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Solving Non-Linear Optimal Stopping Problems by the Method of Time-Change

J. L. PEDERSEN & G. PESKIR

Abstract. Some non-linear optimal stopping problems can be solved explicitly by using a common method which is based on time-change. We describe this method and illustrate its use by considering several examples dealing with Brownian motion. In each of these examples we derive explicit formulas for the value function and display the optimal stopping time. The main emphasis of the paper is on the method of proof and its unifying scope.

1. Introduction

The main goal of this paper is to present a deterministic time-change method which enables one to solve some non-linear optimal stopping problems explicitly. The basic idea is to transform the original (difficult) problem into a new (easier) problem. The method is illustrated through several examples with applications in the next section.

1. To explain the ideas in more detail, let $((X_t), \mathbf{P}_x)$ be a one-dimensional time-homogeneous diffusion associated with the infinitesimal generator

$$\mathbf{L_X} = \mu(x)\,\frac{\partial}{\partial x} + \sigma^2(x)\,\frac{1}{2}\,\frac{\partial^2}{\partial x^2}$$

where $x \mapsto \sigma(x) > 0$ and $x \mapsto \mu(x)$ are continuous. Assume moreover that there exists a standard Brownian motion (B_t) such that (X_t) solves the stochastic differential equation

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t$$

with $X_0 = x$ under $\mathbf{P}_x$. The typical optimal stopping problem which appears under consideration below has the value function given by

$$V_*(t,x) = \sup_{\tau} \mathbf{E}_x\Big(\alpha\big(t+\tau\big)\,X_\tau\Big)$$

where the supremum is taken over a class of stopping times τ for (X_t) and α is a smooth but *non-linear* function. This forces us to take (t, X_t) as the underlying diffusion in the problem, and thus by general optimal stopping theory we know that the value function V_* should solve the following partial differential equation

$$\frac{\partial V}{\partial t}(t,x) + \mathbf{L_X} V(t,x) = 0$$

in the domain of continued observation (see [7]). However, it is generally difficult to find the appropriate solution of the partial differential equation, and the basic idea of the time-change

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method is to transform the original problem into a new optimal stopping problem such that the new value function solves an ordinary differential equation.

2. To do so one is naturally led to find a deterministic time-change $t \mapsto \sigma_t$ satisfying the following two conditions:

(i) $t \mapsto \sigma_t$ is continuous and strictly increasing

(ii) there exists a one-dimensional time-homogeneous diffusion (Z_t) with infinitesimal generator $\mathbf{L_Z}$ such that $\alpha(\sigma_t)\, X_{\sigma_t} = e^{-rt}\, Z_t$ for some $r \in \mathbb{R}$.

From general optimal stopping theory we know that the new (time-changed) value function

$$W_*(z) = \sup_\tau \mathbf{E}_z \Big(e^{-r\tau} Z_\tau \Big)$$

where the supremum is taken over a class of stopping times τ for (Z_t) , should solve the ordinary differential equation

$$\mathbf{L_Z} W_*(z) = r\, W_*(z)$$

in the domain of continued observation. Note that under condition (i) there is a one-to-one correspondence between the original problem and the new problem, i.e. if τ is a stopping time for (Z_t) then σ_τ is a stopping time for (X_t) and vice versa.

3. Given the diffusion (X_t) the crucial point is to find the process (Z_t) and the time-change σ_t fulfilling conditions (i) and (ii) above. Itô formula offers an answer to these questions.

Setting $(Y_t) = (\beta(t) X_t)$ where $\beta \neq 0$ is a smooth function, by Itô formula we get

$$Y_t = Y_0 + \int_0^t \left(\frac{\beta'(u)}{\beta(u)}\, Y_u + \beta(u)\, \mu\Big(\frac{Y_u}{\beta(u)}\Big) \right) du \; + \int_0^t \beta(u)\, \sigma\Big(\frac{Y_u}{\beta(u)}\Big)\, dB_u$$

and hence (Y_t) has the infinitesimal generator

$$\mathbf{L_Y} = \left(\frac{\beta'(t)}{\beta(t)}\, y + \beta(t)\, \mu\Big(\frac{y}{\beta(t)}\Big) \right) \frac{\partial}{\partial y} \; + \beta^2(t)\, \sigma^2\Big(\frac{y}{\beta(t)}\Big)\, \frac{1}{2}\, \frac{\partial^2}{\partial y^2} \; . \tag{1.1}$$

The time-changed process $(Z_t) = (Y_{\sigma_t})$ has the infinitesimal generator (see [4] p.175)

$$\mathbf{L_Z} = \frac{1}{\rho(t)}\, \mathbf{L_Y} \tag{1.2}$$

where σ_t is the time-change given by

$$\sigma_t = \inf \left\{ r > 0 \, : \int_0^r \rho(u)\, du > t \right\}$$

for some $u \mapsto \rho(u) > 0$ (to be found) such that $\sigma_t \to \infty$ as $t \to \infty$.

The process (Z_t) and the time-change σ_t will be fulfilling conditions (i) and (ii) above if the infinitesimal generator $\mathbf{L_Z}$ *does not depend on* t . In view of (1.1) this clearly imposes conditions on β (and α above) which make the method applicable:

$$\mu\Big(\frac{y}{\beta(t)}\Big) = \gamma(t)\, G_1(y) \tag{1.3}$$

$$\sigma^2\Big(\frac{y}{\beta(t)}\Big) = \frac{\gamma(t)}{\beta(t)}\, G_2(y) \tag{1.4}$$

where $\gamma = \gamma(t)$, $G_1 = G_1(y)$ and $G_2 = G_2(y)$ are functions required to exist.

4. In our examples below the diffusion (X_t) is given as Brownian motion $(B_t + x)$ started at x under $\mathbf{P}_x$, and thus its infinitesimal generator is given by

$$\mathbf{L_X} = \frac{1}{2} \frac{\partial^2}{\partial x^2} .$$

By the foregoing observations we shall find a time-change σ_t and a process (Z_t) satisfying conditions (i) and (ii) above. With the notation introduced above we see from (1.1) that the infinitesimal generator of (Y_t) in this case is given by

$$\mathbf{L_Y} = \frac{\beta'(t)}{\beta(t)}\, y \frac{\partial}{\partial y} + \beta^2(t) \frac{1}{2} \frac{\partial^2}{\partial y^2} .$$

Observe that conditions (1.3) and (1.4) are easily realized with $\gamma(t)=\beta(t)$, $G_1=0$ and $G_2=1$. Thus if β solves the differential equation $\beta'(t)/\beta(t) = -\beta^2(t)/2$, and we set $\rho = \beta^2/2$, then from (1.2) we see that $\mathbf{L_Z}$ does not depend on t . Noting that $\beta(t) = 1/\sqrt{1+t}$ solves this equation, and putting $\rho(t) = 1/2(1+t)$, we find that

$$\sigma_t = \inf \left\{ r > 0 : \int_0^r \rho(u)\, du > t \right\} = e^{2t} - 1 . \tag{1.5}$$

Thus the time-changed process (Z_t) has the infinitesimal generator given by

$$\mathbf{L_Z} = -z \frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2}$$

and hence (Z_t) is an Ornstein-Uhlenbeck process. While this fact is well-known, the technique described may be applied in a similar context involving other diffusions (Example 4).

5. We believe that the time-change arguments described above are well-known to the specialists in the field, although we could not find it in the literature on optimal stopping. In the next section we shall apply this method and present solutions to several optimal stopping problems some of which were already treated earlier and solved by means of other techniques. Apart from the time-change arguments just described, the method of proof makes also use of Brownian scaling and the principle of smooth fit in a free boundary problem. Once the guess is performed, Itô calculus is used as a verification tool. The main emphasis of the paper is on the method of proof and its unifying scope.

2. Examples and applications

In this section we explicitly solve some non-linear optimal stopping problems by applying the time-change method described in the first section.

Throughout (B_t) denotes a standard Brownian motion started at zero under $\mathbf{P}$, and the diffusion (X_t) is given as the Brownian motion $B_t + x$ started at x under $\mathbf{P}_x$.

Given the time-change $\sigma_t = e^{2t} - 1$ from (1.5), we know that the time-changed process

$$Z_t = X_{\sigma_t} / \sqrt{1 + \sigma_t} \tag{2.1}$$

is an Ornstein-Uhlenbeck process satisfying

$$dZ_t = -Z_t\, dt + \sqrt{2}\, dB_t \tag{2.2}$$

$$\mathbf{L_Z} = -z \frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2} . \tag{2.3}$$

With this notation we may now enter into the first example.

Example 1. Consider the optimal stopping problem with the value function

$$V_*(t,x) = \sup_\tau \mathbf{E}_x\Big(|X_\tau| - c\sqrt{t+\tau}\,\Big) \tag{2.4}$$

where the supremum is taken over all stopping times τ for (X_t) satisfying $\mathbf{E}_x(\sqrt{\tau}) < \infty$ and $c > 0$ is given and fixed. We shall solve this problem in three steps.

1. In the first step we shall apply Brownian scaling and note that $\tilde{\tau} = \tau/t$ is a stopping time for the Brownian motion $s \mapsto t^{-1/2} B_{ts}$. If we now rewrite (2.4) as

$$V_*(t,x) = \sup_\tau \mathbf{E}\Big(|B_\tau + x| - c\sqrt{t+\tau}\,\Big) = \sqrt{t}\,\sup_{\tau/t} \mathbf{E}\Big(\big|t^{-1/2}B_{t(\tau/t)} + x/\sqrt{t}\,\big| - c\sqrt{1+\tau/t}\,\Big)$$

we clearly see that

$$V_*(t,x) = \sqrt{t}\,V_*(1, x/\sqrt{t}\,) \tag{2.5}$$

and therefore we only need to look at $V_*(1,x)$ in the sequel. By using (2.5) we can also make the following observation on the optimal stopping boundary for the problem (2.4).

Remark 2.1. In the problem (2.4) the gain function equals $g(t,x) = |x| - c\sqrt{t}$ and the diffusion is identified with $\big(t+r, X_r\big)$. If a point (t_0, x_0) belongs to the boundary of the domain of continued observation, i.e. (t_0, x_0) is an instantaneously stopping point ($\tau \equiv 0$ is an optimal stopping time), then we get from (2.5) that $V_*(t_0,x_0) = |x_0| - c\sqrt{t_0} = \sqrt{t_0}\,V_*(1, x_0/\sqrt{t_0}\,)$. Hence $V_*(1, x_0/\sqrt{t_0}\,) = |x_0|/\sqrt{t_0} - c$ and therefore the point $(1, x_0/\sqrt{t_0}\,)$ is also instantaneously stopping. Set now $\gamma_0 = |x_0|/\sqrt{t_0}$ and note that if (t,x) is any point satisfying $|x|/\sqrt{t} = \gamma_0$, then this point is also instantaneously stopping. This offers a heuristic argument that the optimal stopping boundary should be $|x| = \gamma_0\sqrt{t}$ for some $\gamma_0 > 0$ to be found.

