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The British Call Option

G. Peskir & F. Samee

Alongside the British put option [11] we present a new call option where the holder enjoys the early exercise feature of American options whereupon his payoff (deliverable immediately) is the 'best prediction' of the European payoff under the hypothesis that the true drift of the stock price equals a contract drift. Inherent in this is a protection feature which is key to the British call option. Should the option holder believe the true drift of the stock price to be unfavourable (based upon the observed price movements) he can substitute the true drift with the contract drift and minimise his losses. The practical implications of this protection feature are most remarkable as not only can the option holder exercise at or below the strike price to a substantial reimbursement of the original option price (covering the ability to sell in a liquid option market completely endogenously) but also when the stock price movements are favourable he will generally receive high returns. We derive a closed form expression for the arbitrage-free price in terms of the rational exercise boundary and show that the rational exercise boundary itself can be characterised as the unique solution to a nonlinear integral equation. In addition we derive the 'British put-call symmetry' relations which express the arbitrage-free price and the rational exercise boundary of the British call option in terms of the arbitrage-free price and the rational exercise boundary of the British put option where the roles of the contract drift and the interest rate have been swapped. These relations provide a useful insight into the British payoff mechanism that is of both theoretical and practical interest. Using these results we perform a financial analysis of the British call option that leads to the conclusions above and shows that with the contract drift properly selected the British call option becomes a very attractive alternative to the classic European/American call.

1. Introduction

The purpose of the present paper is to introduce a new call option which endogenously provides its holder with a protection mechanism against unfavourable stock price movements. This mechanism is intrinsically built into the option contract using the concept of optimal prediction (see e.g. [4] and the references therein) and we refer to such contracts as 'British' for the reasons outlined in [11] where the British put option was introduced. Similarly to the British put option most remarkable about the British call option is not only that it provides a unique protection against unfavourable stock price movements (endogenously covering the ability of an European call holder to sell his contract in a liquid option market), but also when the stock

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price movements are favourable it enables its holder to obtain high returns. This reaffirms the fact noted in [11] that the British feature of optimal prediction acts as a powerful tool for generating financial instruments which aim at both providing protection against unfavourable price movements as well as securing high returns when these movements are favourable (in effect by enabling the seller/hedger to 'milk' more money out of the stock). We recall that in view of the recent turbulent events in the financial industry and equity markets in particular, these combined features appear to be especially appealing as they address problems of liquidity and return completely endogenously (reducing the need for exogenous regulation).

The paper is organised as follows. In Section 2 we present a basic motivation for the British call option. It should be emphasised that the full financial scope of the option goes beyond these initial considerations (especially regarding the provision of high returns that appears as an additional benefit). In Section 3 we formally define the British call option and present some of its basic properties. This is continued in Section 4 where we derive a closed form expression for the arbitrage-free price in terms of the rational exercise boundary (the early-exercise premium representation) and show that the rational exercise boundary itself can be characterised as the unique solution to a nonlinear integral equation (Theorem 1). Many of these arguments and results stand in parallel to those of the British put option [11] and we follow these leads closely in order to make the present exposition more transparent and self-contained (indicating the parallels explicitly where it is insightful). In Section 5 we derive the 'British put-call symmetry' relations which express the arbitrage-free price and the rational exercise boundary of the British call option in terms of the arbitrage-free price and the rational exercise boundary of the British put option where the roles of the contract drift and the interest rate have been swapped (Theorem 2). These relations provide a useful insight into the British payoff mechanism (at the level of put and call options) that is of both theoretical and practical interest. We note that similar symmetry relations are known to be valid for the American put and call options written on stocks paying dividends (see [2]) where the roles of the dividend yield and the interest rate have been swapped instead. Using these results in Section 6 we present a financial analysis of the British call option (making comparisons with the European/American call option). This analysis provides more detail/insight into the full scope of the conclusions briefly outlined above.

2. Basic motivation for the British call option

We begin our exposition by explaining a basic economic motivation for the British call option. We remark that the full financial scope of the option goes beyond these initial considerations (see Section 6 below for a fuller discussion).

1. Consider the financial market consisting of a risky stock X and a riskless bond B whose prices respectively evolve as

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t \quad (X_0 = x) \tag{2.1}$$

$$dB_t = r B_t \, dt \quad (B_0 = 1) \tag{2.2}$$

where $\mu \in I\!R$ is the appreciation rate (drift), $\sigma > 0$ is the volatility coefficient, $W = (W_t)_{t \geq 0}$ is a standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, and $r > 0$ is the interest rate. Recall that a European call option (cf. [1, 8, 13]) is a financial contract between a seller/hedger and a buyer/holder entitling the latter to sell the underlying stock at a specified strike price $K > 0$ at a specified maturity time $T > 0$. Standard hedging arguments based on self-financing portfolios imply that the arbitrage-free price of the option is given by

$$V = \tilde{\mathsf{E}}\, e^{-rT}(X_T - K)^+ \tag{2.3}$$

where the expectation $\tilde{\mathsf{E}}$ is taken with respect to the (unique) equivalent martingale measure $\tilde{\mathsf{P}}$ (see e.g. [7] for a modern exposition). After receiving the amount V from the buyer, the seller can perfectly hedge his position at time T through trading in the underlying stock and bond, and this enables him to meet his obligation without any risk. On the other hand, since the holder can also trade in the underlying stock and bond, he can perfectly hedge his position in the opposite direction and completely eliminate any risk too (upon exercising at T). Thus, the rational performance is risk free, at least from this theoretical standpoint.

2. There are many reasons, however, why this theoretical risk-free standpoint does not quite translate into the real world markets. Without addressing any of these issues more explicitly, in this section we will analyse the rational performance from the standpoint of a true buyer. By 'true buyer' we mean a buyer who has no ability or desire to sell the option nor to hedge his own position. Thus, every true buyer will exercise the option at time T in accordance with the rational performance. For more details on the motivation and interest for considering a true buyer in this context see [11].

3. With this in mind we now return to the European call holder and recall that he has the right to sell the stock at the strike price K at the maturity time T. Thus his payoff can be expressed as

$$e^{-rT}(X_T(\mu) - K)^+ \tag{2.4}$$

where $X_T = X_T(\mu)$ represents the stock/market price at time T under the actual probability measure P. Recall also that the unique strong solution to (2.1) is given by

$$X_t = X_t(\mu) = x \exp\big(\sigma W_t + \big(\mu - \tfrac{\sigma^2}{2}\big)\, t\big) \tag{2.5}$$

under P for $t \in [0, T]$ where $\mu \in I\!R$ is the actual drift. Note that $\mu \mapsto X_T(\mu)$ is strictly increasing so that $\mu \mapsto e^{-rT}(X_T(\mu) - K)^+$ is (strictly) increasing on $I\!R$ (when non-zero). Moreover, it is well known that $\mathrm{Law}(X(\mu)\,|\,\tilde{\mathsf{P}})$ is the same as $\mathrm{Law}(X(r)\,|\,\mathsf{P})$. Combining this with (2.3) above we see that if $\mu = r$ then the return is 'fair' for the buyer, in the sense that

$$V = \mathsf{E}\, e^{-rT}(X_T(\mu) - K)^+ \tag{2.6}$$

where the left-hand side represents the value of his investment and the right-hand side represents the expected value of his payoff. On the other hand, if $\mu > r$ then the return is 'favourable' for the buyer, in the sense that

$$V < \mathsf{E}\, e^{-rT}(X_T(\mu) - K)^+ \tag{2.7}$$

and if $\mu < r$ then the return is 'unfavourable' for the buyer, in the sense that

$$V > \mathsf{E}\, e^{-rT}(X_T(\mu) - K)^+ \tag{2.8}$$

with the same interpretations as above. Recall that the actual drift μ is unknown at time $t = 0$ and also difficult to estimate at later times $t \in (0, T]$ unless T is unrealistically large.

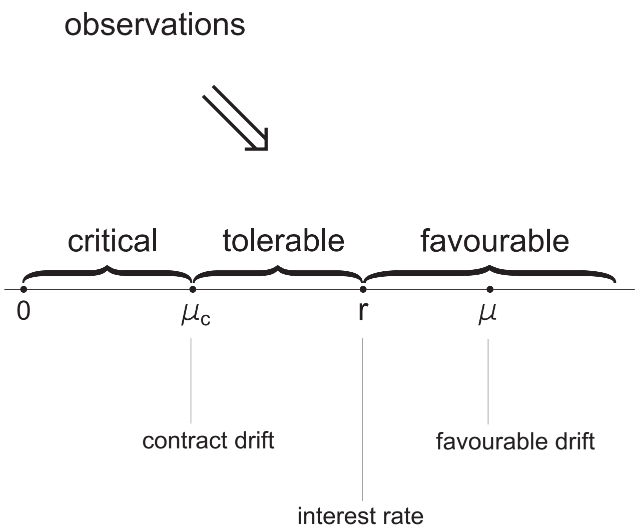

Figure 1. An indication of the economic meaning of the contract drift.

4. The brief analysis above shows that whilst the actual drift μ of the underlying stock price is irrelevant in determining the arbitrage-free price of the option, to a (true) buyer it is crucial, and he will buy the option if he believes that $\mu > r$. If this turns out to be the case then on average he will make a profit. Thus, after purchasing the option, the call holder will be happy if the observed stock price movements reaffirm his belief that $\mu > r$.

The British call option seeks to address the opposite scenario: What if the call holder observes stock price movements which change his belief regarding the actual drift and cause him to believe that $\mu < r$ instead? In this contingency the British call holder is effectively able to substitute this unfavourable drift with a contract drift and minimise his losses. In this way he is endogenously protected from any stock price drift smaller than the contract drift. The value of the contract drift is therefore selected to represent the buyer's expected level of tolerance for the deviation of the actual drift from his original belief (see Figure 1). It will be shown below that the practical implications of this protection feature are most remarkable as not only can the British call holder exercise at or below the strike price to a substantial reimbursement of the original option price (covering the ability to sell in a liquid option market completely endogenously) but also when the stock price movements are favourable he will generally receive high returns (see Section 6 for further details).

5. Releasing now the true-buyer's perspective and considering the real world buyer instead, observe that a call holder who believes (based upon his observations) that $\mu < r$ may choose/attempt to sell his contract. However, in a real financial market the price at which he is able to sell will be determined by the market (as the bid price of the bid-ask spread) and this may also involve additional transaction costs and/or taxes. Moreover, unless the exchange is fully regulated, it may be increasingly difficult to sell the option when out-of-the-money (e.g. in most practical situations of over-the-counter trading such an option will expire worthless). The latter therefore strongly correlates the buyer's risk exposure to the liquidity of the option market. We remark that the liquidity of the option market (alongside possible transaction costs

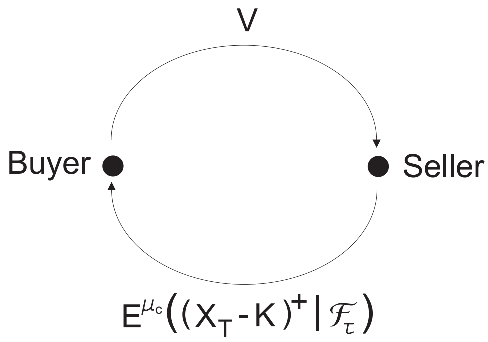

Figure 2. Definition of the British call option in terms of its price and payoff.

and tax considerations) can change during the term of the contract. This for example can be caused by extreme news events either specific (to the underlying stock) or systemic in nature (such as the recent turbulent events in the financial industry). The protection afforded to the British call option holder, on the other hand, is endogenous, i.e. it is always in place regardless of whether the option market is liquid or not.

