Nonstationary Time Series, Priestley's Evolutionary Spectra and Wavelets

Guy Nason,
School of Mathematics, University of Bristol

Summary

- Nonstationary Time Series
- Multitude of Representations
- Possibilities from Applied Computational Harmonic Analysis
- Tests of Stationarity

Nonstationary Time Series, Priestley's Evolutionary Spectra and Wavelets

Guy Nason,
School of Mathematics, University of Bristol

Summary

- Nonstationary Time Series
- Multitude of Representations
- Possibilities from Applied Computational Harmonic Analysis
- Tests of Stationarity

Nonstationary Time Series, Priestley's Evolutionary Spectra and Wavelets

Guy Nason,
School of Mathematics, University of Bristol

Summary

- Nonstationary Time Series
- Multitude of Representations
- Possibilities from Applied Computational Harmonic Analysis
- Tests of Stationarity

Nonstationary Time Series, Priestley's Evolutionary Spectra and Wavelets

Guy Nason,
School of Mathematics, University of Bristol

Summary

- Nonstationary Time Series
- Multitude of Representations
- Possibilities from Applied Computational Harmonic Analysis
- Tests of Stationarity

Nonstationary Time Series, Priestley's Evolutionary Spectra and Wavelets

Guy Nason,
School of Mathematics, University of Bristol

Summary

- Nonstationary Time Series
- Multitude of Representations
- Possibilities from Applied Computational Harmonic Analysis
- Tests of Stationarity

Nonstationary Time Series, Priestley's Evolutionary Spectra and Wavelets

Guy Nason,
School of Mathematics, University of Bristol

Summary

- Nonstationary Time Series
- Multitude of Representations
- Possibilities from Applied Computational Harmonic Analysis
- Tests of Stationarity

Time series are everywhere.

We want to

- model them,
- estimate parameters,
- check model fit, and try other models*,
- forecast future values.

We need models!

* Or change the data, or the sampling mechanism: another story.

Time series are everywhere.

We want to

- model them,
- estimate parameters,
- check model fit, and try other models*.
- forecast future values.

We need models!

* Or change the data, or the sampling mechanism: another story.

Time Series

Time series are everywhere.

We want to

- model them,
- estimate parameters,
- check model fit, and try other models*.
- forecast future values.

We need models!

* Or change the data, or the sampling mechanism: another story.

Time series are everywhere.

We want to

- model them,
- estimate parameters,
- check model fit, and try other models*.
- forecast future values.

We need models!

* Or change the data, or the sampling mechanism: another story.

Time Series

Time series are everywhere.

We want to

- model them,
- estimate parameters,
- check model fit, and try other models*.
- forecast future values.

We need models!

* Or change the data, or the sampling mechanism: another story.

Time series are everywhere.

We want to

- model them,
- estimate parameters,
- check model fit, and try other models*.
- forecast future values.

We need models!

* Or change the data, or the sampling mechanism: another story.

Time Series

Time series are everywhere.

We want to

- model them,
- estimate parameters,
- check model fit, and try other models*.
- forecast future values.

We need models!

* Or change the data, or the sampling mechanism: another story.

Time series are everywhere.

We want to

- model them,
- estimate parameters,
- check model fit, and try other models*.
- forecast future values.

We need models!

* Or change the data, or the sampling mechanism: another story.

Example Time Series: Data Source

MIDAS: Met Office Integrated Data Archive System

Originates from the UK Meteorological Office

Distributed through the NERC British Atmospheric Data Centre

Acknowledgement and thanks is due.

Example Time Series: Data Source

MIDAS: Met Office Integrated Data Archive System

Originates from the UK Meteorological Office

Distributed through the NERC British Atmospheric Data Centre

Acknowledgement and thanks is due.

Example Time Series: Data Source

MIDAS: Met Office Integrated Data Archive System

Originates from the UK Meteorological Office

Distributed through the NERC British Atmospheric Data Centre

Acknowledgement and thanks is due.