2. In the second step we shall apply the time-change $t \mapsto \sigma_t$ from (1.5) to the problem $V_*(1,x)$ and transform it into a new problem. From (2.1) we get

$$|X_{\sigma_\tau}| - c\sqrt{1+\sigma_\tau} = \sqrt{1+\sigma_\tau}\,\big(|Z_\tau| - c\,\big) = e^\tau\big(|Z_\tau| - c\,\big) \tag{2.6}$$

and the problem to determine $V_*(1,x)$ therefore reduces to compute

$$V_*(1,x) = W_*(x) \tag{2.7}$$

where W_* is the value function of the new (time-changed) optimal stopping problem

$$W_*(z) = \sup_\tau \mathbf{E}_z\Big(e^\tau\big(|Z_\tau| - c\,\big)\Big) \tag{2.8}$$

the supremum being taken over all stopping times τ for (Z_t) for which $\mathbf{E}_z(e^\tau) < \infty$. Observe that this problem is one-dimensional.

3. In the third step we shall show how to solve the problem (2.8). From general optimal stopping theory we know that the following stopping time should be optimal

$$\tau_* = \inf\,\big\{\, t > 0 \,:\, |Z_t| \geq z_* \,\big\} \tag{2.9}$$

where $z_* \geq 0$ is the optimal stopping point to be found. Observe that this guess agrees with Remark 2.1. Note that the domain of continued observation $C = (-z_*, z_*)$ is assumed symmetric around zero since the Ornstein-Uhlenbeck process is symmetric, i.e. the process $(-Z_t)$ is also an Ornstein-Uhlenbeck process started at $-z$. By using the same argument we may also argue that the value function W_* should be *even.*

To compute the value function W_* for $z \in (-z_*, z_*)$ and to determine the optimal stopping point z_*, in view of (2.8)+(2.9) it is natural to formulate the following system

$$\mathbf{L_Z} W(z) = -W(z) \quad \text{for } z \in (-z_*, z_*) \tag{2.10}$$

$$W(\pm z_*) = z_* - c \quad \text{(instantaneous stopping)} \tag{2.11}$$

$$W'(\pm z_*) = \pm 1 \quad \text{(smooth fit)} \tag{2.12}$$

with $\mathbf{L_Z}$ in (2.3). The system (2.10)-(2.12) forms a *free boundary problem.* The condition (2.12) is imposed since we expect that *the principle of smooth fit* should hold.

It is known (see Section 3 below) that the equation (2.10) admits the even solution (3.3) and the odd solution (3.4) as two linearly independent solutions. Since the value function should be even, we can forget the odd solution and from (3.3) we see that

$$W(z) = -A\, M(-\tfrac{1}{2}\,, \tfrac{1}{2}\,, \tfrac{z^2}{2})$$

for some $A > 0$ to be found.

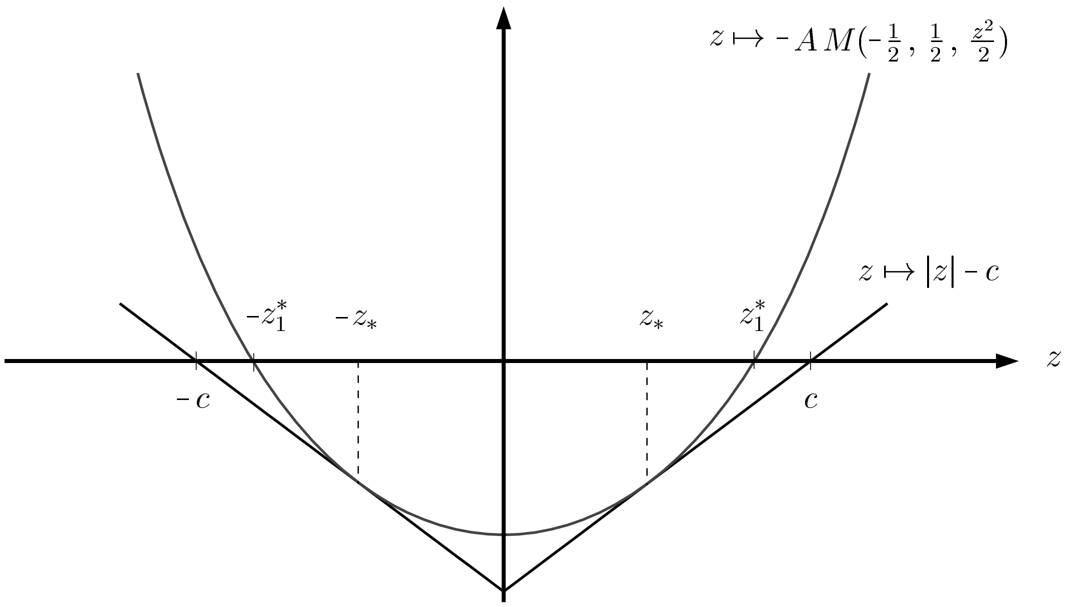

FIGURE 1. A computer drawing of the solution of the free boundary problem (2.10)-(2.12). The solution equals $z \mapsto -A\, M(-1/2, 1/2, z^2/2)$ for $|z| < z_*$ and $z \mapsto |z| - c$ for $|z| \geq z_*$. The constant A is chosen (and z_* is obtained) such that the smooth fit holds at $\pm z_*$ (the first derivative of the solution is continuous at $\pm z_*$).

From FIGURE 1 we clearly see that only for $c \geq z_1^*$ the two boundary conditions (2.11)+(2.12) can be fulfilled, where z_1^* is the unique positive root of $M(-1/2\,, 1/2\,, z^2/2) = 0$. Thus by (2.11)+(2.12) and (3.5) when $c \geq z_1^*$ we find that $A = z_*^{-1}/M(1/2\,, 3/2\,, z_*^2/2)$ and that $z_* \leq z_1^*$ is the unique positive root of the equation

$$z^{-1} M(-\tfrac{1}{2}\,, \tfrac{1}{2}\,, \tfrac{z^2}{2}) = (c - z)\, M(\tfrac{1}{2}\,, \tfrac{3}{2}\,, \tfrac{z^2}{2})\ . \tag{2.13}$$

Note that for $c < z_1^*$ the equation (2.13) have no solution.

In this way we have obtained the following candidate for the value function W_* in the problem (2.8) when $c \geq z_1^*$

$$W(z) = \begin{cases} -z_*^{-1} M(-\frac{1}{2}, \frac{1}{2}, \frac{z^2}{2})/M(\frac{1}{2}, \frac{3}{2}, \frac{z_*^2}{2}) & \text{if } |z| < z_* \\ |z| - c & \text{if } |z| \geq z_* \end{cases} \tag{2.14}$$

and the following candidate for the optimal stopping time τ_* when $c > z_1^*$

$$\tau_{z_*} = \inf \{ t > 0 : |Z_t| \geq z_* \} . \tag{2.15}$$

In the proof below we shall see that $\mathbf{E}_z(e^{\tau_{z_*}}) < \infty$ when $c > z_1^*$ (and thus $z_* < z_1^*$). For $c = z_1^*$ (and thus $z_* = z_1^*$) the stopping time τ_{z_*} fails to satisfy $\mathbf{E}_z(e^{\tau_{z_*}}) < \infty$, but clearly τ_{z_*} are approximately optimal if we let $c \downarrow z_1^*$ (and hence $z_* \uparrow z_1^*$) . For $c < z_1^*$ we have $W(z) = \infty$ and it is never optimal to stop.

4. To verify that these formulas are correct (with $c > z_1^*$ given and fixed) we shall apply Itô formula to the process $(e^t W(Z_t))$. For this note that $z \mapsto W(z)$ is C^2 everywhere but at $\pm z_*$. However, since Lebesgue measure of those u for which $Z_u = \pm z_*$ is zero, the values $W''(\pm z_*)$ matter little in the sequel whatever set to be. In this way by (2.2) we obtain

$$e^t W(Z_t) = W(z) + \int_0^t e^u \big(\mathbf{L_Z} W(Z_u) + W(Z_u) \big) \, du + M_t$$

where (M_t) is a continuous local martingale given by

$$M_t = \sqrt{2} \int_0^t e^u \, W'(Z_u) \, dB_u \ .$$

Using that $\mathbf{L_Z} W(z) + W(z) \leq 0$ for $z \neq \pm z_*$, hence we get

$$e^t W(Z_t) \leq W(z) + M_t \tag{2.16}$$

for all t . Let τ be any stopping time for (Z_t) satisfying $\mathbf{E}_z(e^\tau) < \infty$. Choose a localization sequence (σ_n) of bounded stopping times for (M_t) . Clearly $W(z) \geq |z| - c$ for all z , and hence from (2.16) we find

$$\mathbf{E}_z \Big(e^{\tau \wedge \sigma_n} \big(|Z_{\tau \wedge \sigma_n}| - c \big) \Big) \leq \mathbf{E}_z \Big(e^{\tau \wedge \sigma_n} \, W(Z_{\tau \wedge \sigma_n}) \Big) \leq W(z) + \mathbf{E}_z \big(M_{\tau \wedge \sigma_n} \big) = W(z)$$

for all $n \geq 1$. Letting $n \to \infty$ and using Fatou's lemma, and then taking supremum over all stopping times τ satisfying $\mathbf{E}_z(e^\tau) < \infty$, we obtain

$$W_*(z) \leq W(z) \ . \tag{2.17}$$

Finally, to prove that equality in (2.17) is attained, and that the stopping time (2.15) is optimal, it is enough to verify that

$$W(z) = \mathbf{E}_z \Big(e^{\tau_{z_*}} \big(|Z_{\tau_{z_*}}| - c \big) \Big) = (z_* - c) \, \mathbf{E}_z \big(e^{\tau_{z_*}} \big) \ . \tag{2.18}$$

However, from general Markov process theory we know that $w(z) = \mathbf{E}_z(e^{\tau_{z_*}})$ solves (2.10), and clearly it satisfies $w(\pm z_*) = 1$. Thus (2.18) follows immediately from (2.14) and definition of z_* (see also Remark 2.6 below).

5. In this way we have established that the formulas (2.14) and (2.15) are correct. Recalling by (2.5) and (2.7) that

$$V_*(t,x) = \sqrt{t} \, W_*(x/\sqrt{t}\,)$$

we have therefore proved the following result.