3. The British call option: Definition and basic properties

We begin this section by presenting a formal definition of the British call option. This is then followed by a brief analysis of the optimal stopping problem and the free-boundary problem characterising the arbitrage-free price and the rational exercise strategy. These considerations are continued in Sections 4 and 5 below.

1. Consider the financial market consisting of a risky stock X and a riskless bond B whose prices evolve as (2.1) and (2.2) respectively, where $\mu \in I\!R$ is the appreciation rate (drift), $\sigma > 0$ is the volatility coefficient, $W = (W_t)_{t \geq 0}$ is a standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, and $r > 0$ is the interest rate. We assume that the stock does not pay dividends and there are no transaction costs associated with its trade. Let a strike price $K > 0$ and a maturity time $T > 0$ be given and fixed.

Definition 1. The *British call option* is a financial contract between a seller/hedger and a buyer/holder entitling the latter to exercise at any (stopping) time τ prior to T whereupon his payoff (deliverable immediately) is the 'best prediction' of the European payoff $(X_T - K)^+$ given all the information up to time τ under the hypothesis that the true drift of the stock price equals μ_c (see Figure 2).

The quantity μ_c is defined in the option contract and we refer to it as the 'contract drift'. Recalling our discussion in Section 2 above it is natural that the contract drift satisfies the right-hand inequality in

$$0 < \mu_c < r \tag{3.1}$$

since otherwise the British call holder could beat the interest rate r by simply exercising immediately (a formal argument confirming this economic reasoning will be given shortly below). We will also show below that the left-hand inequality in (3.1) must be satisfied as well since otherwise it would be optimal to exercise at time T and the British call option would reduce to the European call option. Recall also from Section 2 above that the value of the contract drift is selected to represent the buyer's expected level of tolerance for the deviation of the true drift μ from his original belief (see Figure 1). It will be shown in Section 6 below that this protection feature has remarkable implications both in terms of liquidity and return.

2. Denoting by $(\mathcal{F}_t)_{0\le t\le T}$ the natural filtration generated by X (possibly augmented by null sets or in some other way of interest) the payoff of the British call option at a given stopping time τ can be formally written as

$$\mathsf{E}^{\mu_c}\big((X_T-K)^+ \,|\, \mathcal{F}_\tau\big) \tag{3.2}$$

where the conditional expectation is taken with respect to a new probability measure P^{μ_c} under which the stock price X evolves as

$$dX_t = \mu_c X_t\, dt + \sigma X_t\, dW_t \tag{3.3}$$

with $X_0 = x$ in $(0,\infty)$. Comparing (2.1) and (3.3) we see that the effect of exercising the British call option is to substitute the true (unknown) drift of the stock price with the contract drift for the remaining term of the contract.

3. Stationary and independent increments of W governing X imply that

$$\mathsf{E}^{\mu_c}\big((X_T-K)^+ \,|\, \mathcal{F}_t\big) = G^{\mu_c}(t,X_t) \tag{3.4}$$

where the payoff function G^{μ_c} can be expressed as

$$G^{\mu_c}(t,x) = \mathsf{E}(xZ^{\mu_c}_{T-t}-K)^+ \tag{3.5}$$

and $Z^{\mu_c}_{T-t}$ is given by

$$Z^{\mu_c}_{T-t} = \exp\Big(\sigma W_{T-t} + (\mu_c-\tfrac{\sigma^2}{2})(T-t)\Big) \tag{3.6}$$

for $t\in[0,T]$ and $x\in(0,\infty)$. Hence one finds that (3.5) can be rewritten as follows

$$\begin{aligned} G^{\mu_c}(t,x) = {} & x\,\mathrm{e}^{\mu_c(T-t)}\,\Phi\bigg(\frac{1}{\sigma\sqrt{T-t}}\Big[\log\big(\tfrac{x}{K}\big)+(\mu_c+\tfrac{\sigma^2}{2})(T-t)\Big]\bigg) \\ & - K\,\Phi\bigg(\frac{1}{\sigma\sqrt{T-t}}\Big[\log\big(\tfrac{x}{K}\big)+(\mu_c-\tfrac{\sigma^2}{2})(T-t)\Big]\bigg) \end{aligned} \tag{3.7}$$

for $t\in[0,T)$ and $x\in(0,\infty)$ where Φ is the standard normal distribution function given by $\Phi(x) = (1/\sqrt{2\pi})\int_{-\infty}^x \mathrm{e}^{-y^2/2}\,dy$ for $x\in I\!R$.

It may be noted that the expression for $G^{\mu_c}(t,x)$ multiplied by $\mathrm{e}^{-\mu_c(T-t)}$ coincides with the Black-Scholes formula for the arbitrage-free price of the European call option (written for the remaining term of the contract) where the interest rate equals the contract drift μ_c. Hence the British call option can be formally viewed as an American option on the undiscounted European call option written on a stock paying dividends at rate $\delta = r-\mu_c > 0$. Since the payoff (i.e. the European call value) is undiscounted we see that a direct financial interpretation in terms of compound options is not possible (cf. [11, Subsection 3.3] for further details).

4. Standard hedging arguments based on self-financing portfolios (with consumption) imply that the arbitrage-free price of the British call option is given by

$$V = \sup_{0\le\tau\le T} \widetilde{\mathsf{E}}\big[\mathrm{e}^{-r\tau}\,\mathsf{E}^{\mu_c}\big((X_T - K)^+ \,|\, \mathcal{F}_\tau\big)\big] \tag{3.8}$$

where the supremum is taken over all stopping times τ of X with values in $[0,T]$ and $\tilde{\mathsf{E}}$ is taken with respect to the (unique) equivalent martingale measure $\tilde{\mathsf{P}}$. Making use of (3.4) above and the optional sampling theorem, upon enabling the process X to start at any point x in $(0,\infty)$ at any time $t \in [0,T]$, we see that the problem (3.8) extends as follows

$$V(t,x) = \sup_{0\le\tau\le T-t} \widetilde{\mathsf{E}}_{t,x}\big[\mathrm{e}^{-r\tau} G^{\mu_c}(t+\tau, X_{t+\tau})\big] \tag{3.9}$$

where the supremum is taken over all stopping times τ of X with values in $[0,T-t]$ and $\tilde{\mathsf{E}}_{t,x}$ is taken with respect to the (unique) equivalent martingale measure $\tilde{\mathsf{P}}_{t,x}$ under which $X_t = x$. Since the supremum in (3.9) is attained at the first entry time of X to the closed set where V equals G^{μ_c}, and $\mathrm{Law}(X(\mu)\,|\,\tilde{\mathsf{P}})$ is the same as $\mathrm{Law}(X(r)\,|\,\mathsf{P})$, it follows from the well-known flow structure of the geometric Brownian motion X that

$$V(t,x) = \sup_{0\le\tau\le T-t} \mathsf{E}\big[\mathrm{e}^{-r\tau} G^{\mu_c}(t+\tau, xX_\tau)\big] \tag{3.10}$$

for $t \in [0,T]$ and $x \in (0,\infty)$ where the supremum is taken as in (3.9) above and the process $X = X(r)$ under P solves

$$dX_t = rX_t\,dt + \sigma X_t\,dW_t \tag{3.11}$$

with $X_0 = 1$. As it will be clear from the context which initial point of X is being considered, as well as whether the drift of X equals r or not, we will not reflect these facts directly in the notation of X (by adding a superscript or similar).

5. We see from (3.5) that

$$x \mapsto G^{\mu_c}(t,x) \quad \text{is convex} \tag{3.12}$$

and strictly increasing on $(0,\infty)$ with $G^{\mu_c}(t,0) = 0$ and $G^{\mu_c}(t,\infty) = \infty$ for any $t \in [0,T]$ given and fixed. One also sees that $G^{\mu_c}(T,x) = (x-K)^+$ for $x \in (0,\infty)$ showing that the British call payoff coincides with the European/American call payoff at the time of maturity. Moreover, if $\mu_c \le 0$ then from (3.5) and (3.6) we see that

$$G^{\mu_c}(t,x) < \mathrm{e}^{-r(T-t)}\,\mathsf{E}(xZ^r_{T-t} - K)^+ \tag{3.13}$$

for $t \in [0,T)$ and $x \in (0,\infty)$ where the term on the right-hand side can be recognised as the payoff obtained by choosing $\tau = T-t$ in (3.10). From the strict inequality in (3.13), and the fact that the supremum in (3.13) is attained at the first entry time of X to the (closed) set where V equals G^{μ_c}, we therefore see that it is not optimal to stop in (3.10) before the time of maturity when $\mu_c \le 0$ (i.e. the optimal stopping time τ_* in (3.10) equals $T-t$). This establishes the claim about the left-hand inequality following (3.1) above. Denoting by $V_E(t,x)$ the arbitrage-free price of the European call option (given by the term on the right-hand side of

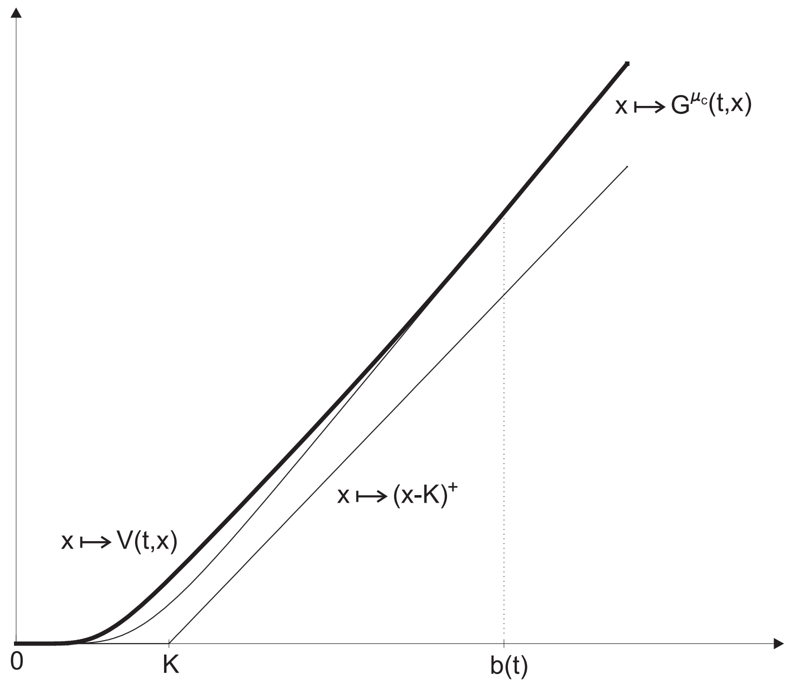

Figure 3. The value/payoff functions of the British/European call options.