Example Time Series: Data Source

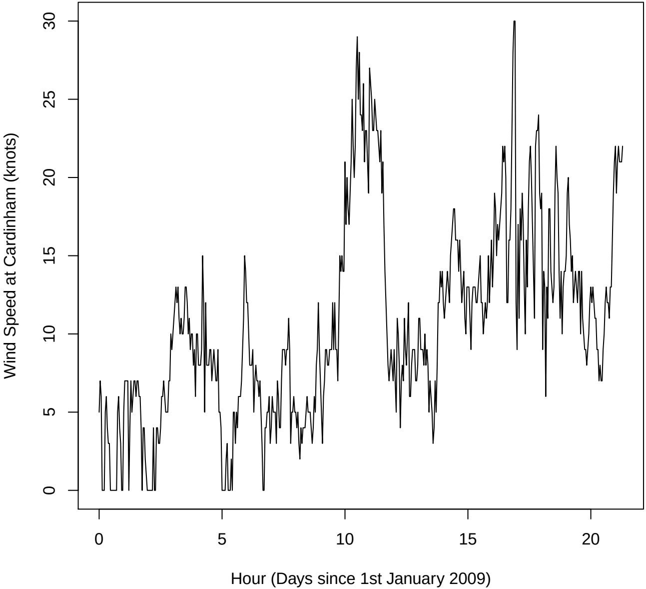
MIDAS: Met Office Integrated Data Archive System

Originates from the UK Meteorological Office

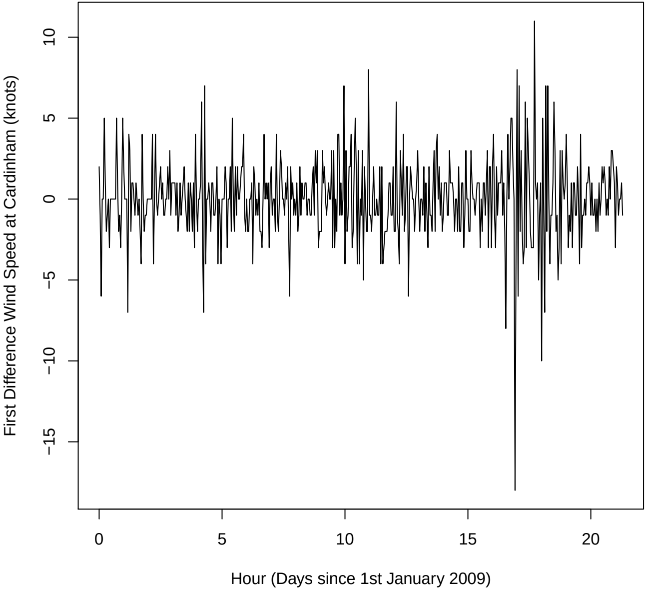
Distributed through the NERC British Atmospheric Data Centre

Acknowledgement and thanks is due.

Hourly Wind Speeds at Cardinham, Bodmin, Cornwall



First Differences of Wind Speed



Time Series Models

"The classical methods of time series analysis . . . are all based on two crucial assumptions, namely that:

(a) *all series are stationary (at least to order 2), or can be reduced to stationarity . . .*

(b) *all models are linear, . . ."*

Priestley (1981), page 816.

"However, stationarity and linearity are
. . . approximations to the real situation."

and

". . . first establish some method of characterizing
. . . non-stationary processes, . . . we describe
. . . non-stationary processes based on the theory of
evolutionary spectra. This approach was developed by
Priestley (1965b, 1966, 1967)"

Priestley (1981), page 816.

"However, stationarity and linearity are
. . . approximations to the real situation."

and

". . . first establish some method of characterizing
. . . non-stationary processes, . . . we describe
. . . non-stationary processes based on the theory of
evolutionary spectra. This approach was developed by
Priestley (1965b, 1966, 1967)"

Priestley (1981), page 816.

Oscillatory Function (Priestley, 1981, Def. 11.2.1)

"The function of t, $\phi_t(\omega)$ will be said to be an oscillatory function if, for some (necessarily unique) $\theta(\omega)$, it may be written in the form

$$\phi_t(\omega) = A_t(\omega) e^{i\theta(\omega)t},$$

where $A_t(\omega)$ is of the form

$$A_t(\omega) = \int_{-\infty}^{\infty} e^{itu} dK_\omega(u),$$

with $|dK_\omega(u)|$ having an absolute maximum at $u = 0$."

Oscillatory Process (Priestley, 1981, Def. 11.2.2)

"If there exists a family of oscillatory functions $\{\phi_t(\omega)\}$ *in terms of which the process* $\{X(t)\}$ *has a representation of the form*

$$X(t) = \int_{-\infty}^{\infty} \phi_t(\omega) dZ(\omega),$$

where $Z(\omega)$ *is an orthogonal process with* $\mathbb{E}[|dZ(\omega)|^2] = d\mu(\omega)$*, then* $\{X(t)\}$ *will be termed an* *oscillatory process."*

Evolutionary Power Spectra (Priestley, 1981, Def. 11.2.3)

"We define the evolutionary power spectrum at time t $dH_t(\omega)$ *by*

$$dH_t(\omega) = |A_t(\omega)|^2 d\mu(\omega)."$$

When $X(t)$ is stationary and $\theta(\omega) = \omega$ then $dH_t(\omega)$ reduces to the regular spectrum, $h(\omega)$.