Theorem 2.2. *Let* z_1^* *denote the unique positive root of* $M(-1/2\,,1/2\,,z^2/2)=0$. *The value function of the optimal stopping problem* (2.4) *for* $c \ge z_1^*$ *is given by*

$$V_*(t,x) = \begin{cases} -\sqrt{t}\, z_*^{-1}\, M(-\frac{1}{2}\,,\frac{1}{2}\,,\frac{x^2}{2t})/M(\frac{1}{2}\,,\frac{3}{2}\,,\frac{z_*^2}{2}) & \text{if } |x|/\sqrt{t} < z_* \\ |x| - c\sqrt{t} & \text{if } |x|/\sqrt{t} \ge z_* \end{cases}$$

where z_* *is the unique positive root of the equation*

$$z^{-1}M(-\tfrac{1}{2}\,,\tfrac{1}{2}\,,\tfrac{z^2}{2}) = (c-z)\,M(\tfrac{1}{2}\,,\tfrac{3}{2}\,,\tfrac{z^2}{2})$$

satisfying $z_* \le z_1^*$. *The optimal stopping time in* (2.4) *for* $c > z_1^*$ *is given by* (see Figure 2)

$$\tau_* = \inf\left\{\, r > 0 \;:\; |X_r| \ge z_*\sqrt{t+r}\,\right\} . \tag{2.19}$$

For $c = z_1^*$ *the stopping times* τ_* *are approximately optimal if we let* $c \downarrow z_1^*$. *For* $c < z_1^*$ *we have* $V_*(t,x) = \infty$ *and it is never optimal to stop.*

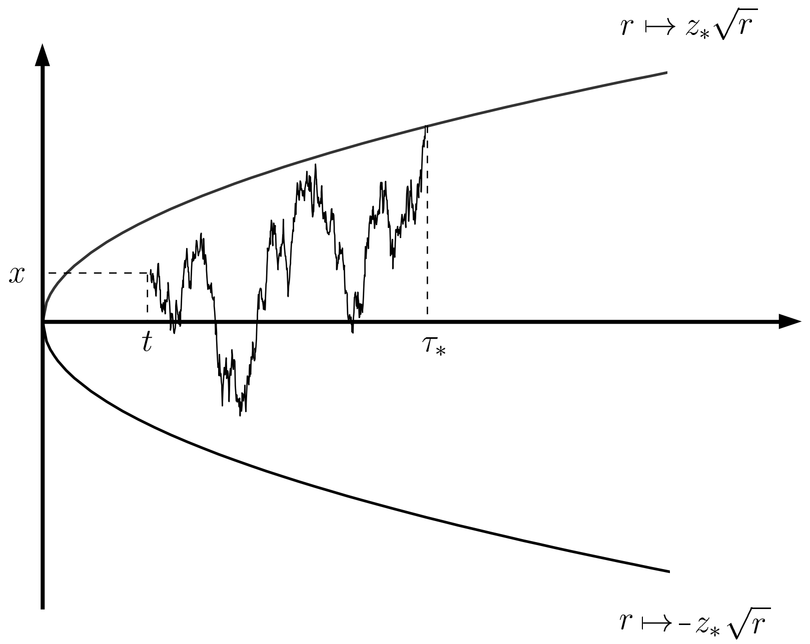

Figure 2. A computer simulation of the optimal stopping time τ_* in the problem (2.4) for $c > z_1^*$ as defined in (2.19). The process above is a standard Brownian motion which at time t starts at x . The optimal time τ_* is obtained by stopping the process as soon as it hits the area above or below the parabolic boundary $r \mapsto \pm z_*\sqrt{r}$.

Using $\sqrt{t+\tau} \le \sqrt{t} + \sqrt{\tau}$ in (2.4) it is easily verified that $V_*(t,0) \to V_*(0,0)$ as $t \downarrow 0$. Hence we see that $V_*(0,0)=0$ with $\tau_* \equiv 0$. Note also that $V_*(0,x) = |x|$ with $\tau_* \equiv 0$.

6. Let τ be any stopping time for (B_t) satisfying $\mathbf{E}(\sqrt{\tau}) < \infty$. Then from Theorem 2.2 we see that $\mathbf{E}(|X_\tau|) \le c\,\mathbf{E}(\sqrt{t+\tau}) + V_*(t,0)$ for all $c > z_1^*$. Letting first $t \downarrow 0$, and then $c \downarrow z_1^*$, we obtain the following sharp inequality which was first derived by Davis [2].

Corollary 2.3. *Let* (B_t) *be a standard Brownian motion started at* 0 , *and let* τ *be any stopping time for* (B_t) . *Then the following inequality is satisfied*

$$\mathbf{E}\big(|B_\tau|\big) \le z_1^*\,\mathbf{E}\big(\sqrt{\tau}\big)$$

with z_1^* *being the unique positive root of* $M(-1/2\,,1/2\,,z^2/2)=0$. *The constant* z_1^* *is best possible. The equality is attained through the stopping times*

$$\tau_* = \inf\,\big\{\, r>0 \;:\; |B_r| \ge z_* \sqrt{t+r}\;\big\}$$

when $t \downarrow 0$ *and* $c \downarrow z_1^*$, *where* z_* *is the unique positive root of the equation*

$$z^{-1}\,M(-\tfrac{1}{2}\,,\tfrac{1}{2}\,,\tfrac{z^2}{2}) = (c-z)\,M(\tfrac{1}{2}\,,\tfrac{3}{2}\,,\tfrac{z^2}{2})$$

satisfying $z_* < z_1^*$. (*Numerical calculations show that* $z_1^* = 1.30693\ldots$)

7. The optimal stopping problem (2.4) can naturally be extended from the power 1 to all other $p>0$. For this consider the optimal stopping problem with the value function

$$V_*(t,x) = \sup_\tau \mathbf{E}_x\Big(|X_\tau|^p - c\,\big(t+\tau\big)^{p/2}\Big) \tag{2.20}$$

where the supremum is taken over all stopping times τ for (X_t) satisfying $\mathbf{E}_x(\tau^{p/2})<\infty$ and $c>0$ is given and fixed.

Note that the case $p=2$ is easily solved directly, since we have

$$V_*(t,x) = \sup_\tau \Big((1-c)\,\mathbf{E}(\tau) + x^2 - c\,t\Big)$$

due to $\mathbf{E}|B_\tau|^2 = \mathbf{E}(\tau)$ whenever $\mathbf{E}(\tau)<\infty$. Hence we see that $V_*(t,x)=+\infty$ if $c<1$ (and it is never optimal to stop), and $V_*(t,x) = x^2-ct$ if $c\ge 1$ (and it is optimal to stop instantly). Thus below we concentrate most to the cases when $p\neq 2$ (although the results formally extend to the case $p=2$ by passing to the limit).

The following extension of Theorem 2.2 and Corollary 2.3 is valid. (Note that in the second part of the results we make use of parabolic cylinder functions $z\mapsto D_p(z)$ which are introduced in Section 3 below.)

Theorem 2.4. (I): *For* $0<p<2$ *given and fixed, let* z_p^* *denote the unique positive root of* $M(-p/2\,,1/2\,,z^2/2)=0$. *The value function of the optimal stopping problem* (2.20) *for* $c\ge (z_p^*)^p$ *is given by*

$$V_*(t,x) = \begin{cases} -t^{p/2}\,z_*^{p-2}\,M(-\frac{p}{2}\,,\frac{1}{2}\,,\frac{x^2}{2t})/M(1-\frac{p}{2}\,,\frac{3}{2}\,,\frac{z_*^2}{2}) & \text{if } |x|/\sqrt{t} < z_* \\ |x|^p - c\,t^{p/2} & \text{if } |x|/\sqrt{t} \ge z_* \end{cases}$$

where z_* *is the unique positive root of the equation*

$$z^{p-2}\,M(-\tfrac{p}{2}\,,\tfrac{1}{2}\,,\tfrac{z^2}{2}) = (c-z^p)\,M(1-\tfrac{p}{2}\,,\tfrac{3}{2}\,,\tfrac{z^2}{2})$$

satisfying $z_*\le z_p^*$. *The optimal stopping time in* (2.20) *for* $c>(z_p^*)^p$ *is given by*

$$\tau_* = \inf\,\big\{\, r>0 \;:\; |X_r| \ge z_*\sqrt{t+r}\;\big\}\,.$$

For $c=(z_p^*)^p$ *the stopping times* τ_* *are approximately optimal if we let* $c\downarrow (z_p^*)^p$. *For* $c<(z_p^*)^p$ *we have* $V_*(t,x)=\infty$ *and it is never optimal to stop.*

(II): *For* $2<p<\infty$ *given and fixed, let* z_p *denote the largest positive root of* $D_p(z)=0$. *The value function of the optimal stopping problem* (2.20) *for* $c\ge (z_p)^p$ *is given by*

$$V_*(t,x) = \begin{cases} t^{p/2}\,z_*^{p-1}\,e^{(x^2/4t)-(z_*^2/4)}\,D_p(|x|/\sqrt{t})/D_{p-1}(z_*) & \text{if } |x|/\sqrt{t} > z_* \\ |x|^p - c\,t^{p/2} & \text{if } |x|/\sqrt{t} \le z_* \end{cases}$$

where z_* *is the unique root of the equation*

$$z^{p-1}\,D_p(z) = (z^p - c)\,D_{p-1}(z)$$

satisfying $z_* \geq z_p$ *. The optimal stopping time in* (2.20) *for* $c > (z_p)^p$ *is given by*

$$\tau_* = \inf \left\{ r > 0 \; : \; |X_r| \leq z_* \sqrt{t+r} \, \right\} .$$

For $c = (z_p)^p$ *the stopping times* τ_* *are approximately optimal if we let* $c \downarrow (z_p)^p$ *. For* $c < (z_p)^p$ *we have* $V_*(t,x) = \infty$ *and it is never optimal to stop.*

Proof. The proof is an easy extension of the proof of Theorem 2.2, and we only present a few steps with differences for convenience.

By Brownian scaling we have

$$\begin{aligned} V_*(t,x) &= \sup_{\tau} \mathbf{E}_x \Big(|B_\tau + x|^p - c \left(t + \tau \right)^{p/2} \Big) \\ &= t^{p/2} \sup_{\tau/t} \mathbf{E}_x \Big(\left| t^{-1/2} B_{t(\tau/t)} + x/\sqrt{t} \, \right|^p - c \, (1 + \tau/t)^{p/2} \Big) \end{aligned}$$

and hence we see that

$$V_*(t,x) = t^{p/2} \, V_*(1, x/\sqrt{t}\,) \; . \tag{2.21}$$

By the time-change $t \mapsto \sigma_t$ from (1.5) we find

$$|X_{\sigma_\tau}|^p - c \, (1 + \sigma_\tau)^{p/2} = (1 + \sigma_\tau)^{p/2} \left(|Z_\tau|^p - c \right) = e^{p\tau} \left(|Z_\tau|^p - c \right)$$

and the problem to determine $V_*(1,x)$ therefore reduces to compute

$$V_*(1,x) = W_*(x) \tag{2.22}$$

where W_* is the value function of the new (time-changed) optimal stopping problem

$$W_*(z) = \sup_{\tau} \mathbf{E}_z \Big(e^{p\tau} \left(|Z_\tau|^p - c \right) \Big) \tag{2.23}$$

the supremum being taken over all stopping times τ for (Z_t) for which $\mathbf{E}_z(e^{p\tau}) < \infty$.

To compute W_* we are naturally led to formulate the following free boundary problem

$$\mathbf{L_Z} W(z) = -p \, W(z) \qquad \text{for } z \in C \tag{2.24}$$

$$W(z) = |z|^p - c \qquad \text{for } z \in \partial C \quad \text{(instantaneous stopping)} \tag{2.25}$$

$$W'(z) = \operatorname{sign}(z) \, p |z|^{p-1} \qquad \text{for } z \in \partial C \quad \text{(smooth fit)} \tag{2.26}$$

where C is the domain of continued observation. Observe again that W_* should be even.

