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The Best Doob-Type Bounds for the Maximum of Brownian Paths

GORAN PESKIR

Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion started at zero. Then the following inequality is shown to be satisfied:

$$E\Big(\max_{0 \leq t \leq \tau} |B_t|^p\Big) \leq \gamma^*_{p,q} \Big(E \int_0^\tau |B_t|^{q-1}\, dt\Big)^{p/(q+1)}$$

for all stopping times τ for B, all $0 < p < 1+q$, and all $q > 0$, with the best possible value for the constant being equal:

$$\gamma^*_{p,q} = (1+\kappa)\Big(\frac{s_*}{\kappa^\kappa}\Big)^{1/(1+\kappa)}$$

where $\kappa = p/(q-p+1)$, and s_* is the zero point of the (unique) maximal solution $s \mapsto g_*(s)$ of the differential equation:

$$g^\alpha(s)\Big(s^\beta - g^\beta(s)\Big)\frac{dg}{ds}(s) = K$$

satisfying $0 < g_*(s) < s$ for all $s > s_*$, where $\alpha = q/p - 1$, $\beta = 1/p$ and $K = p/2$. This solution is also characterized by $g_*(s)/s \to 1$ for $s \to \infty$. The equality above is attained at the stopping time:

$$\tau_* = \inf\{\, t > 0 \mid X_t = g_*(S_t)\,\}$$

where $X_t = |B_t|^p$ and $S_t = \max_{0 \leq r \leq t} |B_r|^p$. In the case $p=1$ the closed form for $s \mapsto g_*(s)$ is found. This yields $\gamma^*_{1,q} = (q(q+1)/2)^{1/(q+1)}(\Gamma(1+(q+1)/q))^{q/(q+1)}$ for all $q > 0$. In the case $p \neq 1$ no closed form for $s \mapsto g_*(s)$ seems to exist. The inequality above holds also in the case $p = q+1$ (Doob's maximal inequality). In this case the equation above (with $K = p/2c$) admits $g_*(s) = \lambda s$ as the maximal solution, and the equality is attained only in the limit through the stopping times $\tau_* = \tau^*(c)$ when c tends to the best value $\gamma^*_{q+1,q} = (q+1)^{q+2}/2q^q$ from above. The method of proof relies upon the principle of smooth fit of Kolmogorov and the maximality principle. The results obtained extend to the case when B starts at any given point, as well as to all non-negative submartingales.

1. Introduction

Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion started at zero. Then the following comparison inequalities are known to be valid:

$$A_p\, E\Big(\int_0^\tau |B_t|^{p-2} dt\Big) \leq E\Big(\max_{0 \leq t \leq \tau} |B_t|^p\Big) \leq B_p\, E\Big(\int_0^\tau |B_t|^{p-2} dt\Big) \tag{1.1}$$

for all stopping times τ for B, and all $p > 1$, where A_p and B_p are some universal constants.

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In this paper we shall address the case $0 < p \leq 1$ when these inequalities fail to hold (note that $E(\int_0^\tau |B_t|^{-1} dt) = \infty$ whenever $B_\tau \not\equiv 0$ P-a.s.) More precisely, for $0 < p \leq 1+q$ and $q > 0$ given and fixed, we shall answer the question on sharpness of the following inequality:

$$E\biggl(\max_{0 \leq t \leq \tau} |B_t|^p \biggr) \leq \gamma_{p,q} E\biggl(\int_0^\tau |B_t|^{q-1} dt \biggr)^{p/q+1} \tag{1.2}$$

where τ is any stopping time for B , and $\gamma_{p,q}$ is a universal constant. In our main result below (Theorem 2.1) we derive (1.2) with the best possible value $\gamma^*_{p,q}$ for the constant $\gamma_{p,q}$, and we find the stopping time τ_* at which the equality in (1.2) is attained (in the limit for $p = 1+q$).

