *is conditionally* S-*regular relative to* $\{B\}$

(3.8) $\quad H^*(B) = H_0^*(B) = \eta^*(B)$, *if* $\mathcal{H}$ *is conditionally* S-*regular relative to* $\{B\}$ *and degenerated*

(3.9) $\quad \bar{H}(\omega, \theta) = \bar{H}_0(\omega, \theta)$ P-*a.s. for any* $\theta \in B^0$ *such that* $\mathcal{H}$ *is conditionally* S-*regular relative to* $B(\theta, r_\theta)$ *for some* $r_\theta > 0$

(3.10) $\bar{H}(\theta) = \bar{H}_0(\theta) = \eta(\theta)$ *for any* $\theta \in B^0$ *such that* $\mathcal{H}$ *is conditionally* S*-regular relative to* $B(\theta, r_\theta)$ *for some* $r_\theta > 0$ *and degenerated*

(3.11) $\eta(\theta) = a.s.\lim_{n\to\infty} \bar{h}_n(\theta)$, *if* $\mathcal{H}$ *is conditionally* S*-regular relative to* $B(\theta, r_\theta)$ *for some* $r_\theta > 0$ *and degenerated*

(3.12) $H^*(\omega, \cup_{j=1}^m A_j) = \max_{1\le j\le m} H^*(\omega, A_j)$ P*-a.s., where* $A_1, \dots, A_m \subset \Theta$ *with* $m \geq 1$

(3.13) $\eta^*(\cup_{j=1}^m A_j) = \max_{1\le j\le m} \eta^*(A_j)$ *, if* $\mathcal{H}$ *is conditionally* S*-regular relative to the family* $\{A_1, \dots, A_m\}$ *and degenerated, where* $A_1, \dots, A_m \in \mathcal{A}(\Theta)$ *with* $m \geq 1$.

Proof. (3.7)-(3.11): It follows by definition of the conditional S-regularity and the reversed submartingale convergence theorems, using the monotone convergence theorem for (3.10).

(3.12): It follows from the fact that limit superior, and maximum over a finite set, commute.

(3.13): It follows from (3.8) and (3.12). □

As in [6] we introduce two sets of points in Θ that play important role towards consistency. Let $\mathcal{H} = (\{h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ be a given family of reversed submartingales. We define *the set of all* L^1*-dominated points* of $\mathcal{H}$ as follows:

$$\Theta_d = \big\{ \ \theta \in \Theta \ \mid \ \exists\, G \in \mathcal{G}(\Theta) \ , \ \theta \in G \ \ \text{with} \ \ \eta^*(G) < \infty \ \big\} \ .$$

Note that Θ_d is an open set in Θ , and a point $\theta \in \Theta$ belongs to Θ_d , if and only if $\exists\, m \geq 1$, $\exists\, \psi \in L^1(P)$ and $\exists\, G \in \mathcal{G}(\Theta)$ with $\theta \in G$ satisfying $h_m(\omega,\theta) \leq \psi(\omega)$ for all $\omega \in \Omega$ and all $\theta \in G \cap \Theta_0$. We define *the set of all upper semicontinuous points* of $\mathcal{H}$ as follows:

$$\Theta_u = \big\{ \ \theta \in \Theta \ \mid \ h_n(\omega, \cdot\,) \ \text{ is } P\text{-}a.s. \text{ upper semicontinuous at } \theta \ \text{ for all } \ n \geq k \ \text{ with some } k \geq 1 \ \big\} \ .$$

Proposition 3.2

Let $\mathcal{H} = (\{h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ *be a given family of reversed submartingales. If* $\mathcal{H}$ *is measurable, then we have:*

(3.14) $\bar{H}(\omega,\theta) = \bar{H}_0(\omega,\theta)$ P*-a.s. for any* $\theta \in \Theta_d$ *such that* $\mathcal{H}$ *is conditionally* S*-regular relative to* $B(\theta, r_\theta)$ *for some* $r_\theta > 0$

(3.15) $\bar{H}(\theta) = \bar{H}_0(\theta) = \eta(\theta) < \infty$ *for any* $\theta \in \Theta_d$ *such that* $\mathcal{H}$ *is conditionally* S*-regular relative to* $B(\theta, r_\theta)$ *for some* $r_\theta > 0$ *and degenerated*

(3.16) $\bar{H}(\theta) = \bar{H}_0(\theta) = \eta(\theta) = I(\theta) < \infty$ *for any* $\theta \in \Theta_u \cap \Theta_d$ *such that* $\mathcal{H}$ *is conditionally* S*-regular relative to* $B(\theta, r_\theta)$ *for some* $r_\theta > 0$ *and degenerated*

(3.17) $\eta^*(K) < \infty$ *for every compact set* $K \subset \Theta_d$ *, if* $\mathcal{H}$ *is conditionally* S*-regular relative to* $\{\ b(\theta,r) \mid \theta \in K \ , \ r \in \mathbf{Q}_+ \ , \ r \leq r_\theta \ \}$ *with some* $r_\theta > 0$ *and degenerated*

(3.18) $\eta^*(\Theta_0) < \infty$ *if and only if* $\Theta_d = \Theta$ *, provided that* $\mathcal{H}$ *is conditionally* S*-regular relative to* $\{\ b(\theta,r) \mid \theta \in \bar{\Theta}_0 \ , \ r \in \mathbf{Q}_+ \ , \ r \leq r_\theta \ \}$ *with some* $r_\theta > 0$ *and degenerated*

(3.19) $\theta \in \Theta_d$ *if and only if there exists an open neighborhood* G *of* θ *satisfying* $\eta^*(A) < \infty$ *for every* $A \subset G \cap \Theta_0$ *such that* $\bar{A} = A \cup \{\theta\}$.

Proof. The statement (3.14) follows from (3.9). The statement (3.15) follows from (3.10). The statement (3.16) follows from (3.15) by definition of I . The statement (3.17) follows from (3.13) by using a compactness argument. The statement (3.18) follows from (3.17) by definition of Θ_d , since Θ is compact. The proof of statement (3.19), which is not used in the sequel, is the same as the proof of the analogous fact given in [6] (see p.40). Observe that the so-called general monotone convergence theorem is used for this purpose. $\square$

Theorem 3.3 (Consistency of Reversed Submartingale Models)
Let $\mathcal{H} = (\{h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ *be a given family of reversed submartingales, and let* Γ *be a subset of* Θ *. Suppose that* $\mathcal{H}$ *is measurable and degenerated.*

(3.20) *If* $\beta = -\infty$ *, then* $\mathcal{H}$ *is* S*-consistent on* Γ *if and only if* $\Gamma \subset \Theta_0 \cup (\Theta \setminus \bar{\Theta}_0) = \Theta \setminus (\bar{\Theta}_0 \setminus \Theta_0)$.

(3.21) *If* $\beta > -\infty$ *, then* $\mathcal{H}$ *is* S*-consistent on* Γ *if and only if* $\Gamma \subset M \cup (\Theta \setminus \hat{M}(\mathcal{H})) \cup (\Theta_u \cap \Theta_d)$, *provided that* $\mathcal{H}$ *is conditionally* S*-regular relative to* $B(\theta, r_\theta)$ *for all* $\theta \in \Gamma \cap \Theta_u \cap \Theta_d$ *with some* $r_\theta > 0$.

(3.22) *If* $\mathcal{H}$ *is* S*-consistent on* Γ *and* $\Gamma \cap M(\mathcal{H}) = \{\theta_0\}$ *for some* $\theta_0 \in \Theta_0$*, then* $\hat{\theta}_n \to \theta_0$ P*-a.s. for every* Γ*-tight sequence of approximating maximums* $\{ \hat{\theta}_n \mid n \geq 1 \}$ *associated with* $\mathcal{H}$ *.*

Proof. (3.20): In this case $M(\mathcal{H}) = \Theta_0$ and $\hat{M}(\mathcal{H}) = \bar{\Theta}_0$, so the statement is obvious.