In the case $0 < p < 2$ we have $C = (-z_*, z_*)$ and the stopping time

$$\tau_* = \inf \left\{ t > 0 \; : \; |Z_t| \geq z_* \right\}$$

is optimal. The proof in this case can be carried out along exactly the same lines as above when $p = 1$. However, in the case $2 < p < \infty$ we have $C = (-\infty, -z_*) \cup (z_*, \infty)$ and thus the following stopping time

$$\tau_* = \inf \left\{ t > 0 \; : \; |Z_t| \leq z_* \right\}$$

is optimal. The proof in this case requires a small modification of the previous argument. The main difference is that the solution of (2.24) used above does not have the power of smooth fit (2.25)+(2.26) any longer. It turns out, however, that the solution $z \mapsto e^{z^2/4} D_p(z)$ has this power (see FIGURE 3 and FIGURE 4), and once this being understood, the proof is again easily completed along the same lines as above (see also Remark 2.6 below). $\square$

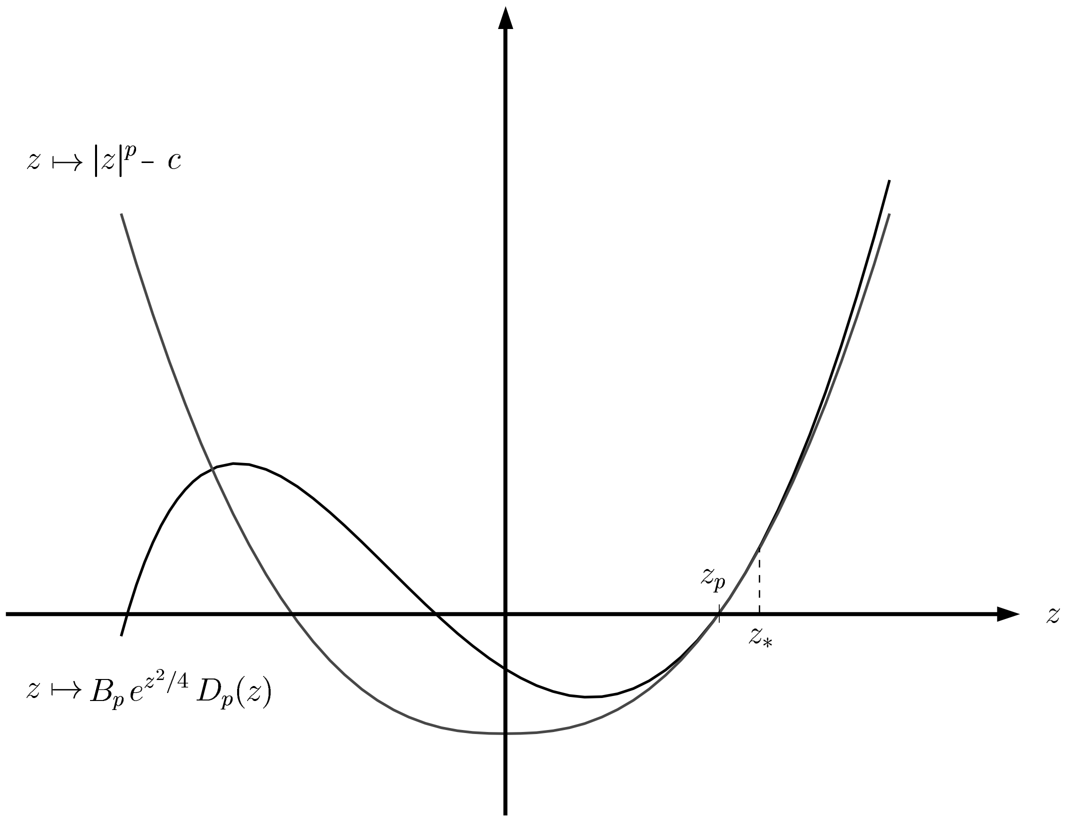

FIGURE 3. A computer drawing of the solution of the free boundary problem (2.24)-(2.26) for positive z when $p = 2.5$. The solution equals $z \mapsto B_p \exp(z^2/4) \, D_p(z)$ for $z > z_*$ and $z \mapsto z^p - c$ for $0 \le z \le z_*$. The solution extends to negative z by mirroring to an even function. The constant B_p is chosen (and z_* is obtained) such that the smooth fit holds at z_* (the first derivative of the solution is continuous at z_*). A similar picture holds for all other $p > 2$ which are not even integers.

Corollary 2.5. *Let* (B_t) *be a standard Brownian motion started at zero, and let* τ *be any stopping time for* (B_t) *.*

(I): *For* $0 < p \le 2$ *the following inequality is satisfied*

$$\mathbf{E}\big(|B_\tau|^p\big) \le (z_p^*)^p \, \mathbf{E}\big(\tau^{p/2}\big)$$

with z_p^* *being the unique positive root of* $M(-p/2\,,1/2\,,z^2/2) = 0$ *. The constant* $(z_p^*)^p$ *is best possible. The equality is attained through the stopping times*

$$\tau_* = \inf\, \big\{\, r > 0 \;:\; |B_r| \ge z_* \sqrt{t+r}\,\big\}$$

when $t \downarrow 0$ *and* $c \downarrow (z_p^*)^p$ *, where* z_* *is the unique positive root of the equation*

$$z^{p-2}\, M(-\tfrac{p}{2}\,,\tfrac{1}{2}\,,\tfrac{z^2}{2}) = (c - z^p)\, M(1 - \tfrac{p}{2}\,,\tfrac{3}{2}\,,\tfrac{z^2}{2})$$

satisfying $z_* < z_p^*$ *.*

(II): *For* $2 \le p < \infty$ *the following inequality is satisfied*

$$\mathbf{E}\big(|B_\tau|^p\big) \le (z_p)^p \, \mathbf{E}\big(\tau^{p/2}\big)$$

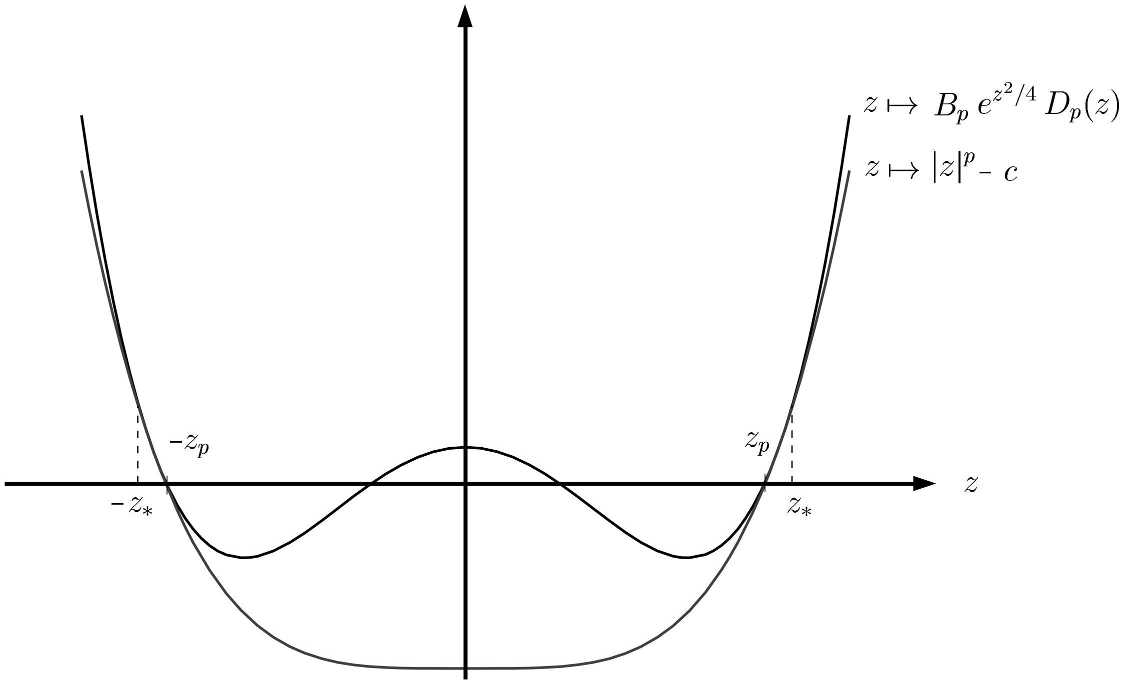

FIGURE 4. A computer drawing of the solution of the free boundary problem (2.24)-(2.26) when $p = 4$. The solution equals $z \mapsto B_p \exp(z^2/4)\, D_p(z)$ for $|z| > z_*$ and $z \mapsto |z|^p - c$ for $|z| \le z_*$. The constant B_p is chosen (and z_* is obtained) such that the smooth fit holds at $\pm z_*$ (the first derivative of the solution is continuous at $\pm z_*$). A similar picture holds for all other $p > 2$ which are even integers.

with z_p *being the largest positive root of* $D_p(z) = 0$ *. The constant* $(z_p)^p$ *is best possible. The equality is attained through the stopping times*

$$\sigma_* = \inf \{ \, r > 0 \; : \; |B_r + x| \le z_* \sqrt{r} \, \}$$

when $x \downarrow 0$ *and* $c \downarrow (z_p)^p$ *, where* z_* *is the unique root of the equation*

$$z^{p-1} D_p(z) = (z^p - c)\, D_{p-1}(z)$$

satisfying $z_* > z_p$ *.*

Remark 2.6. The argument used above to verify (2.18) extends to the general setting of Theorem 2.4 and leads to the following explicit formulas for $0 < p < \infty$. (Note that these formulas are also valid for $-\infty < p < 0$ upon setting $z_p^* = +\infty$ and $z_p = -\infty$.)

1. For $a > 0$ define the following stopping times

$$\tau_a = \inf \left\{ r > 0 \; : \; |Z_r| \ge a \right\}$$
$$\gamma_a = \inf \left\{ r > 0 \; : \; |X_r| \ge a \sqrt{t + r} \, \right\} .$$

By Brownian scaling and the time-change (1.5) it is easily verified that

$$\mathbf{E}_x \Big(\big(\gamma_a + t \big)^{p/2} \Big) = t^{p/2}\, \mathbf{E}_{x/\sqrt{t}} \big(e^{p \tau_a} \big) \; . \tag{2.27}$$

The argument quoted above for $|z| < a$ then gives

$$\mathbf{E}_z \big(e^{p \tau_a} \big) = \begin{cases} M(-\frac{p}{2}\, ,\, \frac{1}{2}\, ,\, \frac{z^2}{2}) / M(-\frac{p}{2}\, ,\, \frac{1}{2}\, ,\, \frac{a^2}{2}) & \text{if } 0 < a < z_p^* \\ \infty & \text{if } a \ge z_p^* \; . \end{cases}$$

Thus by (2.27) for $|x| < a\sqrt{t}$ we obtain

$$\mathbf{E}_x\Big(\big(\gamma_a + t\big)^{p/2}\Big) = \begin{cases} t^{p/2}\, M(-\frac{p}{2}\,,\frac{1}{2}\,,\frac{x^2}{2t})/M(-\frac{p}{2}\,,\frac{1}{2}\,,\frac{a^2}{2}) & \text{if } 0 < a < z_p^* \\ \infty & \text{if } a \ge z_p^* \,. \end{cases}$$

This formula is also derived in [5].