In order to give a more familiar form to the inequality (1.2), note that by Itô formula and the optional sampling theorem we have:

$$E\biggl(\int_0^\tau |B_t|^{q-1} dt \biggr) = \frac{2}{q(q+1)} E|B_\tau|^{q+1} \tag{1.3}$$

whenever τ is a stopping time for B satisfying $E(\tau^{(q+1)/2}) < \infty$ for $q > 0$. Hence the right-hand inequality in (1.1) is the well-known Doob's maximal inequality for non-negative submartingales being applied to $|B| = (|B_t|)_{t \geq 0}$ (see [3]). The advantage of the formulation (1.1) lies in its validity for all stopping times. It is well-known that in the case $p = 1$ the analogue of Doob's inequality fails. In this case the $L \log L$-inequality of Hardy and Littlewood is the adequate substitute (see [7] for the best bounds and [11] for a new probabilistic proof which exhibits the optimal stopping times too). Instead of introducing a log-term as in the Hardy-Littlewood inequality, in the inequality (1.2) we use Doob's bound (1.3) on the right-hand side.

While the inequality (1.2) (with some constant $\gamma_{p,q} > 0$) follows easily from (1.1) and (1.3) by Jensen's inequality, the question of its sharpness is far from being trivial and has gained some interest. The case $p = 1$ was treated independently by Jacka [14] and Gilat [8] both who found the best possible value $\gamma^*_{1,q}$ for $q > 0$. This in particular yields $\gamma^*_{1,1} = \sqrt{2}$ which was obtained independently earlier by Dubins and Schwarz [5], and later again by Dubins, Shepp and Shiryaev [6] who studied a more general case of Bessel processes. (A simple proof for $\gamma^*_{1,1} = \sqrt{2}$ is given in [9]). In the case $p = 1+q$ with $q > 0$, the inequality (1.2) is Doob's maximal inequality (it follows by (1.3) above). The best constants in Doob's maximal inequality and the corresponding optimal stopping times are well-known (see [16]). That the equality in Doob's maximal inequality (for any $p > 1$) cannot be attained by a non-zero (sub)martingale was observed by Cox [2]. The reader should note that this fact also follows from the method and results below (the equality in (1.2) is attained only in the limit when $p = 1+q$).

In this paper we present a proof which gives the best values $\gamma^*_{p,q}$ in (1.2) and the corresponding optimal stopping times τ^* for all $0 < p \leq 1+q$ and all $q > 0$. Our method relies upon the principle of smooth fit of Kolmogorov [6] and the maximality principle [10] (which is the main novelty in this context). It should be noted that the results extend to the case when Brownian motion B starts at any given point (Remark 2.2). Finally, due to its extreme properties, it is clear that the results obtained for reflected Brownian motion $|B|$ extend to all non-negative submartingales. This can be done by using the maximal embedding result of Jacka [13]. For reader's convenience we state this extension and outline the proof (Corollary 2.3).

2. The results and proofs

In this section we present the main results and proofs. In view of the well-known extreme properties in such a context we study the case of Brownian motion. The results obtained are then extended to all non-negative submartingales. The principal result of the paper is contained in the following theorem. It extends the result of Jacka [14] by a different method.

Theorem 2.1

Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion started at zero. Then the following inequality is shown to be satisfied:

$$E\Big(\max_{0 \leq t \leq \tau} |B_t|^p\Big) \leq \gamma_{p,q}^* \Big(E \int_0^\tau |B_t|^{q-1}\, dt\Big)^{p/(q+1)} \tag{2.1}$$

for all stopping times τ for B, all $0 < p < 1+q$, and all $q > 0$, with the best possible value for the constant $\gamma_{p,q}^*$ being equal:

$$\gamma_{p,q}^* = (1+\kappa)\Big(\frac{s_*}{\kappa^\kappa}\Big)^{1/(1+\kappa)} \tag{2.2}$$

where $\kappa = p/(q-p+1)$, and s_* is the zero point of the (unique) maximal solution $s \mapsto g_*(s)$ of the differential equation:

$$g^\alpha(s)\Big(s^\beta - g^\beta(s)\Big)\frac{dg}{ds}(s) = K \tag{2.3}$$

satisfying $0 < g_*(s) < s$ for all $s > s_*$, where $\alpha = q/p - 1$, $\beta = 1/p$ and $K = p/2$. This solution is also characterized by $g_*(s)/s \to 1$ for $s \to \infty$.