(3.21): If $\mathcal{H}$ is S-consistent on Γ , then by (2.43) we have $\Gamma \subset M(\mathcal{H}) \cup (\Theta \setminus \hat{M}(\mathcal{H}))$. Conversely, suppose that $\Gamma \subset M(\mathcal{H}) \cup (\Theta \setminus \hat{M}(\mathcal{H})) \cup (\Theta_u \cap \Theta_d)$, then it is enough to show that $\bar{H}(\theta) < \beta$, $\forall \theta \in \Gamma \setminus M(\mathcal{H})$. For this first note that $\Gamma \setminus M(\mathcal{H}) \subset \Gamma \cap \big\{(\Theta \setminus \hat{M}(\mathcal{H})) \cup (\Theta_u \cap \Theta_d) \setminus M(\mathcal{H})\big\}$, and since $\bar{H}(\theta) < \beta$, $\forall \theta \in \Theta \setminus \hat{M}(\mathcal{H})$, then it is enough to show that $\bar{H}(\theta) < \beta$, $\forall \theta \in \Gamma \cap \big\{(\Theta_u \cap \Theta_d) \setminus M(\mathcal{H})\big\}$. Hence we see that the proof will be completed by showing $\bar{H}(\theta) = I(\theta)$, $\forall \theta \in \Gamma \cap \Theta_u \cap \Theta_d$. For this, since $\mathcal{H}$ is degenerated, we have:

$$\eta(\theta) = \inf_{n \geq 1} E^* \bar{h}_n(\theta) \geq \inf_{n \geq 1} E h_n(\theta) = I(\theta)$$

for all $\theta \in \Theta$. For any $\theta \in \Theta_u$ there exists $k \geq 1$ such that $\bar{h}_n(\theta) = h_n(\theta)$ P-$a.s.$ for all $n \geq k$. Hence we find:

$$I(\theta) = \inf_{n \geq 1} E h_n(\theta) = \inf_{n \geq k} E \bar{h}_n(\theta) \geq \inf_{n \geq 1} E \bar{h}_n(\theta) = \eta(\theta)$$

for all $\theta \in \Theta_u$. Thus we may conclude $I(\theta) = \eta(\theta)$ for all $\theta \in \Theta_u$. Since by our hypotheses $\mathcal{H}$ is conditionally S-regular relative to $B(\theta, r_\theta)$ for all $\theta \in \Gamma \cap \Theta_u \cap \Theta_d$ with some $r_\theta > 0$, then there exist $n_\theta, j_\theta \geq 1$ large enough to satisfy:

$$E h_n^*(b(\theta, 2^{-j})) < \infty$$

for all $n \geq n_\theta$ and all $j \geq j_\theta$, and such that by (i) and (ii) in Corollary 4.2 in [13] and the monotone convergence theorem we may conclude:

$$\eta(\theta) = \inf_{n \geq n_\theta} E \bar{h}_n(\theta) = \inf_{n \geq n_\theta} \inf_{j \geq j_\theta} E h_n^*(b(\theta, 2^{-j})) = \inf_{j \geq j_\theta} \inf_{n \geq n_\theta} E h_n^*(b(\theta, 2^{-j})) =$$

$$= \inf_{j\geq j_\theta} H_0^*(b(\theta,2^{-j})) = \inf_{j\geq j_\theta} H^*(b(\theta,2^{-j})) = \bar{H}_0(\theta) = \bar{H}(\theta)$$

for all $\theta \in \Gamma \cap \Theta_u \cap \Theta_d$. Thus $I(\theta) = \eta(\theta) = \bar{H}(\theta)$ for all $\theta \in \Gamma \cap \Theta_u \cap \Theta_d$, and the proof of (3.21) is complete.

(3.22): This statement follows by definition of S-consistency of $\mathcal{H}$ on Γ . □

4. We continue to examine conditions for consistency by using a different method. As before, we assume that $\mathcal{H} = (\{h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ is a measurable and degenerated family of reversed submartingales. Our main idea is based upon the fact that the set of all possible accumulation points of all possible sequences of approximating maximums $\hat{M}(\mathcal{H})$ is described in terms of the upper information function $\bar{H}$ which is given by:

$$\bar{H}(\theta) = \inf_{r>0} \limsup_{n\to\infty} h_n^*(\omega, b(\theta,r))$$

for all $\omega \in \Omega$ outside some P-null set $N_\theta \in \mathcal{F}$. Hence we see that conditions implying:

$$\limsup_{n\to\infty} h_n^*(\omega, b(\theta,r)) = \sup_{\xi\in b(\theta,r)} I(\xi)$$

for all $\omega \in \Omega$ outside N_θ , and all $r \in \mathbf{Q}_+$, $r \leq r_\theta$, have for a consequence:

$$\bar{H}(\theta) = \bar{I}(\theta)$$

where $\theta \in \Theta$ is a given point and $r_\theta > 0$ is a given number. Since the set:

$$\tilde{M} = \tilde{M}(\mathcal{H}) = \{ \theta \in \bar{\Theta}_0 \mid \bar{I}(\theta) \geq \beta \}$$

is closed and contains $M(\mathcal{H})$, then $cl\ (M(\mathcal{H})) \subset \tilde{M}(\mathcal{H})$. Conversely, if $\theta \in \tilde{M}(\mathcal{H})$, then there exists a sequence $\{ \theta_n \mid n \geq 1 \}$ in Θ satisfying:

$$d(\theta_n,\theta) < 2^{-n} \quad \text{and} \quad I(\theta_n) \geq (\beta \wedge n) - 2^{-n}$$

for all $n \geq 1$. Thus if $\theta_n \to \theta$ with $I(\theta_n) \to \beta$ implies $I(\theta) = \beta$ for all $\theta \in \tilde{M}(\mathcal{H})$, then $\tilde{M}(\mathcal{H}) = M(\mathcal{H}) = cl(M(\mathcal{H}))$. This is for instance true if I has the closed graph, or if I is upper semicontinuous on $\tilde{M}(\mathcal{H})$. It is instructive to observe that I is always upper semicontinuous on $M(\mathcal{H})$, as well as that for every $\theta \in \tilde{M}(\mathcal{H})$ we actually have $\bar{I}(\theta) = \beta$.

5. Our next approach is based upon the basic idea just described. First we consider the separable case in Theorem 3.4, Theorem 3.5 and Theorem 3.6. Then we examine the non-separable case in Theorem 3.7 and Theorem 3.8. All these results are based upon conditions for uniform convergence of families of reversed submartingales established in [15]. In this context we find it convenient to recall some definitions needed in the sequel.

Let $\mathcal{H} = (\{h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ be a given family of reversed submartingales defined on a probability space $(\Omega, \mathcal{F}, P)$ and indexed by an analytic metric space Θ_0 , let $D = \{ \delta_j \mid j \geq 1 \}$ be a countable subset of Θ_0 , and let $D_n = \{ \delta_1, \ldots, \delta_n \}$ for all $n \geq 1$.