(3.13) above) it follows therefore using (3.5) and (3.6) that $V(t,x) = V_E(t,x)$ if $\mu_c \le 0$ and $V(t,x) > V_E(t,x)$ if $\mu_c > 0$ for all $t \in [0,T)$ and $x \in (0,\infty)$. In particular, this confirms that the British call option is more expensive than the European call option whenever (3.1) holds. While this fact is to be expected (due to the presence of the early exercise feature) it may be noted that V and G^{μ_c} tend to stay much closer together than the two functions for the European call option (a financial interpretation of this phenomenon will be addressed in Section 6 below). Finally, from (3.10) and (3.12) we easily find that

$$x \mapsto V(t,x) \quad \text{is convex} \tag{3.14}$$

and (strictly) increasing on $(0,\infty)$ with $V(t,0) = 0$ and $V(t,\infty) = \infty$ for any $t \in [0,T)$ given and fixed, and one likewise sees that $V(T,x) = (x-K)^+$ for $x \in (0,\infty)$. In this sense the value function of the British call option is similar to the value function of the European call option (a snapshot of the former function is shown in Figure 3). The most important technical difference, however, is that whilst the formal American call boundary b_A is trivial (non-existing) this is not the case for the British call boundary b whenever (3.1) holds.

6. To gain a deeper insight into the solution to the optimal stopping problem (3.10), let us note that Itô's formula yields

$$e^{-rs} G^{\mu_c}(t+s, X_{t+s}) = G^{\mu_c}(t,x) + \int_0^s e^{-ru} H^{\mu_c}(t+u, X_{t+u})\, du + M_s \tag{3.15}$$

where the function $H^{\mu_c} = H^{\mu_c}(t,x)$ is given by

$$H^{\mu_c} = G_t^{\mu_c} + r\, x\, G_x^{\mu_c} + \tfrac{\sigma^2}{2} x^2 G_{xx}^{\mu_c} - r G^{\mu_c} \tag{3.16}$$

and $M_s = \sigma \int_0^s e^{-ru} X_u G_x^{\mu_c}(t+u, X_{t+u})\, dW_u$ defines a continuous martingale for $s \in [0, T-t]$ with $t \in [0,T)$. By the optional sampling theorem we therefore find

$$\mathsf{E}\Big[e^{-r\tau} G^{\mu_c}(t+\tau, xX_\tau)\Big] = G^{\mu_c}(t,x) + \mathsf{E}\Big[\int_0^\tau e^{-ru} H^{\mu_c}(t+u, xX_u)\, du\Big] \tag{3.17}$$

for all stopping times τ of X solving (3.11) with values in $[0, T-t]$ with $t \in [0,T)$ and $x \in (0,\infty)$ given and fixed. On the other hand, it is clear from (3.5) that the payoff function G^{μ_c} satisfies the Kolomogorov backward equation (or the undiscounted Black-Scholes equation where the interest rate equals the contract drift)

$$G_t^{\mu_c} + \mu_c x G_x^{\mu_c} + \tfrac{\sigma^2}{2} x^2 G_{xx}^{\mu_c} = 0 \tag{3.18}$$

so that from (3.16) we see that

$$H^{\mu_c} = (r-\mu_c) x G_x^{\mu_c} - r G^{\mu_c}\, . \tag{3.19}$$

This representation shows in particular that if $\mu_c \ge r$ then $H^{\mu_c} < 0$ so that from (3.17) we see that it is always optimal to exercise immediately as pointed out following (3.1) above. Moreover, inserting the expression for G^{μ_c} from (3.7) into (3.19), it is easily verified that

$$\begin{aligned} H^{\mu_c}(t,x) = {} & rK\, \Phi\bigg(\tfrac{1}{\sigma\sqrt{T-t}}\Big[\log\big(\tfrac{x}{K}\big) + (\mu_c - \tfrac{\sigma^2}{2})(T-t)\Big]\bigg) \\ & - \mu_c x\, e^{\mu_c(T-t)}\, \Phi\bigg(\tfrac{1}{\sigma\sqrt{T-t}}\Big[\log\big(\tfrac{x}{K}\big) + (\mu_c + \tfrac{\sigma^2}{2})(T-t)\Big]\bigg) \end{aligned} \tag{3.20}$$

for $t \in [0,T)$ and $x \in (0,\infty)$.

A direct examination of the function H^{μ_c} in (3.20) shows that there exists a continuous (smooth) function $h : [0,T] \to I\!R$ such that

$$H^{\mu_c}(t, h(t)) = 0 \tag{3.21}$$

for all $t \in [0,T)$ with $H^{\mu_c}(t,x) < 0$ for $x > h(t)$ and $H^{\mu_c}(t,x) > 0$ for $x < h(t)$ when $t \in [0,T)$ is given and fixed. In view of (3.17) this implies that no point (t,x) in $[0,T)\times(0,\infty)$ with $x < h(t)$ is a stopping point (for this one can make use of the first exit time from a sufficiently small time-space ball centred at the point). Likewise, it is also clear and can be verified that if $x > h(t)$ and $t < T$ is sufficiently close to T then it is optimal to stop immediately (since the gain obtained from being below h cannot offset the cost of getting there due to the lack of time). This shows that the optimal stopping boundary b separating the continuation set from the stopping set satisfies $b(T) = h(T)$ and this value equals rK/μ_c as is easily seen from (3.20). Note that $rK/\mu_c > K$ since μ_c satisfies (3.1) above.

7. Standard Markovian arguments lead to the following free-boundary problem (for the value function $V = V(t,x)$ and the optimal stopping boundary $b = b(t)$ to be determined):

$$V_t + rxV_x + \tfrac{\sigma^2}{2} x^2 V_{xx} - rV = 0 \quad \text{for } x \in (0, b(t)) \text{ and } t \in [0,T) \tag{3.22}$$

$$V(t,x) = G^{\mu_c}(t,x) \quad \text{for } x = b(t) \text{ and } t \in [0,T] \quad \text{(instantaneous stopping)} \tag{3.23}$$

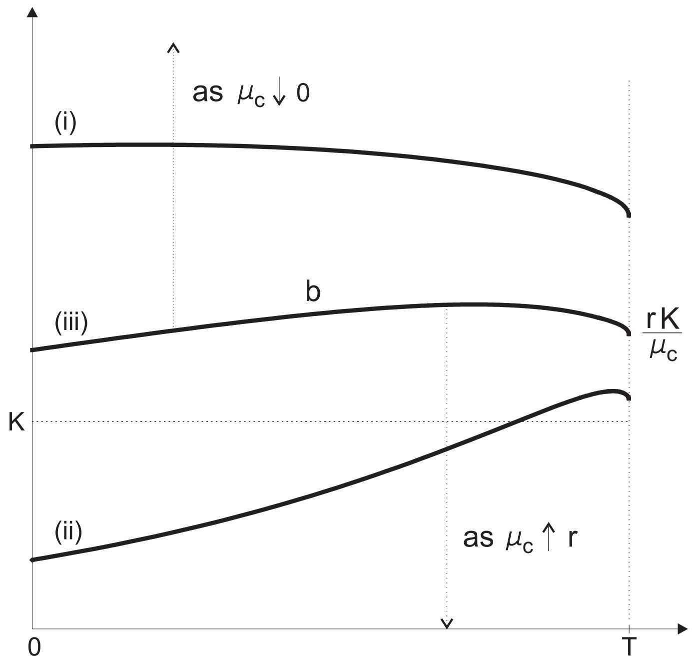

Figure 4. A computer drawing showing how the rational exercise boundary of the British call option changes as one varies the contract drift.

(3.24) $$V_x(t,x) = G_x^{\mu_c}(t,x) \quad \text{for} \;\; x = b(t) \;\; \text{and} \;\; t \in [0,T] \quad \text{(smooth fit)}$$

(3.25) $$V(T,x) = (x-K)^+ \quad \text{for} \;\; x \in (0, b(T)) \;\; \text{with} \;\; b(T) = \tfrac{rK}{\mu_c}$$

(3.26) $$V(t,0) = 0 \quad \text{for} \;\; t \in [0,T]$$

where we also set $V(t,x) = G^{\mu_c}(t,x)$ for $x > b(t)$ and $t \in [0,T]$ (see e.g. [12]). It can be shown that this free-boundary problem has a unique solution V and b which coincide with the value function (3.10) and the optimal stopping boundary respectively. This means that the continuation set is given by $C = \{ V > G^{\mu_c} \} = \{ (t,x) \in [0,T) \times (0,\infty) \mid x < b(t) \}$ and the stopping set is given by $D = \{ V = G^{\mu_c} \} = \{ (t,x) \in [0,T] \times (0,\infty) \mid x \geq b(t) \} \cup \{ (T,x) \mid x \in (0, b(T)) \}$ so that the optimal stopping time in (3.10) is given by

(3.27) $$\tau_b = \inf \{ t \in [0,T] \mid X_t \geq b(t) \} .$$

This stopping time represents the rational exercise strategy for the British call option and plays a key role in financial analysis of the option.

Depending on the size of the contract drift μ_c satisfying (3.1) we distinguish three different regimes for the position and shape of the optimal stopping boundary b (see Figure 4). Firstly, when $\mu_c \in (0,r)$ is close to 0 then b is a decreasing function of time. Secondly, if $\mu_c \in (0,r)$ is close to r then b is a skewed S-shaped function of time. Thirdly, there is an intermediate case where b can take either of the two shapes depending on the size of T. Moreover, the second regime becomes dominant when T is large in the sense that $b(0)$ tends to 0 as $T \to \infty$. These three regimes are not disconnected and if we let μ_c run from 0 to r then the optimal stopping boundary b moves from the ∞ function to the 0 function on $[0,T)$ gradually passing through the three shapes above and always satisfying $b(T) = rK/\mu_c$ (exhibiting also a singular behaviour at T in the sense that $b'(T-) = -\infty$). We will see in Section 6 below that the three regimes have three different economic interpretations and their fuller understanding is important for a correct/desired choice of the contract drift μ_c in relation to the interest rate r and other parameters in the British call option. Note that this structure differs from the three-regime structure in the British put option [11] where the optimal stopping boundary b_p can be skewed U-shaped so that $b_p(0)$ tends to ∞ as $T \to \infty$.

Fuller details of the analysis above go beyond our aims in this paper and for this reason will be omitted. It should be noted however that one of the key elements which makes this analysis more complicated (in comparison with the American put option) is that b is not necessarily a monotone function of time. In the next section we will derive simpler equations which characterise V and b uniquely and can be used for their calculation (Section 6).