The equality in (2.1) is attained at the stopping time:

$$\tau_* = \inf\,\{\, t > 0 \mid X_t = g_*(S_t)\,\} \tag{2.4}$$

where $X_t = |B_t|^p$ and $S_t = \max_{0 \leq r \leq t} |B_r|^p$.

In the case $p = 1$ the closed form for $s \mapsto g_*(s)$ is found:

$$s \exp\Big(-\frac{g_*^q(s)}{Kq}\Big) + \int_0^{g_*(s)} \frac{t^q}{K} \exp\Big(-\frac{t^q}{Kq}\Big)\, dt = q^{1/q}\,\Gamma\big((q+1)/q\big)K^{1/q} \tag{2.5}$$

for $s \geq s_*$. This, in particular, yields:

$$\gamma_{1,q}^* = \Big(q(q+1)/2\Big)^{1/(q+1)} \Big(\Gamma\big(1+(q+1)/q\big)\Big)^{q/(q+1)} \tag{2.6}$$

for all $q > 0$. In the case $p \neq 1$ no closed form for $s \mapsto g_*(s)$ seem to exist.

The inequality (2.1) holds also in the case $p = q+1$. In this case the equation (2.3) (with $K = p/2c$) admits a linear solution $g_*(s) = \lambda s$ as the maximal solution satisfying $0 < g_*(s) < s$ for $s > 0$, where λ is the maximal root of the equation:

$$\lambda^{q/(q+1)} - \lambda = (q+1)/2c \tag{2.7}$$

satisfying $0 < \lambda < 1$. The equality in (2.1) is attained in the limit through the stopping times

$\tau_* = \tau^*(c)$ of the form (2.4) when c tends to the best value:

(2.8) $$\gamma^*_{q+1,q} = (q+1)^{q+2}/2q^q$$

from above.

Proof. 1. Given $0 \le x \le s$ and $c > 0$, consider the optimal stopping problem:

(2.9) $$V_c(x,s) = \sup_\tau E_{x,s}\big(S_\tau - cI_\tau\big)$$

where the supremum is taken over all stopping times τ for B , and the maximum process $S = (S_t)_{t\ge0}$ and the integral process $I = (I_t)_{t\ge0}$ associated with the process $X = (X_t)_{t\ge0}$ are respectively given by:

(2.10) $$S_t = \Big(\max_{0\le r\le t} X_r\Big) \vee s$$

(2.11) $$I_t = \int_0^t (X_r)^{(q-1)/p}\, dr$$

(2.12) $$X_t = |B_t + x^{1/p}|^p$$

with $0 < p < 1+q$ and $q > 0$ given and fixed. Note that under $P_{x,s} := P$ the process (X,S) starts at (x,s) .

2. By the scaling property of Brownian motion it is easily verified:

(2.13) $$V_c(0,0) = c^{-\kappa} V_1(0,0)$$

where $\kappa = p/(q-p+1)$. This yields:

(2.14) $$E(S_\tau) \le \inf_{c>0} \Big(cE(\tau) + c^{-\kappa} V_1(0,0)\Big)$$

for all stopping times τ for B . We shall show below that:

(2.15) $$V_1(0,0) = s_*$$

where s_* is the zero point of the (unique) maximal solution $s \mapsto g_*(s)$ of (2.3) satisfying $0 < g_*(s) < s$ for all $s > s_*$. Then by computing the infimum in (2.14) we get (2.1) with (2.2). Moreover, we shall see below that the equality in (2.15) is attained at τ_* from (2.4). Thus by (2.14) the same holds for (2.1). Finally, we shall prove that:

(2.16) $$\lim_{s\to\infty} \frac{g_*(s)}{s} = 1$$

which leads to the closed form (2.5) when $p = 1$. In this case the equation (2.3) admits an integrating factor ($\mu(x,y) = \exp(-y^q/Kq)$), and the unknown constant from the closed form of its solution is then easily specified by using (2.16). From (2.5) we find the closed expression (2.6). Thus to complete the proof in the case $0 < p < 1+q$ with $q > 0$, it is necessary and sufficient to prove (2.15) with (2.16).

3. We begin by proving (2.15). For this note that (X,S) is a two-dimensional diffusion which changes (increases) in the second coordinate only (instantly) after the process (X,S) hits the diagonal $x=s$. Thus the infinitesimal operator of (X,S) outside the diagonal equals the infinitesimal operator of X, which is easily verified on $]0,\infty[$ to be equal:

$$\mathbf{L_X} = \frac{p(p-1)}{2}\, x^{1-2/p}\, \frac{\partial}{\partial x} + \frac{p^2}{2}\, x^{2-2/p}\, \frac{\partial^2}{\partial x^2}\ . \tag{2.17}$$

Assuming now that the supremum in (2.9) is attained at the exit time by (X,S) from an open set, by the general Markov processes theory we know that $x \mapsto V_c(x,s)$ is to satisfy:

$$(\mathbf{L_X} V_c)(x,s) = c x^{(q-1)/p} \tag{2.18}$$

for $g_*(s) < x < s$, where $s>0$ is given and fixed, while $s \mapsto g_*(s)$ is the optimal stopping boundary to be found. The following boundary conditions seem evident:

$$V_c(x,s)\Big|_{x=g_*(s)+} = s \quad (\text{instantaneous stopping}) \tag{2.19}$$

$$\frac{\partial V_c}{\partial x}(x,s)\Big|_{x=g_*(s)+} = 0 \quad (\text{smooth fit}) \tag{2.20}$$

$$\frac{\partial V_c}{\partial s}(x,s)\Big|_{x=s-} = 0 \quad (\text{normal reflection})\ . \tag{2.21}$$

Note that (2.18)-(2.21) is a Stephan problem with moving (free) boundary. The condition (2.20) is the principle of smooth fit of Kolmogorov (see [6]).

The equation (2.18) is of Cauchy type. Its general solution is given by:

$$V_c(x,s) = A(s)\, x^{1/p} + B(s) + \frac{2c}{q(q+1)}\, x^{(q+1)/p} \tag{2.22}$$

where $s \mapsto A(s)$ and $s \mapsto B(s)$ are unknown functions. By (2.19) and (2.20) we find:

$$A(s) = -\frac{2c}{q}\, g_*^{q/p}(s) \tag{2.23}$$

$$B(s) = s + \frac{2c}{q+1}\, g_*^{(q+1)/p}(s)\ . \tag{2.24}$$

Inserting this into (2.22) and using (2.21) we see that $s \mapsto g_*(s)$ is to satisfy (2.3) with $K = p/2c$. Motivated by the maximality principle [10] we let $s \mapsto g_*(s)$ be the maximal solution of (2.3) satisfying $0 < g_*(s) < s$ for $s > s_*$, where s_* is the (unique) zero point of $s \mapsto g_*(s)$. Thus (2.22) with (2.23)+(2.24) gives $V_c(x,s)$ for $g_*(s) \le x \le s$ only when $s \ge s_*$. Clearly $V_c(x,s) = s$ for $0 \le x \le g_*(s)$ with $s \ge s_*$.