Let us for a given set $A \subset \Theta_0$ denote:

$$M_A(h_n) = \sup_{\theta \in A} \ h_n(\theta)$$

for all $n \geq 1$. Then $\mathcal{H}$ is called *totally bounded in P-mean relative to D*, if any of the following five equivalent conditions is satisfied (see [15]):

(3.23) The double sequence $\{ \ E(M_{D_k}(h_n)) \mid n, k \geq 1 \ \}$ is regularly convergent (in Hardy's sense)

(3.24) The double sequence $\{ \ E(M_{D_k}(h_n)) \mid n, k \geq 1 \ \}$ is convergent (in Pringsheim's sense)

(3.25) $$-\infty < \lim_{k \to \infty} E(M_{D_k}(h_\infty)) = \lim_{n \to \infty} E(M_D(h_n)) < +\infty$$

(3.26) $\forall \varepsilon > 0$, $\exists p_\varepsilon \geq 1$ such that $\forall n, m, k, l \geq p_\varepsilon$ we have:

$$| \ E(M_{D_k}(h_n)) - E(M_{D_l}(h_m)) \ | < \varepsilon$$

(3.27) $\forall \varepsilon > 0$, $\exists p_\varepsilon \geq 1$ such that:

$$E(M_D(h_{p_\varepsilon})) - E(M_{D_{p_\varepsilon}}(h_\infty)) < \varepsilon$$

where $h_\infty(\theta)$ denotes the P-a.s. limit of $h_n(\theta)$ as $n \to \infty$ for all $\theta \in \Theta_0$. In this case the limit of $\{ \ E(M_{D_k}(h_n)) \mid n, k \geq 1 \ \}$ from (3.23) and (3.24) is equal to $E(M_D(h_\infty))$, and we have:

$$\lim_{k \to \infty} \lim_{n \to \infty} E(M_{D_k}(h_n)) \ = \ E(M_D(h_\infty)) \ = \ \lim_{n \to \infty} \lim_{k \to \infty} E(M_{D_k}(h_n)) \ .$$

Moreover, by Theorem 3.1 in [15], then we have:

$$M_D(h_n) \to M_D(h_\infty) \quad P\text{-a.s. and in } \ L^1(P)$$

as $n \to \infty$. For more information in this direction we refer to [15].

We recall that $\bar{I}$ denotes the upper semicontinuous envelope of I on Θ . *The graph* of I is defined by $gr(I) = \{ \ (\theta, I(\theta)) \mid \theta \in \Theta_0 \ \}$. A finite cover of the set T is any family of non-empty subsets $A_1, \ldots, A_n$ of T satisfying $T = \cup_{j=1}^n A_j$. The class of all finite covers of T is denoted by $\Gamma(T)$. Finally, according to [13], we set $\mathcal{C}_{-\infty}(\mathbf{R}) = \{ \ (-\infty, p] \ \mid \ p \in \mathbf{Q} \ \}$.

Theorem 3.4

Let $\mathcal{H} = (\{h_n(\omega, \theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ *be a given family of reversed submartingales, and let* Γ *be a subset of* Θ_0 . *Suppose that* $\mathcal{H}$ *is degenerated and that any of the following three conditions is satisfied:*

(3.28) $\mathcal{H}$ *is separable*

(3.29) $\mathcal{H}$ *is separable relative to* $B(\theta, r_\theta)$ *and* $\mathcal{C}_{-\infty}(\mathbf{R})$ *for all* $\theta \in \Gamma \cap \hat{M}(\mathcal{H})$ *with some* $r_\theta > 0$

(3.30) $\mathcal{H}$ *is* $a.s.$*-lower semicontinuous on* $\Gamma \cap \hat{M}(\mathcal{H})$.

If the family of reversed submartingales:

$$(\{h_n(\omega,\xi), \mathcal{S}_n \mid n \geq 1\} \mid \xi \in b(\theta,r))$$

is totally bounded in P-mean relative to $b(\theta,r) \cap D_\theta$ for all $\theta \in \Gamma \cap \hat{M}(\mathcal{H})$ and all $r \in \mathbf{Q}_+$, $r \leq r_\theta$ with $r_\theta > 0$, where D_θ is a countable subset of Θ_0 satisfying the conditions of the separability definition of $\mathcal{H}$ relative to $B(\theta, r_\theta)$, then we have:

(3.31) $\quad \bar{H}(\theta) = \bar{I}(\theta)\ ,\ \forall \theta \in \Gamma \cap \hat{M}(\mathcal{H})$

(3.32) $\quad \Gamma \cap \hat{M}(\mathcal{H}) = \{\ \theta \in \Gamma \mid \bar{I}(\theta) = \beta\ \}$.

If I in addition satisfies any of the following two equivalent conditions:

(3.33) $\quad$ *I is upper semicontinuous on $\Gamma \cap \hat{M}(\mathcal{H})$*

(3.34) $\quad$ *$cl(gr(I)) \cap ((\Gamma \cap \hat{M}(\mathcal{H})) \times \{\beta\}) \subset gr(I)$, or equivalently if $\theta_n \to \theta$ and $I(\theta_n) \to \beta$ with $\theta \in \Gamma \cap \hat{M}(\mathcal{H})$, then $I(\theta) = \beta$*

then $\mathcal{H}$ is S-consistent on Γ.

Proof. Suppose that $\mathcal{H}$ is degenerated and that any of conditions (3.28)-(3.30) is satisfied. Then by (ii) in Proposition 4.3 in [13] it is no restriction to assume that (3.29) holds. Hence by (i) in Proposition 3.3 in [13] for given $\theta \in \Gamma \cap \hat{M}(\mathcal{H})$ there exists a P-null set $N_\theta \in \mathcal{F}$ such that:

$$\sup_{\xi \in b(\theta,r)} h_n(\omega,\xi) = \sup_{\xi \in b(\theta,r) \cap D_\theta} h_n(\omega,\xi)$$

for all $\omega \in \Omega \setminus N_\theta$, all $r \in \mathbf{Q}_+$, $r \leq r_\theta$ and all $n \geq 1$, where D_θ is a given countable subset of Θ_0 satisfying the conditions of the separability definition of $\mathcal{H}$ relative to $B(\theta, r_\theta)$ with $r_\theta > 0$. Since by our hypotheses the family of reversed submartingales $(\{h_n(\omega,\xi), \mathcal{S}_n \mid n \geq 1\} \mid \xi \in b(\theta,r))$ is totally bounded in P-mean relative to $b(\theta,r) \cap D_\theta$, then by Theorem 3.1 in [13] we have:

$$\sup_{\xi \in b(\theta,r) \cap D_\theta} h_n(\omega,\xi) \longrightarrow \sup_{\xi \in b(\theta,r) \cap D_\theta} I(\xi)$$

as $n \to \infty$, for all $\omega \in \Omega \setminus N_\theta$, and all $r \in \mathbf{Q}_+$, $r \leq r_\theta$, where $N_\theta \in \mathcal{F}$ is a P-null set. Hence by (2.20) we find:

$$\bar{H}(\theta) = \inf_{r>0} \limsup_{n\to\infty} h_n^*(\omega, b(\theta,r)) = \inf_{r>0} \lim_{n\to\infty} h_n^*(\omega, b(\theta,r) \cap D_\theta) =$$

$$= \inf_{r>0} \sup_{\xi \in b(\theta,r) \cap D_\theta} I(\xi) \leq \bar{I}(\theta) \leq \bar{H}(\theta) .$$

These facts imply $\bar{H}(\theta) = \bar{I}(\theta) \geq \beta$ and complete the proof of (3.31) and (3.32).

In addition, given $\theta \in \Gamma \cap \hat{M}(\mathcal{H})$ there exists a sequence $\{\ \theta_n \mid n \geq 1\ \}$ in Θ_0 satisfying:

$$d(\theta_n, \theta) < 2^{-n} \quad \text{and} \quad I(\theta_n) \geq (\beta \wedge n) - 2^{-n}$$

for all $n \geq 1$. Hence we see that $\theta_n \to \theta$ and $I(\theta_n) \to \beta$. Thus if (3.34) is satisfied, we obtain $\Gamma \cap \hat{M}(\mathcal{H}) \subset M(\mathcal{H})$. Moreover, it is straightforward to verify that (3.33) is equivalent to (3.34) under (3.32). These facts complete the proof. □

Remark 3.1

If any of conditions (3.28)-(3.30) in Theorem 3.4 is satisfied, then by Remark 3.2 in [13] and (ii) in Proposition 4.3 in [13] we see that there exists a countable set D in Θ_0 satisfying the conditions of the separability definition relative to all open sets $\mathcal{G}(\Theta_0)$ in Θ_0 and $\mathcal{C}_{-\infty}(\mathbf{R})$. Moreover, if (3.30) is satisfied, then D can be taken as an arbitrary countable dense subset of Θ_0 . Consequently, it might be possible in these cases that in the assumption of totally boundedness in P-mean in Theorem 3.4 we actually have $D_\theta = D$ for all $\theta \in \Gamma \cap \hat{M}(\mathcal{H})$.