8. We conclude this section with a few remarks on the choice of the volatility parameter in the British payoff mechanism. Unlike its classical European or American counterpart, it is seen from (3.7) that the volatility parameter σ appears explicitly in the British payoff (3.2) and hence should be prescribed explicitly in the contract specification. Note that this feature is also present in the British put option (cf. [11]) and its appearance can be seen as a direct consequence of 'optimal prediction'. Considering this from a practical perspective, a natural question (common to all options with volatility-dependent payoffs) is what value of the parameter σ should be used in the contract specification/payoff? Both counterparties to the option trade must agree on the value of this parameter, or at least agree on how it should be calculated from future market observables, at the initiation of the contract. However, whilst this question has many practical implications, it does not pose any conceptual difficulties under the current modelling framework. Since the underlying process is assumed to be a geometric Brownian motion, the (constant) volatility of the stock price is effectively known by all market participants, since one may take any of the standard estimators for the volatility over an arbitrarily small time period prior to the initiation of the contract. This is in direct contrast to the situation with stock price drift μ , whose value/form is inherently unknown, nor can be reasonably estimated (without an impractical amount of data), at the initiation of the contract. In this sense, it seems natural to provide a true buyer with protection from an ambiguous (Knightian) drift rather than a 'known' volatility, at least in this canonical Black-Scholes setting. We remark however that as soon as one departs from the current modelling framework, and gets closer to a more practical perspective, it is clear that the specification of the contract volatility may indeed become very important. In this wider framework one is naturally led to consider the relationship between the realised/implied volatility and the 'contract volatility' using the same/similar rationale as for the actual drift and the 'contract drift' in the text above. Due to the fundamental difference between the drift and the volatility in the canonical (Black-Scholes) setting, and the highly applied nature of such modelling issues, these features of the British payoff mechanism are left as the subject of future development.

4. The arbitrage-free price and the rational exercise boundary

In this section we derive a closed form expression for the arbitrage-free price V in terms of the rational exercise boundary b (the early-exercise premium representation) and show that the rational exercise boundary b itself can be characterised as the unique solution to a nonlinear integral equation (Theorem 1). We note that the former approach was originally applied in the pricing of American options in [6, 5, 3] and the latter characterisation was established in [10] (for more details see e.g. [12]).

We will make use of the following functions in Theorem 1 below:

$$F(t,x) = G^{\mu_c}(t,x) - \mathrm{e}^{-r(T-t)} G^r(t,x) \tag{4.1}$$

$$J(t,x,v,z) = -\,\mathrm{e}^{-r(v-t)} \int_z^\infty H^{\mu_c}(v,y)\, f(v-t,x,y)\, dy \tag{4.2}$$

for $t \in [0,T)$, $x>0$, $v \in (t,T)$ and $z>0$, where the functions G^r and G^{μ_c} are given in (3.5) and (3.7) above (upon identifying μ_c with r in the former case), the function H^{μ_c} is given in (3.16) and (3.20) above, and $y \mapsto f(v-t,x,y)$ is the probability density function of xZ^r_{v-t} from (3.6) above (with μ_c replaced by r and $T-t$ replaced by $v-t$) given by

$$f(v-t,x,y) = \tfrac{1}{\sigma y \sqrt{v-t}}\, \varphi\bigg(\tfrac{1}{\sigma\sqrt{v-t}} \Big[\log\big(\tfrac{y}{x}\big) - (r - \tfrac{\sigma^2}{2})(v-t) \Big] \bigg) \tag{4.3}$$

for $y>0$ (with $v-t$ and x as above) where φ is the standard normal density function given by $\varphi(x) = (1/\sqrt{2\pi})\, \mathrm{e}^{-x^2/2}$ for $x \in I\!\!R$. It should be noted that $J(t,x,v,b(v)) > 0$ for all $t \in [0,T)$, $x>0$ and $v \in (t,T)$ since $H^{\mu_c}(v,y) < 0$ for all $y > b(v)$ as b lies above h (recall (3.21) above). Finally, it can be verified using standard means that

$$\begin{aligned} J(t,x,T-,z) = {} & \mu_c\, x\, \Phi\bigg(\tfrac{1}{\sigma\sqrt{T-t}} \Big[\log\big(\tfrac{x}{z\vee K}\big) + (r + \tfrac{\sigma^2}{2})(T-t) \Big] \bigg) \\ & - rK\, \mathrm{e}^{-r(T-t)}\, \Phi\bigg(\tfrac{1}{\sigma\sqrt{T-t}} \Big[\log\big(\tfrac{x}{z\vee K}\big) + (r - \tfrac{\sigma^2}{2})(T-t) \Big] \bigg) \end{aligned} \tag{4.4}$$

for $t \in [0,T)$, $x>0$ and $z>0$. This expression is useful in a computational treatment of the equation (4.6) below. The main result of this section may now be stated as follows.

Theorem 1. *The arbitrage-free price of the British call option admits the following early-exercise premium representation*

$$V(t,x) = e^{-r(T-t)}\, G^r(t,x) + \int_t^T J(t,x,v,b(v))\, dv \tag{4.5}$$

for all $(t,x) \in [0,T] \times (0,\infty)$ *, where the first term is the arbitrage-free price of the European call option and the second term is the early-exercise premium.*

The rational exercise boundary of the British call option can be characterised as the unique continuous solution $b : [0,T] \rightarrow I\!\!R_+$ *to the nonlinear integral equation*

$$F(t,b(t)) = \int_t^T J(t,b(t),v,b(v))\, dv \tag{4.6}$$

satisfying $b(t) \ge h(t)$ *for all* $t \in [0,T]$ *where* h *is defined by* (3.21) *above.*

Proof. The proof given below is similar to the proof of Theorem 1 in [11] and we present all details for completeness (as well as to demonstrate a full symmetry between the arguments).

Alternatively, one could also exploit the British put-call symmetry relations derived in Section 5 below and deduce either of the two theorems from the existence of the other. In the present proof we first derive (4.5) and show that the rational exercise boundary solves (4.6). We then show that (4.6) cannot have other (continuous) solutions.

1. Let $V : [0,T]\times(0,\infty) \rightarrow I\!R$ and $b : [0,T] \rightarrow I\!R_+$ denote the unique solution to the free-boundary problem (3.22)-(3.26) (where V extends as G^{μ_c} above b), set $C_b = \{\, (t,x) \in [0,T)\times(0,\infty) \mid x < b(t)\,\}$ and $D_b = \{\, (t,x) \in [0,T)\times(0,\infty) \mid x > b(t)\,\}$, and let $I\!L_X V(t,x) = r\,x\,V_x(t,x) + \frac{\sigma^2}{2}\,x^2\,V_{xx}(t,x)$ for $(t,x) \in C_b \cup D_b$. Then V and b are continuous functions satisfying the following conditions: (i) V is $C^{1,2}$ on $C_b \cup D_b$; (ii) b is of bounded variation; (iii) $\mathsf{P}(X_t = c) = 0$ for all $t \in [0,T]$ and $c > 0$; (iv) $V_t + I\!L_X V - rV$ is locally bounded on $C_b \cup D_b$ (recall that V solves (3.22) and coincides with G^{μ_c} on D_b); (v) $x \mapsto V(t,x)$ is convex on $(0,\infty)$ for every $t \in [0,T]$ (recall (3.14) above); and (vi) $t \mapsto V_x(t,b(t)\pm) = G^{\mu_c}_x(t,b(t))$ is continuous on $[0,T]$ (recall that V satisfies the smooth-fit condition (3.24) at b). From these conditions we see that the local time-space formula [9] is applicable to $(s,y) \mapsto \mathrm{e}^{-rs} V(t{+}s,xy)$ with $t \in [0,T)$ and $x > 0$ given and fixed. This yields

$$
\begin{aligned}
(4.7) \qquad \mathrm{e}^{-rs} V(t{+}s, xX_s) &= V(t,x) \\
&+ \int_0^s \mathrm{e}^{-rv} \big(V_t + I\!L_X V - rV\big)(t{+}v, xX_v)\, I(xX_v \neq b(t{+}v))\, dv + M^b_s \\
&+ \frac{1}{2} \int_0^s \mathrm{e}^{-rv} \big(V_x(t{+}v, xX_v{+}) - V_x(t{+}v, xX_v{-})\big)\, I(xX_v = b(t{+}v))\, d\ell^b_v(X^x)
\end{aligned}
$$

where $M^b_s = \sigma \int_0^s \mathrm{e}^{-rv}\, xX_v\, V_x(t{+}v, xX_v)\, I(xX_v \neq b(t{+}v))\, dB_v$ is a continuous local martingale for $s \in [0,T{-}t]$ and $\ell^b(X^x) = (\ell^b_v(X^x))_{0 \le v \le s}$ is the local time of $X^x = (xX_v)_{0 \le v \le s}$ on the curve b for $s \in [0,T{-}t]$. Moreover, since V satisfies (3.22) on C_b and equals G^{μ_c} on D_b , and the smooth-fit condition (3.24) holds at b , we see that (4.7) simplifies to

$$
(4.8) \qquad \mathrm{e}^{-rs} V(t{+}s, xX_s) = V(t,x) + \int_0^s \mathrm{e}^{-rv} H^{\mu_c}(t{+}v, xX_v)\, I(xX_v > b(t{+}v))\, dv + M^b_s
$$

for $s \in [0,T{-}t]$ and $(t,x) \in [0,T)\times(0,\infty)$.

2. We show that $M^b = (M^b_s)_{0 \le s \le T-t}$ is a martingale for $t \in [0,T)$. For this, note that from (3.7), (3.19) and (3.20) (or calculating directly) we find that

$$
(4.9) \qquad G^{\mu_c}_x(t,x) = \mathrm{e}^{\mu_c (T-t)}\, \Phi\bigg(\tfrac{1}{\sigma\sqrt{T-t}} \Big[\log\big(\tfrac{x}{K}\big) + (\mu_c + \tfrac{\sigma^2}{2})(T{-}t) \Big] \bigg)
$$

for $t \in [0,T)$ and $x > 0$. Moreover, by (3.14) and (3.24) we see that

$$
(4.10) \qquad 0 \le V_x(t,x) \le G^{\mu_c}_x(t,b(t))
$$

for all $t \in [0,T)$ and all $x \in (0,b(t))$. Combining (4.9) and (4.10) we conclude that

$$
(4.11) \qquad 0 \le V_x(t,x) \le \mathrm{e}^{\mu_c (T-t)}
$$

for all $t \in [0,T)$ and all $x > 0$ (recall that V equals G^{μ_c} above b). Hence we find that

$$
(4.12) \qquad E\langle M^b, M^b \rangle_{T-t} = \sigma^2 x^2\, \mathsf{E}\bigg(\int_0^{T-t} \mathrm{e}^{-2rv} X^2_v\, V^2_x(t{+}v, xX_v)\, I(xX_v \neq b(t{+}v))\, dv \bigg)
$$

$$\le \sigma^2 x^2 \, e^{\mu_c(T-t)} \int_0^{T-t} \mathsf{E} X_v^2 \, dv = \sigma^2 x^2 \, e^{\mu_c(T-t)} \int_0^{T-t} e^{rv} \, dv$$
$$= \tfrac{\sigma^2 x^2}{r} \, e^{\mu_c(T-t)} \big(e^{r(T-t)} - 1 \big) < \infty$$

from where it follows that M^b is a martingale as claimed.