To get $V_c(x,s)$ for $0 \le x \le s < s_*$, note by the strong Markov property that:

$$V_c(x,s) = V_c(s_*,s_*) - c\,E_{x,s}\big(I_{\sigma_*}\big) \tag{2.25}$$

for all $0 \le x \le s < s_*$, where $\sigma_* = \inf\{\, t>0 \mid X_t = s_* \,\}$. By Itô formula and the optional sampling theorem we find:

$$E_{x,s}\big(I_{\sigma_*}\big) \;=\; \frac{2}{q(q+1)}\Big(s_*^{(q+1)/p} - x^{(q+1)/p}\Big) \tag{2.26}$$

for all $0 \le x \le s < s_*$. Inserting this into (2.25), and using (2.22) with (2.23)+(2.24), we get:

$$V_c(x,s) \;=\; s_* + \frac{2c}{q(q+1)}\, x^{(q+1)/p} \tag{2.27}$$

for all $0 \le x \le s < s_*$. This formula in particular gives (2.15) when $x = s = 0$ and $c = 1$.

4. To deduce (2.16) note (since $g_*(s) \le s \le V_c(s,s)$) from (2.22) with (2.23)+(2.24) that:

$$0 \;\le\; \limsup_{s \mapsto \infty} \frac{V_c(s,s)}{s^{(q+1)/p}} \;\le\; \frac{2c}{q(q+1)} \tag{2.28}$$

for all $c > 0$. The "limsup" in (2.28) is decreasing in c , thus after letting $c \downarrow 0$ we see that the "limsup" must be zero for all $c > 0$. Using this fact and going back to (2.22) with (2.23)+(2.24) we easily obtain (2.16). (Note that this can be proved in a similar way by looking at the definition of $V_c(x,s)$ in (2.9).) By the standard arguments based on Picard's method of successive approximations one can verify that the equation (2.3) admits a (unique) solution satisfying (2.16). This ends the first part of the proof (guess). In the next step we verify its validity by using Itô-Tanaka's formula.

5. To verify that the candidate (2.22) with (2.23)+(2.24) is indeed the payoff (2.9) with the optimal stopping time given by:

$$\tau_* \;=\; \inf\,\{\, t > 0 \mid X_t \,\le\, g_*(S_t) \,\} \tag{2.29}$$

denote this candidate by $V_*(x,s)$, and apply Itô-Tanaka's formula (two-dimensionally) to the process $V_*(X_t, S_t)$. For this note by (2.20) that $x \mapsto V_*(x,s)$ is C^2 on $[0, s\,[$, except at $g_*(s)$ where it is C^1 , while $(x,s) \mapsto V_*(x,s)$ is C^2 away from $\{\,(g_*(s), s) \mid s > 0\,\}$, so that clearly Itô-Tanaka's formula can be applied. In this way by (2.17) and (2.21) we get:

$$\begin{aligned} V_*(X_t,S_t) &= V_*(x,s) + \int_0^t \frac{\partial V_*}{\partial x}(X_r,S_r)\; dX_r + \int_0^t \frac{\partial V_*}{\partial s}(X_r,S_r)\; dS_r \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 V_*}{\partial x^2}(X_r,S_r)\; d\big\langle X,X\big\rangle_r = V_*(x,s) + \int_0^t \mathbf{L_X}(V_*)(X_r,S_r)\; dr \\ &+ \int_0^t \frac{\partial V_*}{\partial x}(X_r,S_r)\,\sigma(X_r)\; dB_r = V_*(x,s) + \int_0^t \mathbf{L_X}(V_*)(X_r,S_r)\; dr + M_t \end{aligned} \tag{2.30}$$

where $\sigma(x) = p x^{1-1/p}$ and $M = (M_t)_{t \ge 0}$ is a continuous local martingale. Hence by (2.18):

$$V_*(X_\tau, S_\tau) \;\le\; V_*(x,s) + cI_\tau + M_\tau \tag{2.31}$$

for all stopping times τ for B , with the equality if $\tau \le \tau_*$. By the optional sampling theorem and Burkholder-Davis-Gundy's inequality for continuous local martingales it is easily verified that:

$$(2.32) \qquad E_{x,s}\big(M_\tau\big) \;=\; 0$$

whenever τ is a stopping time satisfying $E_{x,s}(\tau^{(q+1)/2}) < \infty$. It is well-known that the stopping time τ_* satisfies such an integrability condition (this can be obtained by the methods used in [1] and presented in [15] p. 258-264). Since $s \le V_*(x,s)$ thus from (2.31) then it follows:

$$(2.33) \qquad E_{x,s}\big(S_\tau\big) \;\le\; E_{x,s}\big(V_*(X_\tau,S_\tau)\big) \;\le\; V_*(x,s) + cE_{x,s}\big(I_\tau\big)$$

for all stopping times τ for B , with the equalities if $\tau = \tau_*$. This completes the proof in the case $0 < p < 1+q$ with $q > 0$.

6. The proof just presented extends to the case $p = 1+q$ with $q > 0$ with some minor modifications. In this case the maximal solution of (2.3) with $K = p/2c$ is given by:

$$(2.34) \qquad g_*(s) \;=\; \lambda s$$

where λ is the maximal root (out of two possible ones) of the equation:

$$(2.35) \qquad \lambda^{q/(q+1)} - \lambda \;=\; \frac{q+1}{2c}$$

satisfying $0 < \lambda < 1$. By the standard argument one can verify that this happens if and only if $c \ge \gamma^*_{q+1,q}$ with $\gamma^*_{q+1,q}$ from (2.8). The essential difference from the case $p < 1+q$ is that (2.32) fails for $c \le \gamma^*_{q+1,q}$. Note as above that $E_{x,s}(\tau^{(q+1)/2}) < \infty$ if and only if $c > \gamma^*_{q+1,q}$. Thus the inequality (2.1) for $p = 1+q$ (with the equality) can be obtained in the limit when $c \downarrow \gamma^*_{q+1,q}$. Note that $V_c(0,0) = 0$ for all $c \ge \gamma^*_{q+1,q}$ (compare this with (2.13) above), as well as that (2.22) with (2.23)+(2.24) reads:

$$(2.36) \qquad V_c(x,s) = s + \frac{2c}{(q+1)}\, g_*(s) - \frac{2c}{q}\, g_*^{q/(q+1)}(s)\, x^{1/(q+1)} + \frac{2c}{q(q+1)}\, x$$

for all $0 \le x \le s$, where $s \mapsto g_*(s)$ is given by (2.34). The proof is complete.

□

Remark 2.2

The reader should note that the results stated in Theorem 2.1 above extend to the case when the Brownian motion starts at any given point. The proof above shows that whenever $0 < p \le 1+q$ and $q > 0$ are given and fixed, the following inequality is satisfied:

$$(2.37) \qquad E\Big(\max_{0\le t\le\tau} |B_t + x|^p\Big) \;\le\; c\,E\Big(\int_0^\tau |B_t + x|^{q-1}\,dt\Big) + V_c(x^p, x^p)$$

for all stopping times τ for B and all $c > 0$ (all $c \ge \gamma^*_{q+1,q}$ if $p = 1+q$), where in the case $0 < p < 1+q$ the function $V_c(x,x)$ is given by (2.22) with (2.23)+(2.24) if $x > s_*$ and by (2.27) if $0 \le x \le s_*$ (with $s_* > 0$ as in Theorem 2.1 above), while in the case $p = 1+q$ the function $V_c(x,x)$ is given by (2.36) for $x \ge 0$. Note, moreover, that by Itô formula and the optional sampling theorem, the right-hand side in (2.37) can be modified to read as follows:

$$E\Big(\max_{0\le t\le\tau}|B_t+x|^p\Big)\le\frac{2c}{q(q+1)}\,E|B_\tau+x|^{q+1}-\frac{2c}{q(q+1)}\,x^{q+1}+V_c(x^p,x^p) \tag{2.38}$$

whenever $E(\tau^{(q+1)/2})<\infty$. The inequalities (2.37) and (2.38) are sharp for each fixed c and x (the equality is attained (in the limit if $p=1+q$) at $\tau_*=\tau_*(c;x)$ from (2.4) with $X_t=|B_t+x|^p$ and $S_t=\max_{0\le r\le t}|B_r+x|^p$ where $s\mapsto g_*(s)$ is the maximal solution of (2.3) with $K=p/2c$.)