2. For $a > 0$ define the following stopping times

$$\widetilde{\tau}_a = \inf\,\big\{\, r > 0 \,:\, Z_r \le a \,\big\}$$

$$\widetilde{\gamma}_a = \inf\,\big\{\, r > 0 \,:\, X_r \le a\sqrt{t+r}\,\big\}\,.$$

By precisely the same arguments for $z > a$ we get

$$\mathbf{E}_z\big(e^{p\tilde{\tau}_a}\big) = \begin{cases} e^{(z^2/4)-(a^2/4)}\, D_p(z)/D_p(a) & \text{if } a > z_p \\ \infty & \text{if } a \le z_p \end{cases}$$

and for $x > a\sqrt{t}$ we thus obtain

$$\mathbf{E}_x\Big(\big(\widetilde{\gamma}_a + t\big)^{p/2}\Big) = \begin{cases} t^{p/2}\, e^{(x^2/4t)-(a^2/4)}\, D_p(x/\sqrt{t}\,)/D_p(a) & \text{if } a > z_p \\ \infty & \text{if } a \le z_p \,. \end{cases}$$

This formula is also derived in [3].

Example 2. Consider the optimal stopping problem with the value function

$$V_*(t,x) = \sup_\tau \mathbf{E}_x\Big(X_\tau/(t+\tau)\Big) \tag{2.28}$$

where the supremum is taken over all stopping times τ for (X_t) . This problem was first solved by Shepp [6] and Taylor [8], and it was later extended by Walker [10] and Van Moerbeke [9]. To compute (2.28) we shall use the same arguments as in the proof of Theorem 2.2 above.

1. In the first step we rewrite (2.28) as

$$V_*(t,x) = \sup_\tau \mathbf{E}\Big((B_\tau + x)/(t+\tau)\Big) = \tfrac{1}{\sqrt{t}}\,\sup_{\tau/t} \mathbf{E}\Big(\big(t^{-1/2}\, B_{t(\tau/t)} + x/\sqrt{t}\,\big)/(1+\tau/t)\Big)$$

and note by Brownian scaling that

$$V_*(t,x) = \tfrac{1}{\sqrt{t}}\, V_*(1, x/\sqrt{t}\,) \tag{2.29}$$

so that we only need to look at $V_*(1,x)$ in the sequel. In exactly the same way as in Remark 2.1 above, from (2.29) we can heuristically conclude that the optimal stopping boundary should be $x = \gamma_0\sqrt{t}$ for some $\gamma_0 > 0$ to be found.

2. In the second step we apply the time-change $t \mapsto \sigma_t$ from (1.5) to the problem $V_*(1,x)$ and transform it into a new problem. From (2.1) we get

$$X_{\sigma_\tau}/(1+\sigma_\tau) = Z_\tau/\sqrt{1+\sigma_\tau} = e^{-\tau} Z_\tau$$

and the problem to determine $V_*(1,x)$ therefore reduces to compute

$$V_*(1,x) = W_*(x) \tag{2.30}$$

where W_* is the value function of the new (time-changed) optimal stopping problem

$$W_*(z) = \sup_\tau \mathbf{E}_z\Big(e^{-\tau} Z_\tau\Big) \tag{2.31}$$

the supremum being taken over all stopping times τ for (Z_t) .

3. In the third step we solve the problem (2.31). From general optimal stopping theory we know that the following stopping time should be optimal

$$\tau_* = \inf \left\{ t > 0 \; : \; Z_t \geq z_* \right\} \tag{2.32}$$

where z_* is the optimal stopping point to be found.

To compute the value function W_* for $z < z_*$ and to determine the optimal stopping point z_* , it is natural to formulate the following free boundary problem

$$\mathbf{L_Z} W(z) = W(z) \quad \text{for } z < z_* \tag{2.33}$$

$$W(z_*) = z_* \quad \text{(instantaneous stopping)} \tag{2.34}$$

$$W'(z_*) = 1 \quad \text{(smooth fit)} \tag{2.35}$$

with $\mathbf{L_Z}$ in (2.3).

The equation (2.33) is of the same type as the equation from Example 1. Since the present problem is not symmetrical, we choose its general solution in accordance with (3.6)+(3.7)

$$W(z) = A\, e^{z^2/4} D_{-1}(z) + B\, e^{z^2/4} D_{-1}(-z)$$

where A and B are unknown constants.

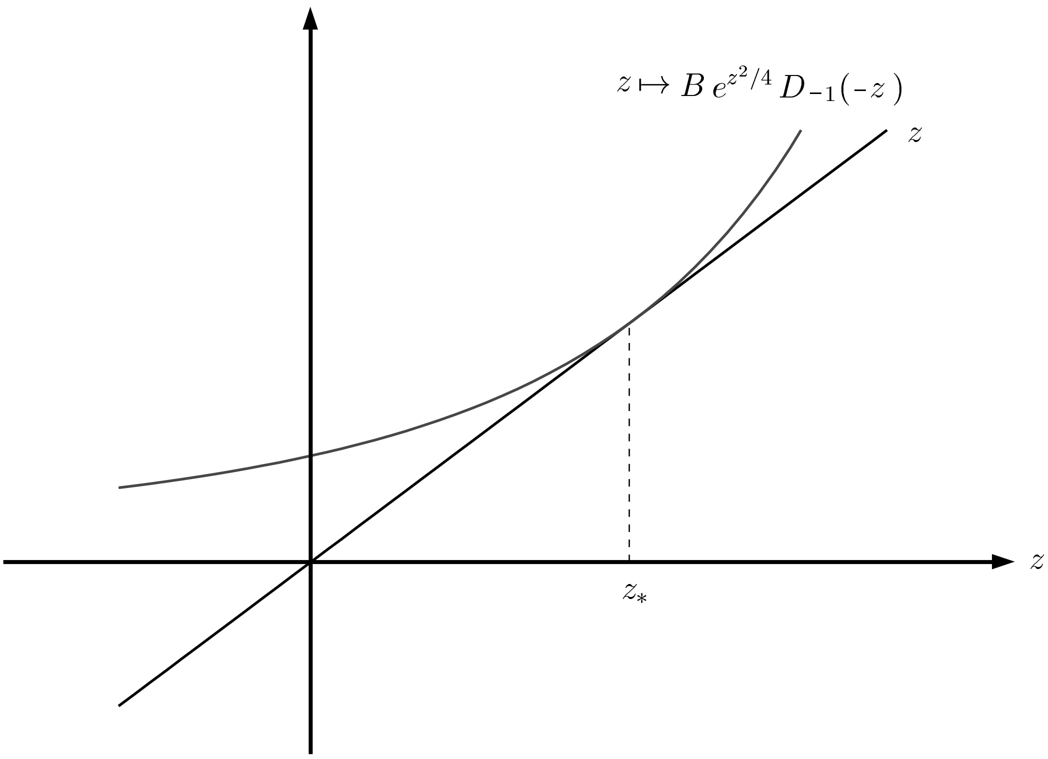

FIGURE 5. A computer drawing of the solution of the free boundary problem (2.33)-(2.35). The solution equals $z \mapsto B \exp(z^2/4)\, D_{-1}(-z)$ for $z < z_*$ and $z \mapsto z$ for $z \geq z_*$. The constant B is chosen (and z_* is obtained) such that the smooth fit holds at z_* (the first derivative of the solution is continuous at z_*).

To determine A and B the following observation is crucial. Letting $z \to -\infty$ above, we see by (3.9) that $e^{z^2/4} D_{-1}(z) \to \infty$ and $e^{z^2/4} D_{-1}(-z) \to 0$. Hence we find that $A > 0$ would contradict the clear fact that $z \mapsto W_*(z)$ is increasing, while $A < 0$ would contradict the fact that $W_*(z) \ge z$ (by observing that $e^{z^2/4}D_{-1}(z)$ converges to ∞ faster than a polynomial). Therefore we must have $A=0$. Moreover, from (3.9) we easily find that

$$e^{z^2/4}\, D_{-1}(-z) = e^{z^2/2} \int_{-\infty}^{z} e^{-u^2/2}\, du$$

and hence $W'(z) = z\, W(z) + B$. The boundary condition (2.35) implies that $1 = W'(z_*) = z_*\, W(z_*) + B = z_*^2 + B$, and hence we obtain $B = 1 - z_*^2$ (see FIGURE 5). Setting this into (2.34), we find that z_* is the root of the equation

$$z = (1-z^2)\, e^{z^2/2} \int_{-\infty}^{z} e^{-u^2/2}\, du\ .$$

In this way we have obtained the following candidate for the value function W_*

$$W(z) = \begin{cases} (1-z_*^2)\, e^{z^2/2} \displaystyle\int_{-\infty}^{z} e^{-u^2/2}\, du & \text{if } z < z_* \\ z & \text{if } z \ge z_* \end{cases} \tag{2.36}$$

and the following candidate for the optimal stopping time

$$\widehat{\tau}_{z_*} = \inf \big\{\, t > 0\ :\ Z_t \ge z_* \,\big\}\ . \tag{2.37}$$

4. To verify that these formulas are correct, we can apply Itô formula to $(e^{-t}\, W(Z_t))$, and in exactly the same way as in the proof of Theorem 2.2 above we can conclude

$$W_*(z) \le W(z)\ .$$

To prove that equality is attained at $\widehat{\tau}_{z_*}$ from (2.37), it is enough to show that

$$W(z) = \mathbf{E}_z\Big(e^{-\hat{\tau}_{z_*}}\, Z_{\hat{\tau}_{z_*}}\Big) = z_*\, \mathbf{E}_z\big(e^{-\hat{\tau}_{z_*}}\big)\ . \tag{2.38}$$

However, from general Markov process theory we know that $w(z) = \mathbf{E}_z(e^{-\hat{\tau}_{z_*}})$ solves (2.33), and clearly it satisfies $w(z_*) = 1$ and $w(-\infty) = 0$. Thus (2.38) follows from (2.36).

5. In this way we have established that formulas (2.36) and (2.37) are correct. Recalling by (2.29) and (2.30) that

$$V_*(t,x) = \tfrac{1}{\sqrt{t}}\, W_*(x/\sqrt{t}\,)$$

we have therefore proved the following result.