3. Replacing s by $T-t$ in (4.8), using that $V(T,x) = G^{\mu_c}(T,x) = (x-K)^+$ for $x > 0$, taking E on both sides and applying the optional sampling theorem, we get

$$\text{(4.13)} \qquad e^{-r(T-t)} \mathsf{E}(xX_{T-t} - K)^+ = V(t,x) + \int_0^{T-t} e^{-rv} \mathsf{E}\big[H^{\mu_c}(t+v, xX_v) \, I(xX_v > b(t+v)) \big] \, dv$$
$$= V(t,x) - \int_t^T J(t,x,v,b(v)) \, dv$$

for all $(t,x) \in [0,T) \times (0,\infty)$. Recognising the left-hand side of (4.13) as $e^{-r(T-t)} G^r(t,x)$ we see that this yields the representation (4.5). Moreover, since $V(t,b(t)) = G^{\mu_c}(t,b(t))$ for all $t \in [0,T]$ we see from (4.5) that b solves (4.6). This establishes the existence of the solution to (4.6). We now turn to its uniqueness.

4. We show that the rational exercise boundary is the unique solution to (4.6) in the class of continuous functions $t \mapsto b(t)$ on $[0,T]$ satisfying $b(t) \ge h(t)$ for all $t \in [0,T]$. For this, take any continuous function $c : [0,T] \to I\!R$ which solves (4.6) and satisfies $c(t) \ge h(t)$ for all $t \in [0,T]$. Motivated by the representation (4.13) above define the function $U^c : [0,T] \times (0,\infty) \to I\!R$ by setting

$$\text{(4.14)} \qquad U^c(t,x) = e^{-r(T-t)} \mathsf{E}\big[G^{\mu_c}(T, xX_{T-t}) \big] - \int_0^{T-t} e^{-rv} \mathsf{E}\big[H^{\mu_c}(t+v, xX_v) \, I(xX_v > c(t+v)) \big] \, dv$$

for $(t,x) \in [0,T] \times (0,\infty)$. Observe that c solving (4.6) means exactly that $U^c(t,c(t)) = G^{\mu_c}(t,c(t))$ for all $t \in [0,T]$ (recall that $G^{\mu_c}(T,x) = (x-K)^+$ for all $x > 0$).

(i) We show that $U^c(t,x) = G^{\mu_c}(t,x)$ for all $(t,x) \in [0,T] \times (0,\infty)$ such that $x \ge c(t)$. For this, take any such (t,x) and note that the Markov property of X implies that

$$\text{(4.15)} \qquad e^{-rs} U^c(t+s, xX_s) - \int_0^s e^{-rv} H^{\mu_c}(t+v, xX_v) \, I(xX_v > c(t+v)) \, dv$$

is a continuous martingale under P for $s \in [0, T-t]$. Consider the stopping time

$$\text{(4.16)} \qquad \sigma_c = \inf \{ \, s \in [0, T-t] \mid xX_s \le c(t+s) \, \}$$

under P. Since $U^c(t,c(t)) = G^{\mu_c}(t,c(t))$ for all $t \in [0,T]$ and $U^c(T,x) = G^{\mu_c}(T,x)$ for all $x > 0$ we see that $U^c(t+\sigma_c, xX_{\sigma_c}) = G^{\mu_c}(t+\sigma_c, xX_{\sigma_c})$. Replacing s by σ_c in (4.15), taking E on both sides and applying the optional sampling theorem, we find that

$$\text{(4.17)} \qquad U^c(t,x) = \mathsf{E}\big[e^{-r\sigma_c} U^c(t+\sigma_c, xX_{\sigma_c}) \big] - \mathsf{E}\bigg(\int_0^{\sigma_c} e^{-rv} H^{\mu_c}(t+v, xX_v) \, I(xX_v > c(t+v)) \, dv \bigg)$$
$$= \mathsf{E}\big[e^{-r\sigma_c} G^{\mu_c}(t+\sigma_c, xX_{\sigma_c}) \big] - \mathsf{E}\bigg(\int_0^{\sigma_c} e^{-rv} H^{\mu_c}(t+v, xX_v) \, dv \bigg) = G^{\mu_c}(t,x)$$

where in the last equality we use (3.17). This shows that U^c equals G^{μ_c} above c as claimed.

(ii) We show that $U^c(t,x) \le V(t,x)$ for all $(t,x) \in [0,T]\times(0,\infty)$. For this, take any such (t,x) and consider the stopping time

$$\tau_c = \inf\,\{\, s \in [0,T-t] \mid xX_s \ge c(t+s)\,\} \tag{4.18}$$

under P . We claim that $U^c(t+\tau_c, xX_{\tau_c}) = G^{\mu_c}(t+\tau_c, xX_{\tau_c})$. Indeed, if $x \ge c(t)$ then $\tau_c = 0$ so that $U^c(t,x) = G^{\mu_c}(t,x)$ by (i) above. On the other hand, if $x < c(t)$ then the claim follows since $U^c(t,c(t)) = G^{\mu_c}(t,c(t))$ for all $t \in [0,T]$ and $U^c(T,x) = G^{\mu_c}(T,x)$ for all $x > 0$. Replacing s by τ_c in (4.15), taking E on both sides and applying the optional sampling theorem, we find that

$$\begin{aligned} U^c(t,x) &= \mathsf{E}\big[\mathrm{e}^{-r\tau_c}\, U^c(t+\tau_c, xX_{\tau_c})\big] - \mathsf{E}\Big(\int_0^{\tau_c} \mathrm{e}^{-rv} H^{\mu_c}(t+v, xX_v)\, I(xX_v > c(t+v))\, dv\Big) \\ &= \mathsf{E}\big[\mathrm{e}^{-r\tau_c}\, G^{\mu_c}(t+\tau_c, xX_{\tau_c})\big] \le V(t,x) \end{aligned} \tag{4.19}$$

where in the second equality we used the definition of τ_c . This shows that $U^c \le V$ as claimed.

(iii) We show that $c(t) \le b(t)$ for all $t \in [0,T]$. For this, suppose that there exists $t \in [0,T)$ such that $c(t) > b(t)$. Take any $x \ge c(t)$ and consider the stopping time

$$\sigma_b = \inf\,\{\, s \in [0,T-t] \mid xX_s \le b(t+s)\,\} \tag{4.20}$$

under P . Replacing s with σ_b in (4.8) and (4.15), taking E on both sides of these identities and applying the optional sampling theorem, we find

$$\mathsf{E}\big[\mathrm{e}^{-r\sigma_b}\, V(t+\sigma_b, xX_{\sigma_b})\big] = V(t,x) + \mathsf{E}\Big(\int_0^{\sigma_b} \mathrm{e}^{-rv} H^{\mu_c}(t+v, xX_v)\, dv\Big) \tag{4.21}$$

$$\mathsf{E}\big[\mathrm{e}^{-r\sigma_b}\, U^c(t+\sigma_b, xX_{\sigma_b})\big] = U^c(t,x) + \mathsf{E}\Big(\int_0^{\sigma_b} \mathrm{e}^{-rv} H^{\mu_c}(t+v, xX_v)\, I(xX_v > c(t+v))\, dv\Big). \tag{4.22}$$

Since $x \ge c(t)$ we see by (i) above that $U^c(t,x) = G^{\mu_c}(t,x) = V(t,x)$ where the last equality follows since x lies above $b(t)$. Moreover, by (ii) above we know that $U^c(t+\sigma_b, xX_{\sigma_b}) \le V(t+\sigma_b, xX_{\sigma_b})$ so that (4.21) and (4.22) imply that

$$\mathsf{E}\Big(\int_0^{\sigma_b} \mathrm{e}^{-rv} H^{\mu_c}(t+v, xX_v)\, I(xX_v \le c(t+v))\, dv\Big) \ge 0\,. \tag{4.23}$$

The fact that $c(t) > b(t)$ and the continuity of the functions c and b imply that there exists $\varepsilon > 0$ sufficiently small such that $c(t+v) > b(t+v)$ for all $v \in [0,\varepsilon]$. Consequently the P-probability of $X^x = (x+X_v)_{0\le v\le\varepsilon}$ spending a strictly positive amount of time (w.r.t. Lebesgue measure) in this set before hitting b is strictly positive. Combined with the fact that b lies above h this forces the expectation in (4.23) to be strictly negative and provides a contradiction. Hence $c \le b$ as claimed.

(iv) We show that $b(t) = c(t)$ for all $t \in [0,T]$. For this, suppose that there exists $t \in [0,T)$ such that $c(t) < b(t)$. Take any $x \in (c(t), b(t))$ and consider the stopping time

$$\tau_b = \inf\,\{\, s \in [0,T-t] \mid xX_s \ge b(t+s)\,\} \tag{4.24}$$

under P . Replacing s with τ_b in (4.8) and (4.15), taking E on both sides of these identities and applying the optional sampling theorem, we find

$$\mathsf{E}\big[\mathrm{e}^{-r\tau_b} V(t+\tau_b, xX_{\tau_b})\big] = V(t,x) \tag{4.25}$$

$$\mathsf{E}\big[\mathrm{e}^{-r\tau_b} U^c(t+\tau_b, xX_{\tau_b})\big] = U^c(t,x) + \mathsf{E}\bigg(\int_0^{\tau_b} \mathrm{e}^{-rv} H^{\mu_c}(t+v, xX_v)\, I(xX_v > c(t+v))\, dv\bigg)\,. \tag{4.26}$$

Since $c \le b$ by (iii) above and U^c equals G^{μ_c} above c by (i) above, we see that $U^c(t+\tau_b, xX_{\tau_b}) = G^{\mu_c}(t+\tau_b, xX_{\tau_b}) = V(t+\tau_b, xX_{\tau_b})$ where the last equality follows since V equals G^{μ_c} above b (recall also that $U^c(T,x) = G^{\mu_c}(T,x) = V(T,x)$ for all $x > 0$). Moreover, by (ii) we know that $U^c \le V$ so that (4.25) and (4.26) imply that

$$\mathsf{E}\bigg(\int_0^{\sigma_b} \mathrm{e}^{-rv} H^{\mu_c}(t+v, xX_v)\, I(xX_v > c(t+v))\, dv\bigg) \ge 0\,. \tag{4.27}$$

But then as in (iii) above the continuity of the functions c and b combined with the fact that c lies above h forces the expectation in (4.27) to be strictly negative and provides a contradiction. Thus $c = b$ as claimed and the proof is complete. □

5. The British put-call symmetry

In this section we derive the 'British put-call symmetry' relations which express the arbitrage-free price and the rational exercise boundary of the British call option in terms of the arbitrage-free price and the rational exercise boundary of the British put option where the roles of the contract drift and the interest rate have been swapped (Theorem 2). We note that similar symmetry relations are known to be valid for the American put and call options written on stocks paying dividends (see [2]) where the roles of the dividend yield and the interest rate have been swapped instead. The idea to examine these relations in the British option setting is due to Kristoffer Glover.