Remark 3.2

Under the hypotheses of Theorem 3.4, let us suppose that $\theta \in \Theta_d$. Then there exists $r_\theta > 0$ such that $Eh_k^*(b(\theta, r_\theta) \cap D_\theta) < \infty$ for some $k \geq 1$, where $D_\theta = \{ \delta_j \mid j \geq 1 \}$ is a countable subset of Θ_0 satisfying the conditions of the separability definition of $\mathcal{H}$ relative to $B(\theta, r_\theta)$. Since D_θ is countable, the family $\{ h_n^*(b(\theta, r) \cap D_\theta) , \mathcal{S}_n \mid n \geq k \}$ forms a reversed submartingale for all $0 < r \leq r_\theta$. Hence we easily find that the family of reversed submartingales:

$$(\{ h_n(\omega, \xi) , \mathcal{S}_n \mid n \geq 1 \} \mid \xi \in b(\theta, r))$$

is totally bounded in P-mean relative to $b(\theta, r) \cap D_\theta$, if and only if the condition is satisfied:

$$-\infty < \lim_{k \to \infty} H^*(b(\theta, r) \cap D_{\theta,k}) = H^*(b(\theta, r) \cap D_\theta) < +\infty \tag{3.35}$$

where $D_{\theta,k} = \{ \delta_1, \ldots, \delta_k \}$ for all $k \geq 1$ and $0 < r \leq r_\theta$. In this case we have:

$$\begin{aligned} H^*(b(\theta, r) \cap D_\theta) = H_0^*(b(\theta, r) \cap D_\theta) = \lim_{n \to \infty} Eh_n^*(b(\theta, r) \cap D_\theta) = \\ = \inf_{n \geq 1} Eh_n^*(b(\theta, r) \cap D_\theta) = \sup_{j \geq 1} I(\delta_j) \end{aligned} \tag{3.36}$$

for all $0 < r \leq r_\theta$. Note also that we have:

$$H^*(b(\theta, r) \cap D_{\theta,k}) = H_0^*(b(\theta, r) \cap D_{\theta,k}) = \lim_{n \to \infty} Eh_n^*(b(\theta, r) \cap D_{\theta,k}) = \sup_{1 \leq j \leq k} I(\delta_j) \tag{3.37}$$

for all $0 < r \leq r_\theta$ and all $k \geq 1$.

6. We continue by examining conditions for consistency of $\mathcal{H}$ that are expressed in terms of an internal (Lipschitz) property of the sequence $\{ h_n(\theta) \mid n \geq 1 \}$ when θ runs over Θ . Our next result in this direction is based upon Theorem 4.7 in [15] and the following simple inequality:

$$\sup_{n \geq 1} (a_n - b_n)^+ \geq \sup_{n \geq 1} a_n - \sup_{n \geq 1} b_n \tag{3.38}$$

where $a_n, b_n \in \mathbf{R}$ for $n \geq 1$, with the convention $\infty - \infty = 0$.

Theorem 3.5

Let $\mathcal{H} = (\{h_n(\omega, \theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ be a given family of reversed submartingales, and let Γ be a subset of Θ_0 . Suppose that $\mathcal{H}$ is degenerated and that any of the following three *conditions is satisfied:*

(3.39) $\quad \mathcal{H}$ *is separable*

(3.40) $\quad \mathcal{H}$ *is separable relative to* $B(\theta, r_\theta)$ *and* $\mathcal{C}_{-\infty}(\mathbf{R})$ *for all* $\theta \in \Gamma \cap \hat{M}(\mathcal{H})$ *with some* $r_\theta > 0$

(3.41) $\quad \mathcal{H}$ *is* $a.s.$*-lower semicontinuous on* $\Gamma \cap \hat{M}(\mathcal{H})$.

Suppose that $\forall \theta \in \Gamma \cap \hat{M}$, $\exists r_\theta > 0$ *such that the following condition is satisfied:*

(3.42) $\quad \forall \varepsilon > 0$, $\exists \Pi = \{ \Delta_1, \ldots, \Delta_{m_\varepsilon} \} \in \Gamma(b(\theta, r_\theta) \cap D_\theta)$ *and* $\exists \delta_1 \in \Delta_1, \ldots, \delta_{m_\varepsilon} \in \Delta_{m_\varepsilon}$ *such that* $\forall N \geq 1$, $\exists n_\varepsilon \geq N$ *and* $\exists \Psi_1, \ldots, \Psi_{m_\varepsilon} \in L^1(P)$ *satisfying:*

(i) $\quad (I(\delta_j) - I(\xi))^+ \leq \varepsilon \ , \forall \xi \in \Delta_j \ , \ \forall j = 1, \ldots, m_\varepsilon$

(ii) $\quad (h_{n_\varepsilon}(\xi) - h_{n_\varepsilon}(\delta_j))^+ \leq \Psi_j \ , \forall \xi \in \Delta_j \ , \ \forall j = 1, \ldots, m_\varepsilon$

(iii) $\quad \max_{1 \leq j \leq m_\varepsilon} E(\Psi_j) \leq \varepsilon$

where D_θ *is a countable subset of* Θ_0 *satisfying the conditions of the separability definition of* $\mathcal{H}$ *relative to* $B(\theta, r_\theta)$. *Then we have:*

(3.43) $\quad \bar{H}(\theta) = \bar{I}(\theta) \ , \forall \theta \in \Gamma \cap \hat{M}(\mathcal{H})$

(3.44) $\quad \Gamma \cap \hat{M}(\mathcal{H}) = \{ \theta \in \Gamma \mid \bar{I}(\theta) = \beta \}$.

If I *in addition satisfies any of the following two equivalent conditions:*

(3.45) $\quad I$ *is upper semicontinuous on* $\Gamma \cap \hat{M}(\mathcal{H})$

(3.46) $\quad cl(gr(I)) \cap ((\Gamma \cap \hat{M}(\mathcal{H})) \times \{\beta\}) \subset gr(I)$ *, or equivalently if* $\theta_n \to \theta$ *and* $I(\theta_n) \to \beta$ *with* $\theta \in \Gamma \cap \hat{M}(\mathcal{H})$ *, then* $I(\theta) = \beta$

then $\mathcal{H}$ *is* S*-consistent on* Γ .

Proof. We showed in the proof of Theorem 3.4 that under hypotheses (3.39)-(3.41) for every $\theta \in \Gamma \cap \hat{M}(\mathcal{H})$ there exists a P-null set $N_\theta \in \mathcal{F}$ such that:

$$\sup_{\xi \in b(\theta,r)} h_n(\omega, \xi) = \sup_{\xi \in b(\theta,r) \cap D_\theta} h_n(\omega, \xi)$$

for all $\omega \in \Omega \setminus N_\theta$, all $r \in \mathbf{Q}_+$, $r \leq r_\theta$ and all $n \geq 1$, where D_θ is a given countable subset of Θ_0 satisfying the conditions of the separability definition of $\mathcal{H}$ relative to $B(\theta, r_\theta)$ and $r_\theta > 0$ is a given number. Let $\theta \in \Gamma \cap \hat{M}(\mathcal{H})$ be a given point. Since for given points $\xi \in b(\theta, r_\theta)$ and $\delta_j \in b(\theta, r_\theta) \cap D_\theta$ with $j \geq 1$ we have:

$$\Big((h_{n_\varepsilon}(\xi) - I(\xi)) - (h_{n_\varepsilon}(\delta_j) - I(\delta_j)) \Big)^+ \leq (h_{n_\varepsilon}(\xi) - h_{n_\varepsilon}(\delta_j))^+ + (I(\delta_j) - I(\xi))^+$$

then by (3.42) and Theorem 4.7 in [15] we may conclude that the family of reversed submartingales:

$$(\{ (h_n(\xi) - I(\xi))^+, \mathcal{S}_n \mid n \geq 1 \} \mid \xi \in b(\theta, r_\theta))$$

is totally bounded in P-mean relative to $b(\theta, r_\theta) \cap D_\theta$. Thus by Theorem 3.1 in [15] we have:

$$\sup_{\xi \in b(\theta, r_\theta) \cap D_\theta} (h_n(\xi) - I(\xi))^+ \longrightarrow 0 \quad P\text{-}a.s. \text{ and in } L^1(P)$$

as $n \to \infty$, for all $r \in \mathbf{Q}_+$, $r \leq r_\theta$. Note that by (ii) in (3.42) for given $r \in \mathbf{Q}_+$, $r \leq r_\theta$ we have $Eh^*_{n_\varepsilon}(b(\theta, r) \cap D_\theta) < \infty$. Thus the family $\{ h^*_n(b(\theta, r) \cap D_\theta) , \mathcal{S}_n \mid n \geq n_\varepsilon \}$ forms a reversed submartingale. Moreover, it is clear that:

$$\sup_{\xi \in b(\theta, r) \cap D_\theta} I(\xi) \leq \lim_{n \to \infty} \sup_{\xi \in b(\theta, r) \cap D_\theta} h_n(\xi) \quad P\text{-}a.s.$$

Hence by (3.38) we obtain:

$$\sup_{\xi \in b(\theta, r) \cap D_\theta} h_n(\omega, \xi) \longrightarrow \sup_{\xi \in b(\theta, r) \cap D_\theta} I(\xi)$$

as $n \to \infty$, for all $\omega \in \Omega \setminus N_\theta$, where $N_\theta \in \mathcal{F}$ is a P-null set. The remaining part of the proof is the same as the last part of the proof of Theorem 3.4. □

The next theorem concerns the martingale case and is based upon Proposition 4.9 in [15].

Theorem 3.6

Let $\mathcal{H} = (\{h_n(\omega, \theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ be a given family of reversed martingales, and let Γ be a subset of Θ_0 . Suppose that $\mathcal{H}$ is degenerated and that any of the following three conditions is satisfied:

(3.47) *$\mathcal{H}$ is separable*

(3.48) *$\mathcal{H}$ is separable relative to $B(\theta, r_\theta)$ and $\mathcal{C}_{-\infty}(\mathbf{R})$ for all $\theta \in \Gamma \cap \hat{M}(\mathcal{H})$ with some $r_\theta > 0$*

(3.49) *$\mathcal{H}$ is $a.s.$-lower semicontinuous.*

Suppose that $\forall \theta \in \Gamma \cap \hat{M}(\mathcal{H})$, $\exists r_\theta > 0$ such that the following condition is satisfied:

(3.50) $\forall \varepsilon > 0$, $\exists n_\varepsilon \geq 1$, $\exists \Pi = \{ \Delta_1, \dots, \Delta_{m_\varepsilon} \} \in \Gamma(b(\theta, r_\theta) \cap D_\theta)$ *and* $\exists \Psi_1, \dots, \Psi_{m_\varepsilon} \in L^1(P)$ *satisfying*

(i) $| h_{n_\varepsilon}(\theta') - h_{n_\varepsilon}(\theta'') | \leq \Psi_j$, $\forall \theta', \theta'' \in \Delta_j$, $\forall j = 1, \dots, m_\varepsilon$

(ii) $\max_{1 \leq j \leq m_\varepsilon} E(\psi_j) \leq \varepsilon$

where D_θ is a countable subset of Θ_0 satisfying the conditions of the separability definition of $\mathcal{H}$ relative to $B(\theta, r_\theta)$. Then we have:

(3.51) $$\sup_{\xi \in b(\theta, r_\theta) \cap D_\theta} | h_n(\xi) - I(\xi) | \longrightarrow 0 \;\; P\text{-}a.s. \text{ and in } L^1(P), \text{ as } n \to \infty , \forall \theta \in \Gamma \cap \hat{M}(\mathcal{H})$$

(3.52) $\bar{H}(\theta) = \bar{I}(\theta)$, $\forall \theta \in \Gamma \cap \hat{M}(\mathcal{H})$

(3.53) $\Gamma \cap \hat{M}(\mathcal{H}) = \{ \theta \in \Gamma \mid \bar{I}(\theta) = \beta \}$.

If I in addition satisfies any of the following two equivalent conditions:

(3.54) *I is upper semicontinuous on $\Gamma \cap \hat{M}(\mathcal{H})$*

(3.55) $cl(gr(I)) \cap ((\Gamma \cap \hat{M}(\mathcal{H})) \times \{\beta\}) \subset gr(I)$ *, or equivalently if* $\theta_n \to \theta$ *and* $I(\theta_n) \to \beta$ *with* $\theta \in \Gamma \cap \hat{M}(\mathcal{H})$ *, then* $I(\theta) = \beta$

then $\mathcal{H}$ *is S-consistent on* Γ *.*

Proof. We showed in the proof of Theorem 3.4 that under hypotheses (3.47)-(3.49) for every $\theta \in \Gamma \cap \hat{M}(\mathcal{H})$ there exists a P-null set $N_\theta \in \mathcal{F}$ such that:

$$\sup_{\xi \in b(\theta,r)} h_n(\omega,\xi) = \sup_{\xi \in b(\theta,r) \cap D_\theta} h_n(\omega,\xi)$$

for all $\omega \in \Omega \setminus N_\theta$, all $r \in \mathbf{Q}_+$, $r \leq r_\theta$ and all $n \geq 1$, where D_θ is a given countable subset of Θ_0 satisfying the conditions of the separability definition of $\mathcal{H}$ relative to $B(\theta, r_\theta)$ and $r_\theta > 0$ is a given number. Take a point $\theta \in \Gamma \cap \hat{M}(\mathcal{H})$, then by (3.50) and Proposition 4.9 in [15] we may conclude that the family of reversed submartingales:

$$(\{ \mid h_n(\xi) - I(\xi) \mid , \mathcal{S}_n \mid n \geq 1 \} , \xi \in b(\theta, r_\theta))$$

is totally bounded in P-mean relative to $b(\theta, r_\theta) \cap D_\theta$ with $r_\theta > 0$. Thus by Theorem 3.1 in [15] we have:

$$\sup_{\xi \in b(\theta,r) \cap D_\theta} \mid h_n(\xi) - I(\xi) \mid \longrightarrow 0 \quad P\text{-}a.s. \text{ and in } L^1(P)$$

as $n \to \infty$, for all $r \in \mathbf{Q}_+$, $r \leq r_\theta$. Hence we find:

$$\sup_{\xi \in b(\theta,r) \cap D_\theta} h_n(\omega,\xi) \longrightarrow \sup_{\xi \in b(\theta,r) \cap D_\theta} I(\xi)$$

as $n \to \infty$, for all $\omega \in \Omega \setminus N_\theta$ and all $r \in \mathbf{Q}_+$, $r \leq r_\theta$, where $N_\theta \in \mathcal{F}$ is a P-null set. These facts complete the proof of (3.51)-(3.53) and the last statement of the theorem in exactly the same way as in the last part of the proof of Theorem 3.4. □

7. We proceed by studying conditions for consistency of not necessarily separable families of reversed submartingales. First we consider the submartingale case in Theorem 3.7. Then we present its martingale version in Theorem 3.8. We find it convenient to recall some definitions from [15].