Theorem 2.7. *The value function of the optimal stopping problem* (2.28) *is given by*

$$V_*(t,x) = \begin{cases} \frac{1}{\sqrt{t}}\, (1-z_*^2)\, e^{x^2/2t} \displaystyle\int_{-\infty}^{x/\sqrt{t}} e^{-u^2/2}\, du & \textit{if } x/\sqrt{t} < z_* \\ x/t & \textit{if } x/\sqrt{t} \ge z_* \end{cases}$$

where z_* *is the unique root of the equation*

$$z = (1 - z^2)\, e^{z^2/2} \int_{-\infty}^{z} e^{-u^2/2}\, du\ . \tag{2.39}$$

The optimal stopping time in (2.28) *is given by* (see FIGURE 6)

$$\tau_* = \inf \big\{\, r > 0\ :\ X_r \ge z_*\, \sqrt{t+r} \,\big\}\ . \tag{2.40}$$

(*Numerical calculations show that* $z_* = 0.83992\ldots$)

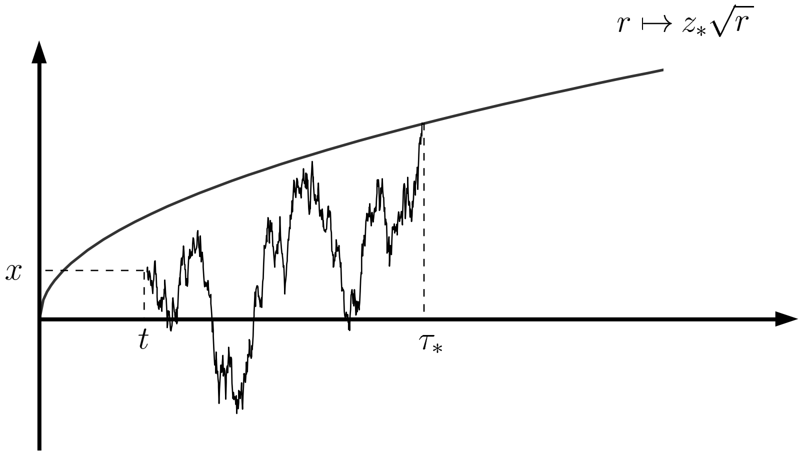

FIGURE 6. A computer simulation of the optimal stopping time τ_* in the problem (2.28) as defined in (2.40). The process above is a standard Brownian motion which at time t starts at x. The optimal time τ_* is obtained by stopping this process as soon as it hits the area above the parabolic boundary $r \mapsto z_* \sqrt{r}$.

6. Since the state space of (X_t) is $\mathbb{R}$ the most natural way to extend the problem (2.28) is to take (X_t) to the power of an odd integer (such that the state space again is $\mathbb{R}$). Consider the optimal stopping problem with the value function

$$V_*(t,x) = \sup_\tau \mathbf{E}_x \Big(X_\tau^{2n-1} / (t+\tau)^q \Big) \tag{2.41}$$

where the supremum is taken over all stopping times τ for (X_t), and $n \geq 1$ and $q > 0$ are given and fixed. This problem was solved by Walker [10] in the case $n = 1$ and $q > 1/2$. We may now further extend Theorem 2.7 as follows.

Theorem 2.8. *Let $n \geq 1$ and $q > 0$ be taken to satisfy $q > n - \frac{1}{2}$. Then the value function of the optimal stopping problem (2.41) is given by*

$$V_*(t,x) = \begin{cases} z_*^{2n-1}\, t^{n-q-1/2}\, e^{(x^2/4t)-(z_*^2/4)}\, D_{2(n-q)-1}(-x/\sqrt{t}\,)/D_{2(n-q)-1}(-z_*) & \textit{if } x/\sqrt{t} < z_* \\ x^{2n-1}/t^q & \textit{if } x/\sqrt{t} \geq z_* \end{cases}$$

where z_* is the unique root of the equation

$$(2n-1)\, D_{2(n-q)-1}(-z) = z\, \big(2(q-n)+1\big)\, D_{2(n-q-1)}(-z)\, .$$

The optimal stopping time in (2.41) is given by

$$\tau_* = \inf\, \Big\{ r > 0 \;:\; X_r \geq z_* \sqrt{t+r}\, \Big\}\, .$$

(*Note that in the case $q \leq n - 1/2$ we have $V_*(t,x) = \infty$ and it is never optimal to stop.*)

Proof. The proof will only be sketched, since the arguments are the same as for the proof of Theorem 2.7. By Brownian scaling and the time-change we find

$$V_*(t,x) = t^{n-q-1/2}\, W_*(x/\sqrt{t}\,) \tag{2.42}$$

where W_* is the value function of the new (time-changed) optimal stopping problem

$$W_*(z) = \sup_\tau \mathbf{E}_z \Big(e^{(2(n-q)-1)\tau}\, Z_\tau^{2n-1} \Big) \tag{2.43}$$

the supremum being taken over all stopping times τ for (Z_t) .

Again the optimal stopping time should be of the form

$$\tau_* = \inf \left\{ t > 0 \; : \; Z_t \ge z_* \right\} \tag{2.44}$$

and therefore the value function W_* and the optimal stopping point z_* should solve the following free boundary problem

$$\mathbf{L_Z} W(z) = \big(1-2(n-q)\big)\, W(z) \quad \text{for } z < z_* \tag{2.45}$$

$$W(z_*) = z_*^{2n-1} \quad \text{(instantaneous stopping)} \tag{2.46}$$

$$W'(z_*) = (2n-1)\, z_*^{2(n-1)} \quad \text{(smooth fit).} \tag{2.47}$$

Arguing like in the proof of Theorem 2.7 we find that the following solution of (2.45) should be taken into consideration

$$W(z) = A\, e^{z^2/4}\, D_{2(n-q)-1}(-z)$$

where A is an unknown constant. The two boundary conditions (2.46)+(2.47) with (3.8) imply that $A = z_*^{2n-1} e^{-z_*^2/4} / D_{2(n-q)-1}(-z_*)$ where z_* is the root of the equation

$$(2n-1)\, D_{2(n-q)-1}(-z) = z\, (2(q-n)+1)\, D_{2(n-q-1)}(-z) \; .$$

Thus the candidate guessed for W_* is

$$W(z) = \begin{cases} z_*^{2n-1}\, e^{(z^2/4)-(z_*^2/4)}\, D_{2(n-q)-1}(-z)/D_{2(n-q)-1}(-z_*) & \text{if } z < z_* \\ z^{2n-1} & \text{if } z \ge z_* \end{cases}$$

and the optimal stopping time is given by (2.44). By applying Itô formula like in the proof of Theorem 2.7 one can verify that these formulas are correct. Finally, inserting this back into (2.42) one obtains the result. □

Remark 2.9. By exactly the same arguments as in Remark 2.6 above, we can extend the verification of (2.38) to the general setting of Theorem 2.8, and this leads to the following explicit formulas for $0 < p < \infty$.

For $a > 0$ define the following stopping times

$$\widehat{\tau}_a = \inf \left\{ r > 0 \; : \; Z_r \ge a \right\}$$
$$\widehat{\gamma}_a = \inf \left\{ r > 0 \; : \; X_r \ge a \sqrt{t+r} \right\} .$$

Then for $z < a$ we get

$$\mathbf{E}_z\big(e^{-p\hat{\tau}_a}\big) = e^{(z^2/4)-(a^2/4)}\, D_{-p}(-z)/D_{-p}(-a)$$

and for $x < a\sqrt{t}$ we thus obtain

$$\mathbf{E}_x\Big(\big(\widehat{\gamma}_a + t\big)^{-p/2}\Big) = t^{-p/2}\, e^{(x^2/4t)-(a^2/4)}\, D_{-p}(-x/\sqrt{t}\,)/D_{-p}(-a) \; .$$

Example 3. Consider the optimal stopping problem with the value function

$$V_*(t,x) = \sup_\tau \mathbf{E}_x\Big(|X_\tau|/(t+\tau)\Big) \tag{2.48}$$

where the supremum is taken over all stopping times τ for (X_t) . This problem is a natural extension of the problem (2.28) and can be solved likewise.

By Brownian scaling and the time-change we find

$$V_*(t,x) = \tfrac{1}{\sqrt{t}}\, W_*(x/\sqrt{t}\,) \tag{2.49}$$

where W_* is the value function of the new (time-changed) optimal stopping problem

$$W_*(z) = \sup_\tau \mathbf{E}_z \Big(e^{-\tau} \, |Z_\tau| \Big) \tag{2.50}$$

the supremum being taken over all stopping times for (Z_t) .

The problem (2.50) is symmetrical (recall the discussion about (2.9) above), and therefore the following stopping time should be optimal

$$\tau_* = \inf \Big\{ \, t > 0 \; : \; |Z_t| \geq z_* \, \Big\} \, . \tag{2.51}$$

Thus it is natural to formulate the following free boundary problem

$$\mathbf{L_Z} W(z) = W(z) \quad \text{for } z \in (-z_*, z_*) \tag{2.52}$$

$$W(\pm z_*) = |z_*| \qquad \text{(instantaneous stopping)} \tag{2.53}$$

$$W'(\pm z_*) = \pm 1 \qquad \text{(smooth fit).} \tag{2.54}$$

From the proof of Theorem 2.2 we know that the equation (2.52) admits an even and an odd solution which are linearly independent. Since the value function should be even, we can forget the odd solution, and therefore we must have

$$W(z) = A \, M(\tfrac{1}{2} \, , \tfrac{1}{2} \, , \tfrac{z^2}{2})$$

for some $A > 0$ to be found. Note from Section 3 below that $M(1/2 \, , 1/2 \, , z^2/2) = \exp(z^2/2)$. The two boundary conditions (2.53) and (2.54) imply that $A = 1/\sqrt{e}$ and $z_* = 1$, and in this way we obtain the following candidate for the value function

$$W(z) = e^{(z^2/2)-(1/2)}$$

for $z \in (-1, 1)$, and the following candidate for the optimal stopping time

$$\tau = \inf \Big\{ \, t > 0 \; : \; |Z_t| \geq 1 \, \Big\} \, .$$

By applying Itô formula (as in Example 2) one can prove that these formulas are correct. Inserting this back into (2.49) we obtain the following result.

Theorem 2.10. *The value function of the optimal stopping problem* (2.48) *is given by*

$$V_*(t,x) = \begin{cases} \frac{1}{\sqrt{t}} \, e^{(x^2/2t)-(1/2)} & \text{if } |x| < \sqrt{t} \\ |x|/t & \text{if } |x| \geq \sqrt{t} \, . \end{cases}$$

The optimal stopping time in (2.48) *is given by*

$$\tau_* = \inf \Big\{ \, r > 0 \; : \; |X_r| \geq \sqrt{t+r} \, \Big\} \, .$$

As in Example 2 above, we can further extend (2.48) by considering the optimal stopping problem with the value function

$$V_*(t,x) = \sup_\tau \mathbf{E}_x \Big(|X_\tau|^p / (t+\tau)^q \Big) \tag{2.55}$$

where the supremum is taken over all finite stopping times τ for (X_t) , and $p \, , q > 0$ are given and fixed. The arguments used to solve the problem (2.48) can be repeated, and in this way we obtain the following result.