1. In the setting of Section 3 let us put

$$G_c^{\mu_c}(t,x;K) = \mathsf{E}(xZ_{T-t}^{\mu_c} - K)^+ \tag{5.1}$$

$$G_p^{\mu_c}(t,x;K) = \mathsf{E}(K - xZ_{T-t}^{\mu_c})^+ \tag{5.2}$$

to denote the payoff functions of the British call and put options respectively, where the contract drift $\mu_c > 0$ and the strike price $K > 0$ are given and fixed, $(t,x) \in [0,T]\times(0,\infty)$, and $Z_{T-t}^{\mu_c}$ is given in (3.6) above. Let us also put

$$V_c^{\mu_c}(t,x;K,r) = \sup_{0\le\tau\le T-t} \mathsf{E}\big[\mathrm{e}^{-r\tau} G_c^{\mu_c}(t+\tau, xX_\tau^r;K)\big] \tag{5.3}$$

$$V_p^{\mu_c}(t,x;K,r) = \sup_{0\le\tau\le T-t} \mathsf{E}\big[\mathrm{e}^{-r\tau} G_p^{\mu_c}(t+\tau, xX_\tau^r;K)\big] \tag{5.4}$$

to denote the arbitrage-free prices of the British call and put options respectively, where the interest rate $r > 0$ is given and fixed, $(t,x) \in [0,T]\times(0,\infty)$, and the process X^r solves (3.11) above with $X_0^r = 1$. Recall that the supremums in (5.3) and (5.4) are taken over all stopping times τ of X^r with values in $[0, T-t]$. The following lemma lists basic properties of the payoff functions (5.1) and (5.2) that will be useful in the proof of Theorem 2 below.

Lemma 1. *The following relations are valid:*

$$G_a^{\mu_c}(t,x;K) = x\, G_a^{\mu_c}(t,1;\tfrac{K}{x}) \tag{5.5}$$

$$G_a^{\mu_c}(t,x;K) = K\, G_a^{\mu_c}(t,\tfrac{x}{K};1) \tag{5.6}$$

$$G_a^{\mu_c}(t,x;K) = G_a^{0}(t,x\, e^{\mu_c(T-t)};K) \tag{5.7}$$

$$G_c^{0}(t,x;1) = x\, G_p^{0}(t,\tfrac{1}{x};1) \tag{5.8}$$

$$G_p^{0}(t,x;1) = x\, G_c^{0}(t,\tfrac{1}{x};1) \tag{5.9}$$

for all values of the parameters, where the subscript a *in* (5.5)-(5.7) *stands for either* c *or* p *.*

Proof. Since (5.5)-(5.7) are evident from the definitions (5.1) and (5.2) we focus on (5.8). For this, note that by the Girsanov theorem we have

$$\begin{aligned} G_c^0(t,x;1) &= \mathsf{E}(xZ_{T-t}^0-1)^+ = x\,\mathsf{E}\Big[Z_{T-t}^0\big(1-\tfrac{1}{xZ_{T-t}^0}\big)^+\Big] \\ &= x\,\mathsf{E}\Big[\mathrm{e}^{\sigma B_{T-t}-\frac{\sigma^2}{2}(T-t)}\big(1-\tfrac{1}{x}\,\mathrm{e}^{-\sigma B_{T-t}+\frac{\sigma^2}{2}(T-t)}\big)^+\Big] \\ &= x\,\widetilde{\mathsf{E}}\Big(1-\tfrac{1}{x}\,\mathrm{e}^{-\sigma(B_{T-t}-\sigma(T-t))-\sigma^2(T-t)+\frac{\sigma^2}{2}(T-t)}\Big)^+ \\ &= x\,\widetilde{\mathsf{E}}\Big(1-\tfrac{1}{x}\,\mathrm{e}^{-\sigma\widetilde{B}_{T-t}-\frac{\sigma^2}{2}(T-t)}\Big)^+ = x\,\mathsf{E}(1-\tfrac{1}{x}Z_{T-t}^0)^+ = x\,G_p^0(t,\tfrac{1}{x};1) \end{aligned} \tag{5.10}$$

where $\tilde{\mathsf{E}}$ is taken with respect to $\tilde{\mathsf{P}}$ given by $d\tilde{\mathsf{P}} = \mathrm{e}^{\sigma B_{T-t}-\frac{\sigma^2}{2}(T-t)}\,d\mathsf{P}$ and $\tilde{B}_s = B_s - \sigma s$ is a standard Brownian motion under $\tilde{\mathsf{P}}$ for $s \in [0,T-t]$ (recall also that $-\tilde{B}_{T-t} =^{\text{law}} \tilde{B}_{T-t}$ under $\tilde{P}$). This establishes (5.8) from where (5.9) follows as well. □

The main result of this section may now be stated as follows.

Theorem 2. *The arbitrage-free price of the British call option with the contract drift* μ_c *and the interest rate* r *satisfying* (3.1) *can be expressed in terms of the arbitrage-free price of the British put option with the contract drift* r *and the interest rate* μ_c *as*

$$V_c^{\mu_c}(t,x;K,r) = \tfrac{x}{K}\, e^{\mu_c(T-t)}\, V_p^r(t,\tfrac{K^2}{x}\, e^{-(r+\mu_c)(T-t)};K,\mu_c) \tag{5.11}$$

for all $(t,x) \in [0,T]\times(0,\infty)$ *.*

The rational exercise boundary of the British call option with the contract drift μ_c *and the interest rate* r *satisfying* (3.1) *can be expressed in terms of the rational exercise boundary of the British put option with the contract drift* r *and the interest rate* μ_c *as*

$$b_c^{\mu_c,r}(t) = e^{-(r+\mu_c)(T-t)}\frac{K^2}{b_p^{r,\mu_c}(t)} \tag{5.12}$$

for all $t \in [0,T]$ *.*

Proof. We first derive the identity (5.11). For this, fix any $(t,x) \in [0,T]\times(0,\infty)$ and take any stopping time τ of X^r with values in $[0,T-t]$. Using the facts from Lemma 1 indicated below and applying the Girsanov theorem we find that

$$\mathsf{E}\big[\mathrm{e}^{-r\tau}G_c^{\mu_c}(t+\tau,xX_\tau^r;K)\big] \overset{(5.6)}{=} K\,\mathsf{E}\big[\mathrm{e}^{-r\tau}G_c^{\mu_c}(t+\tau,\tfrac{x}{K}X_\tau^r;1)\big] \tag{5.13}$$

$$\overset{(5.7)}{=} K\,\mathsf{E}\big[\mathrm{e}^{-r\tau}G_c^0(t{+}\tau,\tfrac{x}{K}\,\mathrm{e}^{\mu_c(T-t-\tau)}X_\tau^r;1)\big]$$
$$\overset{(5.8)}{=} K\,\mathsf{E}\big[\mathrm{e}^{-r\tau}\tfrac{x}{K}\,\mathrm{e}^{\mu_c(T-t-\tau)}X_\tau^r\,G_p^0(t{+}\tau,\tfrac{K}{x}\,\mathrm{e}^{-\mu_c(T-t-\tau)}\tfrac{1}{X_\tau^r};1)\big]$$
$$= x\,\mathrm{e}^{\mu_c(T-t)}\,\mathsf{E}\big[\mathrm{e}^{\sigma B_\tau-\frac{\sigma^2}{2}\tau}\,\mathrm{e}^{-\mu_c\tau}\,G_p^0(t{+}\tau,\tfrac{K}{x}\,\mathrm{e}^{-\mu_c(T-t)}\mathrm{e}^{-\sigma B_\tau+(\mu_c-r+\frac{\sigma^2}{2})\tau};1)\big]$$
$$= x\,\mathrm{e}^{\mu_c(T-t)}\,\widetilde{\mathsf{E}}\big[\mathrm{e}^{-\mu_c\tau}\,G_p^0(t{+}\tau,\tfrac{K}{x}\,\mathrm{e}^{-\mu_c(T-t)}\mathrm{e}^{-\sigma(B_\tau-\sigma\tau)-\sigma^2\tau+(\mu_c-r+\frac{\sigma^2}{2})\tau};1)\big]$$
$$= x\,\mathrm{e}^{\mu_c(T-t)}\,\widetilde{\mathsf{E}}\big[\mathrm{e}^{-\mu_c\tau}\,G_p^0(t{+}\tau,\tfrac{K}{x}\,\mathrm{e}^{-\mu_c(T-t)}\mathrm{e}^{-\sigma\widetilde{B}_\tau+(\mu_c-r-\frac{\sigma^2}{2})\tau};1)\big]$$
$$= x\,\mathrm{e}^{\mu_c(T-t)}\,\widetilde{\mathsf{E}}\big[\mathrm{e}^{-\mu_c\tau}\,G_p^0(t{+}\tau,\tfrac{K}{x}\,\mathrm{e}^{-(r+\mu_c)(T-t)}\,\mathrm{e}^{r(T-t-\tau)}\mathrm{e}^{-\sigma\widetilde{B}_\tau+(\mu_c-\frac{\sigma^2}{2})\tau};1)\big]$$
$$\overset{(5.7)}{=} x\,\mathrm{e}^{\mu_c(T-t)}\,\widetilde{\mathsf{E}}\big[\mathrm{e}^{-\mu_c\tau}\,G_p^r(t{+}\tau,\tfrac{K}{x}\,\mathrm{e}^{-(r+\mu_c)(T-t)}\,\widetilde{X}_\tau^{\mu_c};\tfrac{K}{K})\big]$$
$$\overset{(5.6)}{=} \tfrac{x}{K}\,\mathrm{e}^{\mu_c(T-t)}\,\widetilde{\mathsf{E}}\big[\mathrm{e}^{-\mu_c\tau}\,G_p^r(t{+}\tau,\tfrac{K^2}{x}\,\mathrm{e}^{-(r+\mu_c)(T-t)}\,\widetilde{X}_\tau^{\mu_c};K)\big]$$

where $\tilde{\mathsf{E}}$ is taken with respect to $\tilde{\mathsf{P}}$ given by $d\tilde{\mathsf{P}} = \mathrm{e}^{\sigma B_\tau-\frac{\sigma^2}{2}\tau}\,d\mathsf{P}$ and $\tilde{B}_t = B_t-\sigma t$ is a standard Brownian motion under $\tilde{\mathsf{P}}$ for $t\in[0,T]$ (recall also that $-\tilde{B} =^{\text{law}} \tilde{B}$ under $\tilde{P}$). Taking the supremum over all stopping times τ of X^r with values in $[0,T{-}t]$ on both sides of (5.13) we obtain the identity (5.11) as claimed.