Let Θ_0 be an analytic metric space, let (Θ, d) be a compact metric space containing Θ_0 , and let f be a real valued function defined on Θ . Then we define *the lower, upper and absolute jump* of f at a given point $\theta \in \Theta$ as follows:

$$\partial^+(\theta, f) = \inf_{r>0} \sup_{\xi \in b(\theta,r)} [f(\theta) - f(\xi)]$$

$$\partial^-(\theta, f) = \inf_{r>0} \sup_{\xi \in b(\theta,r)} [f(\xi) - f(\theta)]$$

$$\partial(\theta, f) = \max \{ \partial^+(\theta, f) , \partial^-(\theta, f) \} = \inf_{r>0} \sup_{\xi \in b(\theta,r)} \mid f(\theta) - f(\xi) \mid .$$

In addition, we introduce the following notation:

$$\Delta^+(f) = \sup \{ \partial^+(\theta, f) \mid \theta \in \Theta \}$$

$$\Delta^-(f) \;=\; \sup \,\{\; \partial^-(\theta, f) \;|\; \theta \in \Theta \;\}$$

$$\Delta(f) \;=\; \sup \,\{\; \partial(\theta, f) \;|\; \theta \in \Theta \;\} \;.$$

Theorem 3.7

Let $\mathcal{H} = (\; \{\; h_n(\omega, \theta) \;,\; \mathcal{S}_n \;|\; n \geq 1 \;\} \;|\; \theta \in \Theta_0 \;)$ *be a given family of reversed submartingales, and let* Γ *be a subset of* Θ_0 *. Suppose that* $\mathcal{H}$ *is degenerated and* $\forall \theta \in \Gamma \cap \hat{M}(\mathcal{H})$ *,* $\exists r_\theta > 0$ *such that the following condition is satisfied:*

(3.56) $\forall \varepsilon > 0 \;,\; \exists \Pi = \{\; \Delta_1, \ldots, \Delta_{m_\varepsilon} \;\} \in \Gamma(b(\theta, r_\theta) \cap D)$ *and* $\exists \delta_1 \in \Delta_1, \ldots, \delta_{m_\varepsilon} \in \Delta_{m_\varepsilon}$ *such that* $\forall N \geq 1 \;,\; \exists n_\varepsilon \geq N$ *and* $\exists \Psi_1, \ldots \Psi_{m_\varepsilon} \in L^1(P)$ *satisfying*

(i) $(I(\delta_j) - I(\xi))^+ \leq \varepsilon \;,\quad \forall \xi \in \Delta_j \;,\; \forall j = 1, \ldots, m_\varepsilon$

(ii) $(h_{n_\varepsilon}(\xi) - h_{n_\varepsilon}(\delta_j))^+ \leq \Psi_j \;,\quad \forall \xi \in \Delta_j \;,\; \forall j = 1, \ldots, m_\varepsilon$

(iii) $\max\limits_{1 \leq j \leq m_\varepsilon} E(\Psi_j) \leq \varepsilon$

where D *is a countable subset of* Θ_0 *. Then we have:*

(3.57) *If (3.56) is satisfied for each countable* D *subset of* Θ_0 *and* $\Delta^+(h_n) \to 0$ P*-a.s. as* $n \to \infty$ *, then we have:*

(i) $\bar{H}(\theta) = \bar{I}(\theta) \;,\; \forall \theta \in \Gamma \cap \hat{M}(\mathcal{H})$

(ii) $\Gamma \cap \hat{M}(\mathcal{H}) = \{\; \theta \in \Gamma \;|\; \bar{I}(\theta) = \beta \;\}$.

If I *in addition satisfies any of the following two equivalent conditions:*

(iii) I *is upper semicontinuous on* $\Gamma \cap \hat{M}(\mathcal{H})$

(iv) $cl(gr(I)) \cap ((\Gamma \cap \hat{M}(\mathcal{H})) \times \{\beta\}) \subset gr(I)$ *, or equivalently if* $\theta_n \to \theta$ *and* $I(\theta_n) \to \beta$ *with* $\theta \in \Gamma \cap \hat{M}(\mathcal{H})$ *, then* $I(\theta) = \beta$

then $\mathcal{H}$ *is S-consistent on* Γ .

(3.58) *If (3.56) is satisfied for some countable dense* D *subset of* Θ_0 *,* I *is upper semicontinuous on* $\cup_{\theta \in \Gamma \cap \hat{M}(\mathcal{H})} b(\theta, r_\theta)$ *, and* $\Delta^+(h_n) \to 0$ P*-a.s. as* $n \to \infty$ *, then we have:*

(i) $\bar{H}(\theta) = I(\theta) \;,\; \forall \theta \in \Gamma \cap \hat{M}(\mathcal{H})$

(ii) $\Gamma \cap \hat{M}(\mathcal{H}) = \{\; \theta \in \Gamma \;|\; I(\theta) = \beta \;\}$

and $\mathcal{H}$ *is* S*-consistent on* Γ .

Proof. By Theorem 4.7 in [15] we see that condition (3.56) is equivalent to the fact that the family of reversed submartingales:

$$(\, \{\, (h_n(\xi) - I(\xi))^+,\, \mathcal{S}_n \;|\; n \geq 1 \,\} \;|\; \xi \in b(\theta, r_\theta) \,)$$

is totally bounded in P-mean relative to $b(\theta, r_\theta) \cap D$, where $r_\theta > 0$ is a given number. Hence by the first hypothesis in (3.57), or the first two hypotheses in (3.58), and by Theorem 4.1 in [15] there exists a sequence of random variables $\{ V_n \mid n \geq 1 \}$ satisfying $V_n \to 0$ P-$a.s.$ as $n \to \infty$ such that:

$$\sup_{\xi \in b(\theta, r_\theta)} (h_n(\omega,\xi) - I(\xi))^+ \leq \Delta^+(h_n(\omega)) + V_n(\omega)$$

for all $\omega \in \Omega$ and all $n \geq 1$. Thus the assumption $\Delta^+(h_n) \to 0$ P-$a.s.$ for $n \to \infty$ implies:

$$\sup_{\xi \in b(\theta, r)} (h_n(\xi) - I(\xi))^+ \longrightarrow 0 \quad P\text{-}a.s.$$

as $n \to \infty$, for all $r \in \mathbf{Q}_+$, $r \leq r_\theta$. Since the following two inequalities are satisfied:

$$\sup_{\xi \in b(\theta, r)} (h_n(\omega,\xi) - I(\xi))^+ \geq \sup_{\xi \in b(\theta, r)} h_n(\omega,\xi) - \sup_{\xi \in b(\theta, r)} I(\xi)$$

$$\limsup_{n \to \infty} \sup_{\xi \in b(\theta, r)} h_n(\xi) \geq \sup_{\xi \in b(\theta, r)} I(\xi) \quad P\text{-}a.s.$$

whenever $\omega \in \Omega$, we may conclude:

$$\sup_{\xi \in b(\theta, r)} h_n(\omega,\xi) \longrightarrow \sup_{\xi \in b(\theta, r)} I(\xi)$$

as $n \to \infty$, for all $\omega \in \Omega \setminus N_\theta$ and all $r \in \mathbf{Q}_+$, $r \leq r_\theta$, where N_θ is a P-null set in $\mathcal{F}$. The remaining part of the proof is the same as the last part of the proof of Theorem 3.4. □

Theorem 3.8

Let $\mathcal{H} = (\{ h_n(\omega,\theta) , \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \Theta_0)$ be a given family of reversed martingales, and let Γ be a subset of Θ_0. Suppose that $\mathcal{H}$ is degenerated and $\forall \theta \in \Gamma \cap \hat{M}(\mathcal{H})$, $\exists r_\theta > 0$ such that the following condition is satisfied:

(3.59) $\forall \varepsilon > 0$, $\exists n_\varepsilon \geq 1$, $\exists \Pi = \{ \Delta_1, \dots, \Delta_{m_\varepsilon} \} \in \Gamma(b(\theta, r_\theta) \cap D)$ *and* $\exists \Psi_1, \dots \Psi_{m_\varepsilon} \in L^1(P)$ *satisfying*

(i) $| h_{n_\varepsilon}(\theta') - h_{n_\varepsilon}(\theta'') | \leq \Psi_j$, $\forall \theta', \theta'' \in \Delta_j$, $\forall j = 1, \dots, m_\varepsilon$

(ii) $\max_{1 \leq j \leq m_\varepsilon} E(\Psi_j) \leq \varepsilon$

where D is a countable subset of Θ_0. Then we have:

(3.60) *If (3.59) is satisfied for each countable D subset of Θ_0 and $\Delta(h_n) \to 0$ P-$a.s.$ as $n \to \infty$, then we have:*

(i) $\bar{H}(\theta) = \bar{I}(\theta)$, $\forall \theta \in \Gamma \cap \hat{M}(\mathcal{H})$

(ii) $\Gamma \cap \hat{M}(\mathcal{H}) = \{ \theta \in \Gamma \mid \bar{I}(\theta) = \beta \}$.