Theorem 2.11. *Let* $p\,,\,q>0$ *be taken to satisfy* $q>p/2$. *Then the value function of the optimal stopping problem* (2.55) *is given by*

$$V_*(t,x)=\begin{cases} z_*^p\,t^{p/2-q}\,M(q-\frac{p}{2}\,,\frac{1}{2}\,,\frac{x^2}{2t})/M(q-\frac{p}{2}\,,\frac{1}{2}\,,\frac{z_*^2}{2}) & \textit{if } |x|/\sqrt{t}<z_* \\ |x|^p/t^q & \textit{if } |x|/\sqrt{t}\geq z_*\,. \end{cases}$$

where z_* *is the unique root of the equation*

$$p\,M(q-\tfrac{p}{2}\,,\tfrac{1}{2}\,,\tfrac{z^2}{2})=z^2\,(2q-p)\,M(q+1-\tfrac{p}{2}\,,\tfrac{3}{2}\,,\tfrac{z^2}{2})\,.$$

The optimal stopping time in (2.55) *is given by*

$$\tau_*=\inf\,\big\{\,r>0\,:\,X_r\geq z_*\sqrt{t+r}\,\big\}\,.$$

(*Note that in the case* $q\leq p/2$ *we have* $V_*(t,x)=\infty$ *and it is never optimal to stop.*)

Example 4. In this example we indicate how the problem and the results in Example 1 and Example 3 above can be extended from reflected Brownian motion to *Bessel processes* of arbitrary dimension $\alpha\geq 0$. The avoid the computational complexity which arises, we shall only indicate the essential steps towards solution.

1. *The case* $\alpha>1$. The Bessel process of dimension $\alpha>1$ is a unique (non-negative) strong solution of the stochastic differential equation

$$dX_t=\frac{\alpha-1}{2X_t}\,dt+dB_t \tag{2.56}$$

satisfying $X_0=x$ for some $x\geq 0$. The boundary point 0 is *instantaneously reflecting* if $\alpha<2$, and is an *entrance* boundary point if $\alpha\geq 2$. (When $\alpha\in\mathbb{N}$ the process (X_t) may be realized as the *radial part* of the α-dimensional Brownian motion.)

In the notation of Section 1 consider the process $(Y_t)=(\beta(t)X_t)$ and note that $\mu(x)=(\alpha-1)/2x$ and $\sigma(x)=1$. Thus conditions (1.3) and (1.4) may be realized with $\gamma(t)=\beta(t)$, $G_1(y)=(\alpha-1)/2y$ and $G_2(y)=1$. Noting that $\beta(t)=1/\sqrt{1+t}$ solves $\beta'(t)/\beta(t)=-\beta^2(t)/2$ and setting $\rho=\beta^2/2$, we see from (1.2) that

$$\mathbf{L_Z}=\left(-z+\frac{\alpha-1}{z}\right)\frac{\partial}{\partial z}+\frac{\partial^2}{\partial z^2}$$

where $(Z_t)=(Y_{\sigma_t})$ with $\sigma_t=e^{2t}-1$. Thus (Z_t) solves the equation:

$$dZ_t=\left(-Z_t+\frac{\alpha-1}{Z_t}\right)dt+\sqrt{2}\,dB_t\,.$$

Observe that the diffusion (Z_t) may be seen as *the Euclidean velocity* of the α-dimensional Brownian motion whenever $\alpha\in\mathbb{N}$, and thus may be interpreted as the Euclidean velocity of the Bessel process (X_t) of any dimension $\alpha>1$.

The Bessel process (X_t) of any dimension $\alpha\geq 0$ satisfies the *Brownian scaling* property $Law\big((c^{-1}X_{c^2t})\,|\,\mathbf{P}_{x/c}\big)=Law\big((X_t)\,|\,\mathbf{P}_x\big)$ for all $c>0$ and all x . Thus the initial arguments used in Example 1 and Example 3 can be repeated, and the crucial point in the formulation of the corresponding free boundary problem is the analogue of the equations (2.10) and (2.52)

$$\mathbf{L_Z}W(z)=\rho\,W(z)$$

where $\rho\in\mathbb{R}$. In comparison with the equation (3.1) this reads as follows

$$y''(x)-\Big(x-\tfrac{\alpha-1}{x}\Big)\,y'(x)-\rho\,y(x)=0 \tag{2.57}$$

where $\rho \in \mathbb{R}$. By substituting $y(x) = x^{-(\alpha-1)/2} \exp(x^2/4)\, u(x)$ the equation (2.57) reduces to the following equation

$$u''(x) - \Big(\tfrac{x^2}{4} + \big(\rho - \tfrac{\alpha}{2}\big) + \tfrac{\alpha-1}{2}\big(\tfrac{\alpha-1}{2} - 1\big)\tfrac{1}{x^2} \Big)\, u(x) = 0 \; . \tag{2.58}$$

The unpleasant term in this equation is $1/x^2$, and the general solution is not immediately found in the list of special functions in [1]. Motivated by our considerations below when $0 \le \alpha \le 1$, we may substitute $\bar{y}(x^2) = y(x)$ and observe that the equation (2.57) is equivalent to:

$$4z\,\bar{y}''(z) + 2(\alpha - z)\,\bar{y}'(z) - \rho\,\bar{y}(z) = 0 \tag{2.59}$$

where $z = x^2$. This equation now can be reduced to the *Whittaker's equation* (see [1]) as described in (2.60) and (2.61) below. The general solution of the Whittaker's equation is given by Whittaker's functions which are expressed in terms of Kummer's functions. This establishes a basic fact about the extension of the free boundary problem from the reflected Brownian motion to the Bessel process of the dimension $\alpha > 1$. The problem then can be solved in exactly the same manner as before. It is interesting to observe that if the dimension α of the Bessel process (X_t) equals 3 , then the equation (2.58) is of the form (3.2), and thus the optimal stopping problem is solved immediately by using the corresponding closed form solution given in Example 1 and Example 3 above.

2. *The case* $0 \le \alpha \le 1$. The Bessel process of dimension $0 \le \alpha \le 1$ does not solve a stochastic differential equation in the sense of (2.56), and therefore it is convenient to look at the *squared Bessel process* $(\bar{X}_t)$ which is a unique (non-negative) strong solution of the stochastic differential equation

$$d\bar{X}_t = \alpha\, dt + 2\sqrt{\bar{X}_t}\, dB_t$$

satisfying $\bar{X}_0 = \bar{x}$ for some $\bar{x} \ge 0$. (This is true for all $\alpha \ge 0$.) The Bessel process (X_t) is then defined as the square root of $(\bar{X}_t)$. Thus

$$X_t = \sqrt{\bar{X}_t}\; .$$

The boundary point 0 is *instantaneously reflecting* if $0 < \alpha \le 1$, and is a *trap* if $\alpha = 0$. (The Bessel process (X_t) may be realized as a reflected Brownian motion when $\alpha = 1$.)

In the notation of Section 1 consider the process $(\bar{Y}_t) = (\beta(t)\bar{X}_t)$ and note that $\mu(x) = \alpha$ and $\sigma(x) = 2\sqrt{x}$. Thus conditions (1.3) and (1.4) may be realized with $\gamma(t) = 1$, $G_1(y) = \alpha$ and $G_2(y) = 4y$. Noting that $\beta(t) = 1/(1+t)$ solves $\beta'(t)/\beta(t) = -\beta(t)$ and setting $\rho = \beta/2$, we see from (1.2) that

$$\mathbf{L}_{\bar{\mathbf{Z}}} = 2\big(-z + \alpha\big)\, \frac{\partial}{\partial z} + 4z\, \frac{\partial^2}{\partial z^2}$$

where $(\bar{Z}_t) = (\bar{Y}_{\sigma_t})$ with $\sigma_t = e^{2t} - 1$. Thus $(\bar{Z}_t)$ solves the equation:

$$d\bar{Z}_t = 2\big(-\bar{Z}_t + \alpha\big)\, dt + 2\sqrt{2\bar{Z}_t}\, dB_t\; .$$

It is interesting to observe that

$$\bar{Z}_t = \bar{Y}_{\sigma_t} = \frac{\bar{X}_{\sigma_t}}{1+\sigma_t} = \left(\frac{X_{\sigma_t}}{\sqrt{1+\sigma_t}} \right)^2$$

and thus the process $\big(\sqrt{\bar{Z}_t}\big)$ may be seen as *the Euclidean velocity* of the α-dimensional Brownian motion for $\alpha \in [0,1]$.

This enables us to reformulate the initial problem about (X_t) in terms of $(\bar{X}_t)$ and then after Brownian scaling and time-change $t \mapsto \sigma_t$ in terms of the diffusion $(\bar{Z}_t)$. The pleasant fact is hidden in the formulation of the corresponding free boundary problem for $(\bar{Z}_t)$:

$$\mathbf{L}_{\bar{\mathbf{Z}}} W = \rho W$$

which in comparison with the equation (3.1) reads as follows

$$4x\, y''(x) + 2(\alpha - x)\, y'(x) - \rho\, y(x) = 0 \ . \tag{2.60}$$

Observe that this equation is of the same type as the equation (2.59). By substituting $y(x) = x^{-\alpha/4} \exp(x/4)\, u(x)$ the equation (2.60) reduces to

$$u''(x) + \Big(-\frac{1}{16} + \frac{1}{4}\Big(\rho + \frac{\alpha}{2}\Big)\frac{1}{x} + \frac{\alpha}{4}\Big(1 - \frac{\alpha}{4}\Big)\frac{1}{x^2} \Big)\, u(x) = 0 \tag{2.61}$$

which may be recognized as a *Whittaker's equation* (see [1]). The general solution of the Whittaker's equation is given by Whittaker's functions which are expressed in terms of Kummer's functions. This again establishes a basic fact about the extension of the free boundary problem from the reflected Brownian motion to the Bessel process of the dimension $0 \leq \alpha < 1$. The problem then can be solved in exactly the same manner as before. Note also that the arguments about the passage to the squared Bessel process just presented are valid for all $\alpha \geq 0$. When $\alpha > 1$ it is a matter of taste which way to choose.

Example 5. In this example we show how to solve some *path-dependent* optimal stopping problems (i.e. problems with the gain function depending on the entire path of the underlying process up to the time of observation).

Given an Ornstein-Uhlenbeck process (Z_t) satisfying (2.2), started at z under $\mathbf{P}_z$, consider the optimal stopping problem with the value function

$$\widetilde{W}_*(z) = \sup_\tau \mathbf{E}_z \Big(\int_0^\tau e^{-u} Z_u \, du \Big) \tag{2.62}$$

where the supremum is taken over all stopping time τ for (Z_t) . This problem is motivated by the fact that the integral appearing above may be viewed as a measure of the accumulated gain (up to the time of observation) which is assumed proportional to the velocity of the Brownian particle being discounted. We will first verify by Itô formula that this problem is in fact equivalent to the one-dimensional problem (2.31). Then by using the time-change σ_t we shall show that these problems are also equivalent to yet another path-dependent optimal stopping problem which is given in (2.64) below.

1. Applying Itô formula to the process $(e^{-t} Z_t)$, we find by using (2.2) that

$$e^{-t} Z_t = z + M_t - 2 \int_0^t e^{-u} Z_u \, du$$

where (M_t) is a continuous local martingale given by

$$M_t = \sqrt{2} \int_0^t e^{-u} \, dB_u \ .$$

If τ is a bounded stopping time for (Z_t) , then by the optional sampling theorem we get

$$\mathbf{E}_z \Big(\int_0^\tau e^{-u} Z_u \, du \Big) = \frac{1}{2} \Big(z + \mathbf{E}_z \Big(e^{-\tau} (-Z_\tau) \Big) \Big) \ .$$

Taking supremum over all bounded stopping times τ for (Z_t), and using that $(-Z_t)$ is an Ornstein-Uhlenbeck process starting from $-z$ under $\mathbf{P}_z$, we obtain

$$\widetilde{W}_*(z) = \frac{1}{2}\Big(z + W_*(-z)\Big) \tag{2.63}$$

where W_* is the value function from (2.31). The explicit expression for W_* is given in (2.36), and inserting it in (2.63), we immediately obtain the following result.