To prove (5.12) note that by setting $\tau\equiv 0$ in (5.13) we find that

$$G_c^{\mu_c}(t,x;K) = \tfrac{x}{K}\,\mathrm{e}^{\mu_c(T-t)}\,G_p^r(t,\tfrac{K^2}{x}\,\mathrm{e}^{-(r+\mu_c)(T-t)};K) \tag{5.14}$$

for all $(t,x)\in[0,T]\times(0,\infty)$. A direct comparison of (5.11) and (5.14) shows that

$$V_c^{\mu_c}(t,x;K,r) > G_c^{\mu_c}(t,x;K) \tag{5.15}$$

if and only if the following inequality is satisfied

$$V_p^r(t,\tfrac{K^2}{x}\,\mathrm{e}^{-(r+\mu_c)(T-t)};K,\mu_c) > G_p^r(t,\tfrac{K^2}{x}\,\mathrm{e}^{-(r+\mu_c)(T-t)};K) \tag{5.16}$$

for all $(t,x)\in[0,T]\times(0,\infty)$. As the two relations (5.15) and (5.16) characterise the continuation sets for the optimal stopping problems (5.3) and (5.4) respectively, it follows that (5.12) holds and the proof is complete. □

Remark 3. Note that the previous arguments extend to the case when the strike price in (5.1)+(5.3) is not necessarily equal to the strike price in (5.2)+(5.4). More precisely, replacing the strike price K in (5.1)+(5.3) by $K_c>0$ and the strike price K in (5.2)+(5.4) by $K_p>0$, and noting that the final equality in (5.13) remains valid if we replace K/K by K_p/K_p, we obtain the following extensions of (5.11) and (5.12) respectively:

$$V_c^{\mu_c}(t,x;K_c,r) = \tfrac{x}{K_p}\,\mathrm{e}^{\mu_c(T-t)}\,V_p^r(t,\tfrac{K_cK_p}{x}\,\mathrm{e}^{-(r+\mu_c)(T-t)};K_p,\mu_c) \tag{5.17}$$

$$b_c^{\mu_c,r}(t) = \mathrm{e}^{-(r+\mu_c)(T-t)}\,\frac{K_cK_p}{b_p^{r,\mu_c}(t)} \tag{5.18}$$

for all $(t,x)\in[0,T]\times(0,\infty)$. This can be rewritten in a more symmetric form as follows:

$$V_c^{\mu_c}(t,xK_p\,\mathrm{e}^{-\mu_c(T-t)};yK_c,r) = V_p^r(t,yK_c\,\mathrm{e}^{-r(T-t)};xK_p,\mu_c) \tag{5.19}$$

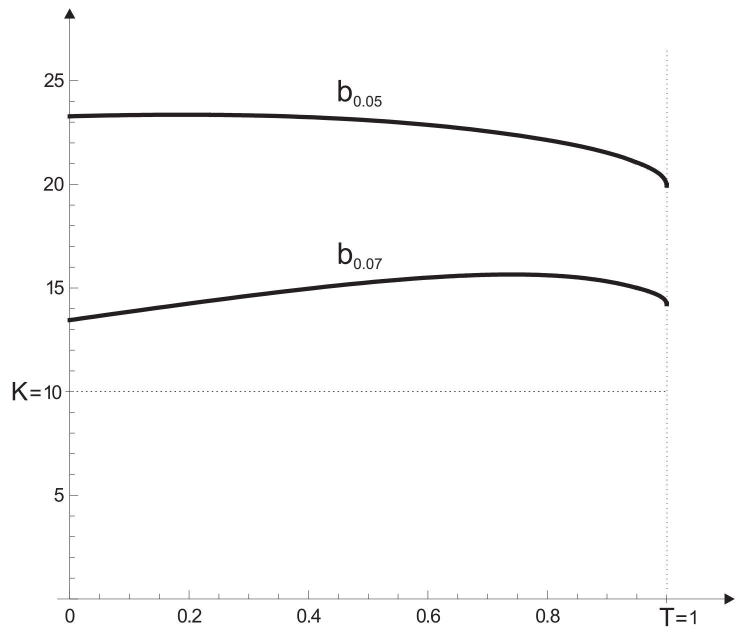

Figure 5. A computer drawing showing the rational exercise boundaries of the British call option with $K = 10$, $T = 1$, $r = 0.1$, $\sigma = 0.4$ when the contract drift μ_c equals 0.05 and 0.07 .

$$b_c^{\mu_c,r}(t)\, b_p^{r,\mu_c}(t) = \mathrm{e}^{-(r+\mu_c)(T-t)}\, K_c K_p \tag{5.20}$$

for all $t \in [0,T]$ and all $x, y \in (0,\infty)$.

6. Financial analysis of the British call option

In the present section we firstly discuss the rational exercise strategy of the British call option, and then a numerical example is presented to highlight the practical features of the option. We draw comparisons with the European call option in particular because the latter option is widely traded and well understood (recalling that the American call option written on a stock paying no dividends reduces to the European call option since it is rational to exercise at maturity). In the financial analysis of the option returns presented below we mainly address the question as to what the return would be if the stock price enters the given region at a given time (i.e. we do not discuss the probability of the latter event nor do we account for any risk associated with its occurrence). Such a 'skeleton analysis' is both natural and practical since it places the question of probabilities and risk under the subjective assessment of the option holder (irrespective of whether the stock price model is correct or not).

1. In Section 3 above we saw that the rational exercise strategy of the British call option (the optimal stopping time (3.27) in the problem (3.10) above) changes as one varies the contract drift μ_c . This is illustrated in Figure 4 above. To explain the economic meaning of the three regimes appearing on this figure, let us first recall that it is natural to set $\mu_c < r$. Indeed, if one sets $\mu_c \geq r$ then it is always optimal to stop at once in (3.10). In this case the buyer

Time (months)	0	2	4	6	8	10	12
Exercise at K with $\mu_c = 0.07$	99%	88%	77%	65%	51%	35%	0%
Exercise at K with $\mu_c = 0.05$	93%	84%	74%	62%	50%	34%	0%
Exercise at 9 with $\mu_c = 0.07$	68%	58%	49%	38%	27%	13%	0%
Exercise at 9 with $\mu_c = 0.05$	63%	55%	46%	36%	26%	13%	0%
Exercise at 8 with $\mu_c = 0.07$	42%	35%	27%	19%	11%	03%	0%
Exercise at 8 with $\mu_c = 0.05$	39%	32%	25%	18%	10%	03%	0%
Exercise at 7 with $\mu_c = 0.07$	23%	18%	12%	07%	03%	0.4%	0%
Exercise at 7 with $\mu_c = 0.05$	21%	16%	12%	07%	03%	0.3%	0%
Exercise at 6 with $\mu_c = 0.07$	11%	07%	04%	02%	01%	0.0%	0%
Exercise at 6 with $\mu_c = 0.05$	10%	07%	04%	02%	01%	0.0%	0%
Exercise at 5 with $\mu_c = 0.07$	04%	02%	01%	0.3%	0.0%	0.0%	0%
Exercise at 5 with $\mu_c = 0.05$	03%	02%	01%	0.3%	0.0%	0.0%	0%

Table 6. Returns observed upon exercising the British call option at and below the strike price K . The returns are expressed as a percentage of the original option price paid by the buyer (rounded to the nearest integer), i.e. $R(t,x)/100 = G^{\mu_c}(t,x)/V(0,K)$. The parameter set is the same as in Figure 5 above ($K = 10$, $T = 1$, $r = 0.1$, $\sigma = 0.4$) and the initial stock price equals K .

is overprotected. By exercising immediately he will beat the interest rate r and moreover he will avoid any discounting of his payoff. We also noted above that the rational exercise boundary b in Figure 4 satisfies $b(T) = rK/\mu_c$. In particular, when $\mu_c = r$ then $b(T) = K$ and b extends backwards in time to zero. The interest rate r represents a borderline case and any $\mu_c > 0$ strictly smaller than r will inevitably lead to a non-trivial rational exercise boundary (when the initial stock price is sufficiently low). It is clear however that not all of these situations will be of economic interest and in practice μ_c should be set further away from r in order to avoid overprotection (note that most generally overprotection refers to the case where the initial stock price lies above $b(0)$). On the other hand, when $\mu_c \downarrow 0$ then $b(T)$ increases up to infinity and b disappears in the limit, so that it is never optimal to stop before the maturity time T and the British call option reduces to the European call option. In the latter case a zero contract drift represents an infinite tolerance of unfavourable drifts and the British call holder will never exercise the option before the time of maturity. This brief analysis shows that the contract drift μ_c should not be too close to r (since in this case the buyer is overprotected) and should not be too close to zero (since in this case the British call option effectively reduces to the European call option). We remark that the latter possibility is not fully excluded, however, especially if sharp increases in the stock price are possible or likely.

2. Further to our comments above, consider the position and shape of the rational exercise boundaries in Figure 4, and assume that the initial stock price equals K . In this case the boundary (ii) represents overprotection. Indeed, although not larger than r , the contract drift μ_c is favourable enough so that the additional incentive of avoiding discounting makes it rational to exercise immediately. On the other hand, it may be noted that the position of the boundaries (i) and (iii) suggests that the buyer should exercise the British call option rationally when observed price movements are favourable. As this effect is analogous to the same effect in the British put option we refer to [11] for further details. In addition we remark that the

Time (months)	0	2	4	6	8	10	12
Exercise at 20 (British call)	561%	548%	535%	522%	510%	499%	487%
Selling at 20 (European call)	541%	533%	525%	516%	508%	500%	492%
Exercise at 18 (British call)	459%	446%	434%	422%	411%	400%	390%
Selling at 18 (European call)	445%	436%	427%	419%	410%	402%	394%
Exercise at 16 (British call)	359%	347%	335%	323%	312%	302%	292%
Selling at 16 (European call)	350%	341%	331%	322%	312%	304%	295%
Exercise at b (British call)	238%	257%	274%	287%	292%	278%	209%
Selling at b (European call)	235%	254%	272%	287%	293%	280%	211%
Exercise at 12 (British call)	175%	163%	151%	138%	124%	109%	98%
Selling at 12 (European call)	174%	163%	152%	140%	126%	111%	98%
Exercise at 11 (British call)	135%	124%	112%	99%	84%	68%	49%
Selling at 11 (European call)	135%	125%	113%	101%	87%	70%	49%

Table 7. Returns observed upon exercising the British call option (with $\mu_c = 0.07$) above the strike price K compared with returns received upon selling the European call option in the same contingency. The returns are expressed as a percentage of the original option price paid by the buyer (rounded to the nearest integer), i.e. $R(t,x)/100 = G^{\mu_c}(t,x)/V(0,K)$ and $R_E(t,x)/100 = V_E(t,x)/V_E(0,K)$ respectively. The parameter set is the same as in Figure 5 above ($K = 10$, $T = 1$, $r = 0.1$, $\sigma = 0.4$) and the initial stock price equals K .

position of the boundaries in Figure 4 is largely determined by the presence of discounting in the problem. As the stock price increases, the payoff function (3.5) increases as well (for fixed time), and the effect of discounting impacts more heavily on any decision to continue. The British call holder thus becomes more inclined to exercise simply to beat the discounting effect and the higher (more favourable) the contract drift the more so this is true.