If I in addition satisfies any of the following two equivalent conditions:

(iii) *I is upper semicontinuous on $\Gamma \cap \hat{M}(\mathcal{H})$*

(iv) $cl(gr(I)) \cap ((\Gamma \cap \hat{M}(\mathcal{H})) \times \{\beta\}) \subset gr(I)$, *or equivalently if $\theta_n \to \theta$ and $I(\theta_n) \to \beta$ with $\theta \in \Gamma \cap \hat{M}(\mathcal{H})$, then $I(\theta) = \beta$*

then $\mathcal{H}$ is S-consistent on Γ.

(3.61) *If (3.59) is satisfied for some countable dense* D *subset of* Θ_0 , I *is continuous on* $\cup_{\theta\in\Gamma\cap\hat{M}(\mathcal{H})} b(\theta, r_\theta)$, *and* $\Delta(h_n) \to 0$ P*-a.s. as* $n \to \infty$, *then we have:*

(i) $\bar{H}(\theta) = \bar{I}(\theta)$, $\forall\theta \in \Gamma \cap \hat{M}(\mathcal{H})$

(ii) $\Gamma \cap \hat{M}(\mathcal{H}) = \{ \theta \in \Gamma \mid \bar{I}(\theta) = \beta \}$

and $\mathcal{H}$ *is* S*-consistent on* Γ .

Proof. The proof can be carried out as the proof of Theorem 3.7 upon using Proposition 4.9 in [15] instead of Theorem 4.7 in [15], and Theorem 4.3 in [15] instead of Theorem 4.1 in [15]. □

4. Examples of application and concluding remarks

There is a large number of *statistical models* that are covered by the preceding results. We cannot review them all here, but will instead refer the reader to [1]-[3], [5]-[12], [16]-[18]. Of course, there are examples of statistical models which stay out of this scope, but they usually require individual treatments. Our main aim, however, was to unify as many examples as possible, under common and simple conditions.

1. To obtain a better feeling for applications in general, we find it convenient to restate and clarify the result of Theorem 3.3 in a less formal setting. Let $\mathcal{H} = (\{h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0)$ be a family of reversed submartingales defined on the probability space $(\Omega, \mathcal{F}, P)$ and indexed by the analytic metric space Θ_0 (with the Borel σ-algebra $\mathcal{B}_0$). Let Θ be a compact metric space containing Θ_0 , and let $h_n(\omega,\theta) = -\infty$ for $n \geq 1$, $\omega \in \Omega$ and $\theta \in \Theta \setminus \Theta_0$. Suppose that:

(4.1) $(\omega,\theta) \mapsto h_n(\omega,\theta)$ is $\mathcal{S}_n \times \mathcal{B}_0$-measurable for all $n \geq 1$

(4.2) $\mathcal{S}_\infty = \cap_{n=1}^\infty \mathcal{S}_n$ is degenerated.

Let $\Gamma \subset \Theta$ be given. Then $\mathcal{H}$ is S-consistent on Γ as soon as the conditions are fulfilled:

(4.3) $\int \sup_{\xi\in b(\theta,r_\theta)} h_1(\omega,\xi)\ P(d\omega) < \infty$, $\forall\theta \in \Gamma$ with some $r_\theta > 0$

(4.4) $h_n(\omega,\cdot)$ is P-a.s. upper semicontinuous at θ , $\forall n \geq 1$ and $\forall\theta \in \Gamma$

(4.5) $\mathcal{H}$ is conditionally S-regular relative to:

$$B(\theta, r_\theta) = \{ b(\theta, r) \mid r \in \mathbf{Q}_+ , r \leq r_\theta \}$$

for all $\theta \in \Gamma$ with some $r_\theta > 0$.

In other words, whenever (4.3)-(4.5) is satisfied, every accumulation point of any sequence of approximating (asymptotic, empirical) maximums $\{ \hat{\theta}_n \mid n \geq 1 \}$ associated with $\mathcal{H}$ which belongs to Γ is a maximum point on Θ_0 of the information function I associated with $\mathcal{H}$.

2. We find it useful to explain condition (4.5) in more detail. For this first recall definition (2.6). Note that if (4.1) and (4.3) above are fulfilled, then (i) and (ii) from this definition are satisfied with $B = b(\theta, r)$ whenever $\theta \in \Gamma$ and $r \in \mathbf{Q}_+$, $r \leq r_\theta$. By definition of a reversed submartingale and monotonicity of the conditional expectation, we can clearly select a P-null set

$N_\theta \in \mathcal{F}$ depending on the given $\theta \in B$ such that:

$$E\big\{h_n^*(B)|\mathcal{S}_{n+1}\big\}(\omega) \geq E\big\{h_n(\theta)|\mathcal{S}_{n+1}\big\}(\omega) \geq h_{n+1}(\omega,\theta) \tag{4.6}$$

for all $\omega \in \Omega \setminus N_\theta$. However B might be uncountable, and therefore we cannot generally pass to the supremum in (4.6) over all $\theta \in B$ (see [13] for a counter-example). This is a crucial fact to be understood about the property of conditional S-regularity of $\mathcal{H}$ relative to $\{B\}$. Note that this property says that such a passage to the supremum is possible. In this context we find it instructive once again to direct reader's attention to the definition (2.6).

Generally, the condition (4.5) is fulfilled in any of the following cases:

(4.7) Process $\{\, h_n(\omega,\theta)\,\}_{\theta\in\Theta_0}$ is separable for $n \geq 1$ (see Proposition 4.1 in [13])

(4.8) Trajectory $h_n(\omega,\cdot)$ is lower semicontinuous (on the neighborhood of Γ) for P-a.s. $\omega \in \Omega$ and $n \geq 1$ (see Proposition 4.3 in [13])

(4.9) Any U-process:

$$h_n(\omega,\theta) = \frac{1}{n!}\sum_{\sigma\in\mathcal{P}_n} h(X_\sigma(\omega),\theta) \quad (\ n \geq 1\ ,\ \omega \in \Omega\ ,\ \theta \in \Theta_0\ )$$

satisfies (4.5) whenever $X = (X_1, X_2, \dots)$ is exchangeable and $Eh(X,\theta) < \infty$ for all $\theta \in \Theta_0$ (see Example 4.4 in [13]) . We recall that $\mathcal{P}_n$ denotes the set of all permutations of $\{\, 1, 2, \dots, n\,\}$, and that $X_\sigma = (X_{\sigma_1}, \dots, X_{\sigma_n}, X_{n+1}, \dots)$ where X_j take values in any measurable space. The map $h(\cdot,\theta)$ is real valued for all $\theta \in \Theta_0$.

We think that (4.9) is of theoretical and practical interest. In this way we see that the preceding results cover a variety of important examples. Note also that Theorems 3.4-3.8 offer a different type conditions for S-consistency of $\mathcal{H}$. These results are particularly useful when condition (4.4) fails, but the information function I associated with $\mathcal{H}$ is still upper semicontinuous.

3. In the remainder we explain the role of the preceding results in the area of *stochastic processes*. In this context the following problem appears worthy of consideration.