Moreover, in the case $0<p<1+q$ by taking the infimum over all $c>0$ on the right-hand side in either (2.37) or (2.38) we get a sharp inequality for each fixed x (the equality will be attained at each $\tau_*=\tau_*(c;x)$ for all $c>0$). For simplicity we omit the explicit expressions. Note, however, that in the case $p=1+q>1$ such a procedure (when $c\downarrow\gamma^*_{q+1,q}$) gives:

$$E\Big(\max_{0\le t\le\tau}|B_t+x|^p\Big)\le\Big(\frac{p}{p-1}\Big)^pE|B_\tau+x|^p-\Big(\frac{p}{p-1}\Big)\,x^p \tag{2.39}$$

for all $x\ge 0$ and all stopping times τ for B satisfying $E(\tau^{p/2})<\infty$.

Observe that Cox [2] derived inequality (2.39) for discrete non-negative submartingales by a different method. Our proof above shows that this inequality is sharp for each fixed x (the equality is attained at $\tau_*=\tau_*(c;x)$ from (2.4) with $X_t=|B_t+x|^p$ and $S_t=\max_{0\le r\le t}|B_r+x|^p$ where $s\mapsto g_*(s)$ is from (2.34) above). $\square$

Due to the extreme properties of Brownian motion, the inequalities (2.38) and (2.39) extend to all non-negative submartingales. This can be obtained by using the maximal embedding result of Jacka [13]. For reader's convenience, we state the result and outline the proof.

Corollary 2.3

Let $X=(X_t)_{t\ge0}$ be a non-negative cadlag (right continuous with left limits) uniformly integrable submartingale started at $x\ge0$ under P . Let X_∞ denote the P-a.s. limit of X for $t\to\infty$. Then the following inequality is satisfied:

$$E\Big(\sup_{t>0}X_t^p\Big)\le\frac{2c}{q(q+1)}\,E\big(X_\infty^{q+1}\big)-\frac{2c}{q(q+1)}\,x^{q+1}+V_c(x^p,x^p) \tag{2.40}$$

for all $0<p<1+q$ and all $q>0$, where $V_c(x,x)$ is given by (2.22) with (2.23)+(2.24) if $x>s_*$ and by (2.27) if $0\le x\le s_*$ (with $s_*>0$ as in Theorem 2.1 above). This inequality is sharp.

Similarly, the following inequality is satisfied:

$$E\Big(\sup_{t>0}X_t^p\Big)\le\Big(\frac{p}{p-1}\Big)^pE\big(X_\infty^p\big)-\Big(\frac{p}{p-1}\Big)\,x^p \tag{2.41}$$

for all $p>1$. This inequality is sharp.

Proof. Given such a submartingale $X=(X_t)_{t\ge0}$ satisfying $E(X_\infty)<\infty$, and a Brownian motion $B=(B_t)_{t\ge0}$ started at $X_0=x$ under P_x , by the result of Jacka [13] we know that there exists a stopping time τ for B , such that $|B_\tau|\sim X_\infty$ and $P\{\sup_{t\ge0}X_t\ge\lambda\}\le P_x\{\max_{0\le t\le\tau}|B_t|\ge\lambda\}$ for all $\lambda>0$, with $(B_{t\wedge\tau})_{t\ge0}$ being uniformly integrable. The result then easily follows from the proof of Theorem 2.1 as indicated in Remark 2.2 by using the integration by parts formula. $\square$

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Goran Peskir
Department of Mathematical Sciences
University of Aarhus, Denmark
Ny Munkegade, DK-8000 Aarhus
home.imf.au.dk/goran
goran@imf.au.dk