3. Figure 5 shows the rational exercise boundaries of the British call option for a fixed parameter set (chosen to present the practical features of the option in a fair and representative way). The values of the contract drift μ_c have been selected to produce boundaries of type (i) and (iii) in Figure 4. As already indicated above, these values also depend strongly upon the volatility coefficient σ . When σ is large then the British call holder will have a greater tolerance for unfavourable drifts since the high volatility is more likely to drown out the effect of the drift and as such the buyer can still record favourable returns. In this case the contract drift μ_c must be set further away from the interest rate r in order to avoid overprotection. This is also seen from the fact that the rational exercise boundary b becomes more S-skewed as the volatility coefficient σ increases so that $b(0)$ can become small. Assuming again that the initial stock price equals K , the price of the European call option is 2.032 . The price of the British call option is 2.034 if $\mu_c = 0.05$ and 2.052 if $\mu_c = 0.07$. Note that the closer the contract drift gets to r , the stronger is the protection feature provided, and the more expensive the British call option becomes. Moreover, in terms of the price sizes it can be seen that this example is not isolated since the price of the British call option stays very close to the price of the European call option unless the contract drift μ_c is unrealistically close to the interest rate r (so that the buyer is overprotected and the situation is uninteresting economically). Recall also that when $\mu_c \downarrow 0$ then in the limit it is not rational to exercise before the time

Time (months)	0	2	4	6	8	10	12
Exercise at K (British call)	99%	88%	77%	65%	51%	35%	0%
Selling at K (European call)	100%	90%	79%	67%	53%	36%	0%
Exercise at 9 (British call)	68%	58%	49%	38%	27%	13%	0%
Selling at 9 (European call)	69%	60%	50%	40%	28%	14%	0%
Exercise at 8 (British call)	42%	35%	27%	19%	11%	03%	0%
Selling at 8 (European call)	44%	36%	28%	20%	12%	04%	0%
Exercise at 7 (British call)	23%	18%	12%	07%	03%	0.4%	0%
Selling at 7 (European call)	24%	19%	13%	08%	03%	0.5%	0%
Exercise at 6 (British call)	11%	07%	04%	02%	01%	0.0%	0%
Selling at 6 (European call)	11%	08%	05%	02%	01%	0.0%	0%
Exercise at 5 (British call)	04%	02%	01%	0.3%	0.0%	0.0%	0%
Selling at 5 (European call)	04%	02%	01%	0.4%	0.1%	0.0%	0%

Table 8. Returns observed upon exercising the British call option (with $\mu_c = 0.07$) at and below the strike price K compared with returns received upon selling the European call option in the same contingency. The returns are expressed as a percentage of the original option price paid by the buyer (rounded to the nearest integer), i.e. $R(t,x)/100 = G^{\mu_c}(t,x)/V(0,K)$ and $R_E(t,x)/100 = V_E(t,x)/V_E(0,K)$ respectively. The parameter set is the same as in Figure 5 above ($K=10$, $T=1$, $r=0.1$, $\sigma=0.4$) and the initial stock price equals K .

of maturity and the price of the British call option reduces to the price of the European call option. The fact that the price of the British call option is very comparable to the price of the European call option (in most of situations that are of interest for trading) is of considerable practical value (given the additional benefits to be discussed shortly below). It also implies (as a rule of thumb) that the deltas for both options (hedge ratios) stay very close to each other as well as that many of the sensitivities of the British call option (option greeks) will be very similar to those of the European call option. Further details of such an analysis can be readily generated using the formulae derived above.

4. Table 6 shows the power of the protection feature in practice. For example, if the stock price is at K halfway to maturity (clearly representative of unfavourable price movements) then the British call holder can exercise immediately to a payoff which represents a reimbursement of 62-65% of his original investment. Compare this with a 'formal American call' holder who in this contingency is out-of-the-money and would receive zero payoff upon exercise. We also see that the size of the reimbursement received by the British call holder depends upon the contract drift. The closer the contract drift is to r , the more protection the British call holder is afforded, and thus the greater his reimbursement will be. We note in addition that setting $V = V_K$ and $G^{\mu_c} = G^{\mu_c}_K$ to indicate dependence on the strike price K , we have $V_K(t,x) = KV_1(t,x/K)$ and $G^{\mu_c}_K(t,x) = KG^{\mu_c}_1(t,x/K)$, so that the return does not depend on the size of the strike price (when properly scaled). The same fact is also valid for the returns of the European call option considered in Tables 7 and 8.

5. Tables 7 and 8 show the returns that the British call holder can extract when the price movements are favourable or unfavourable respectively. We focus on the British call option with $\mu_c = 0.07$ since in this case the rational exercise boundary is closer to the strike price

Time (months)	0	2	4	6	8	10	12
Exercise at b	238%	257%	274%	287%	292%	278%	209%
Selling at b	238%	257%	274%	287%	292%	278%	209%
Exercise at 12	175%	163%	151%	138%	124%	109%	98%
Selling at 12	175%	164%	152%	139%	125%	110%	98%
Exercise at 11	135%	124%	112%	99%	84%	68%	49%
Selling at 11	136%	125%	113%	100%	86%	69%	49%
Exercise at K	99%	88%	77%	65%	51%	35%	0%
Selling at K	100%	90%	78%	66%	53%	36%	0%
Exercise at 9	68%	58%	49%	38%	27%	13%	0%
Selling at 9	69%	60%	50%	39%	28%	14%	0%
Exercise at 8	42%	35%	27%	19%	11%	03%	0%
Selling at 8	44%	36%	28%	20%	11%	04%	0%
Exercise at 7	23%	18%	12%	07%	03%	0.4%	0%
Selling at 7	24%	19%	13%	08%	03%	0.5%	0%
Exercise at 6	11%	07%	04%	02%	01%	0.0%	0%
Selling at 6	11%	08%	05%	02%	01%	0.0%	0%
Exercise at 5	04%	02%	01%	0.3%	0.0%	0.0%	0%
Selling at 5	04%	02%	01%	0.4%	0.1%	0.0%	0%

Table 9. Returns observed upon (i) exercising and (ii) selling the British call option (with $\mu_c = 0.07$) at and below the rational exercise boundary. The returns are expressed as a percentage of the original option price paid by the buyer (rounded to the nearest integer), i.e. $R_e(t,x)/100 = G^{\mu_c}(t,x)/V(0,K)$ and $R_s(t,x)/100 = V(t,x)/V(0,K)$. The parameter set is the same as in Figure 5 above ($K = 10$, $T = 1$, $r = 0.1$, $\sigma = 0.4$) and the initial stock price equals K .

K and this makes the comparison more interesting economically. Tables 7 and 8 also show the returns observed upon selling the European call option (at the arbitrage-free price) in the same contingency. This is motivated by the fact that in practice the option holder may also choose to sell his option at any time during the term of the contract, and in this case one may view his 'payoff' as the price he receives upon selling. In particular, apart from being able to 'cash in' immediately upon favourable stock price movements (Table 7), the European call holder is to some extent protected from unfavourable price movements by his ability to sell the contract (Table 8). Upon inspection of Tables 7 and 8 we see that exercising the British call option produces remarkably similar returns to selling the European call option (at the arbitrage-free price) irrespective of whether stock price movements are favourable or unfavourable. However, in a real financial market the option holder's ability and/or desire to sell his contract may depend upon a number of exogenous factors. These include his ability to access the option market, the transaction costs and/or taxes involved in selling the option (i.e. friction costs), and in particular the liquidity of the option market itself (which in turn determines the market/liquidation price of the option). Recall also from Section 2 that these factors can change during the term of the contract and that these changes can be difficult to predict. We want to make it clear however that the British call option (as considered in this paper) does not include any of the exogenous factors explicitly in the pricing model (nor aims at modelling their changes). Rather, and crucially, the protection feature of the British call option is *intrinsic* to it, that is, it is completely endogenous. It is inherent in the payoff function itself (obtained as a consequence of optimal prediction), and as such it is independent of any exogenous factors. From this point of view the British call option is a particularly attractive financial instrument for the buyer who is unable to access the market freely to sell his contract, when the friction costs involved in doing so are significant, or when the market for the contract is not perfectly liquid.

6. Finally in Table 9 we highlight a remarkable and peculiar aspect of the British call option (also seen in the British put option [11]) which was touched upon in Section 3 above. We observed there (see Figure 3) that the value function and the payoff function of the British call option stay close together in the continuation set. The extent to which this is true is made apparent by Table 9. For this particular choice of contract drift we see that the value function stays above the payoff function (both expressed as percentage returns) within a margin of two percent. The tightness of this relationship will be affected by the choice of the contract drift (indeed the closer the contract drift is to r the stronger the protection and the tighter the relationship will be). From Table 9 we see that (i) exercising in the continuation set produces a remarkably comparable return to selling the contract in a liquid option market; and (ii) even if the option market is perfectly liquid it may still be more profitable to exercise rather than sell (when the friction costs exceed the margin of two percent for instance).

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References

[1] Black, F. *and* Scholes, M. S. (1973). The pricing of options and corporate liabilities. *J. Political Econ.* 83 (637–659).

[2] Carr, P. *and* Chesney, M. (1996). American put call symmetry. Working Paper 13/11, Morgan Stanley & Groupe HEC.

[3] Carr, P., Jarrow, R. *and* Myneni, R. (1992). Alternative characterizations of American put options. *Math. Finance* 2 (78–106).

[4] Du Toit, J. *and* Peskir, G. (2007). The trap of complacency in predicting the maximum. *Ann. Probab.* 35 (340–365).

[5] Jacka, S. D. (1991). Optimal stopping and the American put. *Math. Finance* 1 (1–14).

[6] Kim, I. J. (1990). The analytic valuation of American options. *Rev. Financial Stud.* 3 (547–572).

[7] Karatzas, I. and Shreve, S. E. (1998). *Methods of Mathematical Finance.* Springer.

[8] Merton, R. C. (1973). Theory of rational pricing. *Bell J. Econom. Manag. Sci.* 4 (141–183).

[9] Peskir, G. (2005). A change-of-variable formula with local time on curves. *J. Theoret. Probab.* 18 (499–535).

[10] Peskir, G. (2005). On the American option problem. *Math. Finance* 15 (169–181).

[11] Peskir, G. *and* Samee, F. (2011). The British put option. *Appl. Math. Finance.* 18 (537–563).

[12] Peskir, G. *and* Shiryaev, A. N. (2006). *Optimal Stopping and Free-Boundary Problems.* Lectures in Mathematics, ETH Zürich, Birkhäuser.

[13] Samuelson, P. A. (1965). Rational theory of warrant pricing. *Indust. Manag. Rev.* 6 (13–32).

Goran Peskir
School of Mathematics
The University of Manchester
Oxford Road
Manchester M13 9PL
United Kingdom
goran@maths.man.ac.uk

Farman Samee
School of Mathematics
The University of Manchester
Oxford Road
Manchester M13 9PL
United Kingdom
Farman.Samee@postgrad.manchester.ac.uk