Let $\mathcal{Z} = \{(Z_n(t))_{t\in T} \mid n \geq 1\}$ be a sequence of stochastic processes defined on the probability space $(\Omega, \mathcal{F}, P)$ with the common time set T . Let $\hat{t}_n(\omega)$ be a maximum point of $Z_n(\omega,\cdot)$ on T :

$$Z_n(\omega,\hat{t}_n(\omega)) = \sup_{t\in T} Z_n(\omega,t) \tag{4.10}$$

for $\omega \in \Omega$ and $n \geq 1$. The problem is to *describe the asymptotic behavior of* $\hat{t}_n(\omega)$ *for* $n \to \infty$.

Under the hypotheses in this paper we have:

$$Z_n(\cdot,t) \to L(t) \quad P\text{-a.s.} \tag{4.11}$$

as $n \to \infty$, where $L(t)$ is degenerated (a constant) for all $t \in T$. It indicates that maximums $\hat{t}_n(\omega)$ could approach the set $M \subset T$ of all maximum points of the limit L on T . Since it may happen that the supremum in (4.10) is not attainable, we relax this condition by requiring:

$$Z_n(\omega,\hat{t}_n(\omega)) \geq \Big(\sup_{t\in T} Z_n(\omega,t) - \varepsilon_n(\omega)\Big) \wedge n \tag{4.12}$$

for $\omega \in \Omega$ and $n \geq 1$ with $\varepsilon_n(\omega) \to 0$ as $n \to \infty$. We assume that the time set T

is an analytic metric space, and for any compact metric space $\hat{T}$ which contains T we put $Z_n(\omega, \theta) = -\infty$ for $n \geq 1$, $\omega \in \Omega$ and $t \in \hat{T} \setminus T$. We moreover suppose that:

(4.13) $$(\omega, t) \rightarrow Z_n(\omega, t) \text{ is measurable}$$

as a map from $\Omega \times T$ into $\mathbf{R}$ for all $n \geq 1$. Therefore by passing to the limit in (4.12) we get:

(4.14) $$\liminf_{n \to \infty} Z_n(\omega, \hat{t}_n(\omega)) \geq \sup_{t \in T} L(t) \quad P\text{-a.s.}$$

In this way a sequence of approximating maximums $\{\hat{t}_n \mid n \geq 1\}$ associated with $\mathcal{Z}$ is obtained. We may then ask when the *consistency* statement is satisfied:

(4.15) *Every accumulation point of any sequence of approximating maximums* $\{ \hat{t}_n \mid n \geq 1 \}$ *associated with* $\mathcal{Z}$ *, which belongs to the given set* $\Gamma \subset \hat{T}$ *, is a maximum point on* T *of the limiting process* L *of* $\mathcal{Z}$ *.*

4. In this paper we find a solution of this problem under the additional hypothesis:

(4.16) $$\{ Z_n(t) \, , \, \mathcal{F}_n \mid n \geq 1 \} \text{ is a reversed submartingale}$$

for all $t \in T$.

The following conditions (see Theorem 3.3) are sufficient for (4.15):

(4.17) $\int \sup_{s \in b(t, r_t)} Z_1(\omega, s) \, P(d\omega) < \infty \, , \forall t \in \Gamma$ with some $r_t > 0$

(4.18) $Z_n(\omega, \cdot)$ is P-a.s. upper semicontinuous at t , $\forall n \geq 1$ and $\forall t \in \Gamma$

(4.19) $\mathcal{Z}$ is conditionally S-regular relative to $\{ b(t,r) \mid r \in \mathbf{Q}_+ \, , \, r < r_t \}$, $\forall t \in \Gamma$ with some $r_t > 0$.

It is moreover known (see [13]) that any of the following conditions is sufficient for (4.19):

(4.20) Process $(Z_n(t))_{t \in T}$ is separable for $n \geq 1$

(4.21) Trajectory $Z_n(\omega, \cdot)$ is lower semicontinuous (on the neighborhood of Γ) for P-a.s. $\omega \in \Omega$ and $n \geq 1$

(4.22) Any U-process:

$$Z_n(\omega, t) = \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} z(X_\sigma(\omega), t) \quad (\, n \geq 1 \, , \, \omega \in \Omega \, , \, t \in T \,)$$

satisfies (4.19) whenever $X = (X_1, X_2, \dots)$ is exchangeable and $Ez(X, t) < \infty$ for all $t \in T$.

Finally, note that Theorems 3.4-3.8 offer a different type of conditions for (4.15). These results are useful when condition (4.18) fails, but the limiting process L is still upper semicontinuous.

5. We conclude the paper by giving two examples of application which follow the same pattern and can easily be modified to treat new cases. We are unaware of similar results.

Throughout $\{ X_j \mid j \geq 1 \}$ denotes an i.i.d. sequence of random variables, and the processes $Z_n(\omega, t)$ are of the form (4.22).

Example 4.1

Let $X_1 \sim N(0,1)$ be from the standard Gaussian distribution with density function $f(x) = \exp(-x^2/2)/\sqrt{2\pi}$ for $x \in \mathbf{R}$. Let T be a compact set in $\mathbf{R}$, and let $\alpha = \min(T)$ and $\beta = \max(T)$. If $\hat{t}_n(\omega)$ maximizes the process:

$$(n-1)\, Z_n(\omega,t) = \sum_{i=1}^{n} \Big(\cos\big(tX_i(\omega)\big) - \frac{1}{n}\sum_{j=1}^{n} \cos\big(tX_j(\omega)\big)\Big)^2$$

over $t \in T$ (in the sense of (4.12) or (4.14) above), then we have:

1° $(|\alpha| > |\beta|) \Rightarrow \hat{t}_n \rightarrow \alpha$ P-a.s.
2° $(|\alpha| < |\beta|) \Rightarrow \hat{t}_n \rightarrow \beta$ P-a.s.
3° $(|\alpha| = |\beta|) \Rightarrow \hat{t}_n \rightarrow \{\alpha, \beta\}$ P-a.s.

as $n \rightarrow \infty$. We clarify that $\hat{t}_n \rightarrow \{\alpha, \beta\}$ P-a.s. means that every accumulation point of $\{\, \hat{t}_n(\omega) \mid n \geq 1 \,\}$ is either α or β for P-a.s. $\omega \in \Omega$.

These facts readily follow from (4.17)-(4.19) by putting $z(x,t) = (\cos(tx_1) - \cos(tx_2))^2/2$ in (4.22) and using that $\int_{-\infty}^{\infty} \exp(-x^2/2)\cos(tx)\, dx = \sqrt{2\pi}\exp(-t^2/2)$ for $t \in \mathbf{R}$. It should be noted that L from (4.11) takes the form:

$$L(t) = Var(\cos(tX_1)) = \tfrac{1}{2} + \tfrac{1}{2}\exp(-2t^2) - \exp(-t^2)$$

for $t \in T$.

Example 4.2

Let $X_1 \sim U(0,1)$ be from the uniform distribution on $[0,1]$. Let $T = [-\alpha, \beta]$ for $\alpha \geq 0$ and $\beta \geq \pi$. If $\hat{t}_n(\omega)$ maximizes the process:

$$n(n-1)\, Z_n(\omega,t) = \sum_{i,j=1}^{n}\sum_{i\neq j}^{n} X_i(\omega)\sin(tX_j(\omega))$$

over $t \in T$ (in the sense of (4.12) or (4.14) above), then $\hat{t}_n \rightarrow \zeta$ P-a.s. as $n \rightarrow \infty$. The given ζ is a unique number from $]0,\pi[$ that satisfies $\zeta\sin(\zeta) + \cos(\zeta) = 1$.

This fact readily follows from (4.17)-(4.19) by putting $z(x,t) = x_1\sin(tx_2)$ in (4.22). It should be noted that L from (4.11) takes the form:

$$L(t) = E(X_1\sin(tX_2)) = \tfrac{1}{2t}(1-\cos(t))$$

for $t \in T$, as well as that ζ is a unique maximum point of L on $\mathbf{R}$.

6. The problem of asymptotic normality in these and similar examples appears worthy of consideration and is postponed for further research reports.

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