Corollary 2.12. *The value function of the optimal stopping problem* (2.62) *is given by*

$$\widetilde{W}_*(z) = \begin{cases} \frac{1}{2}\Big(z+(1-z_*^2)\, e^{z^2/2} \displaystyle\int_z^\infty e^{-u^2/2}\, du\Big) & \text{if } z > -z_* \\ 0 & \text{if } z \le -z_* \end{cases}$$

where $z_* > 0$ *is the unique root of* (2.39). *The optimal stopping time in* (2.62) *is given by*

$$\tau_* = \inf\ \big\{\, t > 0 \,:\, Z_t \le -z_* \,\big\}\ .$$

2. Given the Brownian motion $X_t = B_t + x$ started at x under $\mathbf{P}_x$, consider the optimal stopping problem with the value function

$$\widetilde{V}_*(t,x) = \sup_\tau \mathbf{E}_x\bigg(\int_0^\tau \frac{X_u}{(t+u)^2}\, du\bigg) \tag{2.64}$$

where the supremum is taken over all stopping times τ for (X_t). It is easily verified by Brownian scaling that we have

$$\widetilde{V}_*(t,x) = \tfrac{1}{\sqrt{t}}\, \widetilde{V}_*(1, x/\sqrt{t}\,)\ . \tag{2.65}$$

Moreover, by time-change (1.5) we get

$$\begin{aligned}\int_0^{\sigma_\tau} X_u/(1+u)^2\, du &= \int_0^\tau X_{\sigma_u}/(1+\sigma_u)^2\, d\sigma_u \\ &= 2\int_0^\tau e^{2u}(1+\sigma_u)^{-3/2} Z_u\, du = 2\int_0^\tau e^{-u} Z_u\, du\end{aligned}$$

and the problem to determine $\widetilde{V}_*(1,x)$ therefore reduces to compute

$$\widetilde{V}_*(1,x) = \widetilde{W}_*(x) \tag{2.66}$$

where $\widetilde{W}_*$ is given by (2.62). From (2.65) and (2.66) we thus obtain the following result as an immediate consequence of Corollary 2.12.

Corollary 2.13. *The value function of the optimal stopping problem* (2.64) *is given by*

$$\widetilde{V}_*(t,x) = \begin{cases} \frac{x}{t} + \frac{1}{\sqrt{t}}(1-z_*^2)\, e^{x^2/2t} \displaystyle\int_{x/\sqrt{t}}^\infty e^{-u^2/2}\, du & \text{if } x/\sqrt{t} > -z_* \\ 0 & \text{if } x/\sqrt{t} \le -z_* \end{cases}$$

where $z_* > 0$ *is the unique root of* (2.39). *The optimal stopping time in* (2.64) *is given by*

$$\tau_* = \inf\ \big\{\, r > 0 \,:\, X_r \le -z_*\sqrt{t+r}\,\big\}\ .$$

3. The optimal stopping problem (2.62) can be naturally extended by considering the optimal stopping problem with the value function

$$\widetilde{W}_*(z) = \sup_\tau \mathbf{E}_z\Big(e^{-p\tau} He_n(Z_u)\, du\Big) \tag{2.67}$$

where the supremum is taken over all stopping times τ for (Z_t) and $x \mapsto He_n(x)$ is the Hermite polynomial given by (3.10), with $p > 0$ given and fixed. The crucial fact is that $x \mapsto He_n(x)$ solves the differential equation (3.1), and by Itô formula and (2.2) this implies

$$\begin{aligned} e^{-pt} He_n(Z_t) &= He_n(z) + M_t + \int_0^t e^{-pu}\Big(\mathbf{L_Z}\big(He_n\big)(Z_u) - pHe_n(Z_u)\Big)\, du \\ &= He_n(z) + M_t - (n{+}p)\int_0^t e^{-pu} He_n(Z_u)\, du \end{aligned}$$

where (M_t) is a continuous local martingale given by

$$M_t = \sqrt{2}\int_0^t e^{-pu} (He_n)'(Z_u)\, du\ .$$

Again as above we find that

$$\widetilde{W}_*(z) = \tfrac{1}{n+p}\big(He_n(z) + W_*(z)\big)$$

with W_* being the value function of the optimal stopping problem

$$W_*(z) = \sup_\tau \mathbf{E}_z\Big(e^{-p\tau}\big(-He_n(Z_u)\big)\Big)$$

where the supremum is taken over all stopping times τ for (Z_t) . This problem is one-dimensional and can be solved by the method used in Example 1.

4. Observe that the problem (2.67) with the arguments just presented can be extended from the Hermite polynomial to any solution of the differential equation (3.1).

3. Appendix: Auxiliary results

In the examples above we need the general solution of the second-order differential equation

$$y''(x) - x\, y'(x) - \rho\, y(x) = 0 \tag{3.1}$$

where $\rho \in \mathbb{R}$. By substituting $y(x) = \exp(x^2/4)\, u(x)$ the equation (3.1) reduces to

$$u''(x) - \Big(\tfrac{x^2}{4} + \big(\rho - \tfrac{1}{2}\big)\Big)\, u(x) = 0\ . \tag{3.2}$$

The general solution of (3.2) is well-known, and in the text above we make use of the following two pairs of linearly independent solutions (see [1]).

1. The *Kummer confluent hypergeometric* function is defined by

$$M(a,b,x) = 1 + \tfrac{a}{b}\, x + \tfrac{a(a+1)}{b(b+1)}\, \tfrac{x^2}{2!} + \ \ldots$$

Two linearly independent solutions of (3.2) can be expressed as

$$u_1(x) = e^{-x^2/4}\, M(\tfrac{\rho}{2}\,,\, \tfrac{1}{2}\,,\, \tfrac{x^2}{2}) \quad \text{and} \quad u_2(x) = x\, e^{-x^2/4}\, M(\tfrac{\rho}{2} + \tfrac{1}{2}\,, \tfrac{3}{2}\,, \tfrac{x^2}{2})$$

and therefore two linearly independent solutions of (3.1) are given by

$$y_1(x) = M(\tfrac{\rho}{2}\,,\tfrac{1}{2}\,,\tfrac{x^2}{2}) \tag{3.3}$$

$$y_2(x) = x\,M(\tfrac{\rho}{2}+\tfrac{1}{2}\,,\tfrac{3}{2}\,,\tfrac{x^2}{2})\ . \tag{3.4}$$

Observe that y_1 is *even* and y_2 is *odd*. Note also that

$$M'(a,b,x) = \tfrac{a}{b}\,M(a+1,b+1,x)\ . \tag{3.5}$$

2. The *parabolic cylinder* function is defined by

$$D_\nu(x) = A_1\,e^{-x^2/4}\,M(-\tfrac{\nu}{2}\,,\tfrac{1}{2}\,,\tfrac{x^2}{2}) + A_2\,x\,e^{-x^2/4}\,M(-\tfrac{\nu}{2}+\tfrac{1}{2}\,,\tfrac{3}{2}\,,\tfrac{x^2}{2})$$

where $A_1 = 2^{\nu/2}\,\pi^{-1/2}\cos(\nu\pi/2)\,\Gamma((1+\nu)/2)$ and $A_2 = 2^{(1+\nu)/2}\,\pi^{-1/2}\sin(\nu\pi/2)\,\Gamma(1+\nu/2)$. Two linearly independent solutions of (3.2) can be expressed as

$$\widetilde{u}_1(x) = D_{-\rho}(x) \quad \text{and} \quad \widetilde{u}_2(x) = D_{-\rho}(-x)$$

and therefore two linearly independent solutions of (3.1) are given by

$$\widetilde{y}_1(x) = e^{x^2/4}\,D_{-\rho}(x) \tag{3.6}$$

$$\widetilde{y}_2(x) = e^{x^2/4}\,D_{-\rho}(-x) \tag{3.7}$$

whenever $-\rho \notin \mathbb{N}\cup\{0\}$. Note that $\widetilde{y}_1$ and $\widetilde{y}_2$ are not symmetric around zero unless $-\rho \in \mathbb{N}\cup\{0\}$. Note also that

$$\frac{d}{dx}\big(\,e^{x^2/4}D_\nu(x)\,\big) = \nu\,e^{x^2/4}D_{\nu-1}(x)\ . \tag{3.8}$$

Moreover, the following integral representation is valid

$$D_\nu(x) = \frac{e^{-x^2/4}}{\Gamma(-\nu)}\int_0^\infty u^{-\nu-1}\,e^{-xu-u^2/2}\,du \tag{3.9}$$

whenever $\nu < 0$.

3. To identify zero points of the solutions above, it is useful to note that

$$M(-n\,,\tfrac{1}{2}\,,\tfrac{x^2}{2}) = He_{2n}(x)/He_{2n}(0)$$

$$e^{x^2/4}D_n(x) = He_n(x)$$

where $x \mapsto He_n(x)$ is the *Hermite polynomial*

$$He_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}\Big(e^{-x^2/2}\Big) \tag{3.10}$$

for $n \geq 0$. For more information on the facts presented in this section we refer to [1].

References

[1] Abramowitz, M. *and* Stegun, I. A. (1964). *Handbook of Mathematical Functions.* National Bureau of Standards.

[2] Davis, B. (1976). On the L^p norms of stochastic integrals and other martingales. *Duke Math. J.* 43 (697-704).

[3] Novikov, A. A. (1971). On stopping times for a Wiener process. *Theory Probab. Appl.* 16 (449-456).

[4] Rogers, L. C. G. *and* Williams, D. (1987). *Diffusions, Markov Processes, and Martingales; Volume 2: Itô's Calculus.* John Wiley & Sons.

[5] Shepp, L. A. (1967). A first passage time for the Wiener process. *Ann. Math. Statist.* 38 (1912-1914).
[6] Shepp, L. A. (1969). Explicit solutions to some problems of optimal stopping. *Ann. Math. Statist.* 40 (993-1010).
[7] Shiryaev, A. N. (1978). *Optimal stopping rules.* Springer-Verlag, Berlin-New York.
[8] Taylor, H. M. (1968). Optimal stopping in a Markov process. *Ann. Math. Statist.* 39 (1333-1344).
[9] Van Moerbeke, P. (1974). Optimal stopping and free boundary problems. *Rocky. Mountain. J. Math.* 4 (539-578).
[10] Walker, L. H. (1974). Optimal stopping variables for Brownian motion. *Ann. Probab.* 2 (317-320).

Jesper Lund Pedersen
Department of Mathematical Sciences
University of Aarhus, Denmark
Ny Munkegade, DK-8000 Aarhus
jesperl@imf.au.dk

Goran Peskir
Department of Mathematical Sciences
University of Aarhus, Denmark
Ny Munkegade, DK-8000 Aarhus
home.imf.au.dk/goran
goran@imf.au.dk