$H_t(\omega)$ is the integrated time-frequency spectrum.

Assuming smoothness the *evolutionary spectral density* function is

$$h_t(\omega) = H_t'(\omega).$$

Evolutionary Power Spectra (Priestley, 1981, Def. 11.2.3)

"We define the evolutionary power spectrum at time t $dH_t(\omega)$ by

$$dH_t(\omega) = |A_t(\omega)|^2 d\mu(\omega)."$$

When $X(t)$ is stationary and $\theta(\omega) = \omega$ then $dH_t(\omega)$ reduces to the regular spectrum, $h(\omega)$.

$H_t(\omega)$ is the integrated time-frequency spectrum.

Assuming smoothness the *evolutionary spectral density* function is

$$h_t(\omega) = H_t'(\omega).$$

Evolutionary Power Spectra (Priestley, 1981, Def. 11.2.3)

"We define the evolutionary power spectrum at time t $dH_t(\omega)$ *by*

$$dH_t(\omega) = |A_t(\omega)|^2 d\mu(\omega)."$$

When $X(t)$ is stationary and $\theta(\omega) = \omega$ then $dH_t(\omega)$ reduces to the regular spectrum, $h(\omega)$.

$H_t(\omega)$ is the integrated time-frequency spectrum.

Assuming smoothness the *evolutionary spectral density* function is

$$h_t(\omega) = H_t'(\omega).$$

Evolutionary Power Spectra (Priestley, 1981, Def. 11.2.3)

"We define the evolutionary power spectrum at time t $dH_t(\omega)$ by

$$dH_t(\omega) = |A_t(\omega)|^2 d\mu(\omega)."$$

When $X(t)$ is stationary and $\theta(\omega) = \omega$ then $dH_t(\omega)$ reduces to the regular spectrum, $h(\omega)$.

$H_t(\omega)$ is the integrated time-frequency spectrum.

Assuming smoothness the *evolutionary spectral density* function is

$$h_t(\omega) = H_t'(\omega).$$

1. Use Priestley's (1965) 'double-window' estimator: $\hat{h}_t(\omega)$.
2. Define $Y(t,\omega) = \log \hat{h}_t(\omega)$.
3. Then, approximately, $\mathbb{E}\{Y(t,\omega)\} = \log h_t(\omega)$,
4. And, crucially, $\text{var}\{Y(t,\omega)\} = \sigma^2$.

In other words

$$Y(t,\omega) = \log h_t(\omega) + \epsilon(t,\omega),$$

which we can discretize over a set of times $t_1, \ldots, t_I$ and frequencies $\omega_1, \ldots, \omega_J$ to get the nice linear model:

$$H : Y_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ij},$$

in an obvious way.

Approximately $\epsilon_{ij} \overset{iid}{\sim} N(0,\sigma^2)$ if t_i, ω_j spaced out enough.

1. Use Priestley's (1965) 'double-window' estimator: $\hat{h}_t(\omega)$.
2. Define $Y(t, \omega) = \log \hat{h}_t(\omega)$.
3. Then, approximately, $\mathbb{E}\{Y(t, \omega)\} = \log h_t(\omega)$,
4. And, crucially, $\mathrm{var}\{Y(t, \omega)\} = \sigma^2$.

In other words

$$Y(t, \omega) = \log h_t(\omega) + \epsilon(t, \omega),$$

which we can discretize over a set of times $t_1, \ldots, t_I$ and frequencies $\omega_1, \ldots, \omega_J$ to get the nice linear model:

$$H: Y_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ij},$$

in an obvious way.

Approximately $\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$ if t_i, ω_j spaced out enough.

Stationarity Tests (Priestley & Subba Rao, 1969, JRSS-B)

1. Use Priestley's (1965) 'double-window' estimator: $\hat{h}_t(\omega)$.
2. Define $Y(t,\omega) = \log \hat{h}_t(\omega)$.
3. Then, approximately, $\mathbb{E}\{Y(t,\omega)\} = \log h_t(\omega)$,
4. And, crucially, $\mathrm{var}\{Y(t,\omega)\} = \sigma^2$.

In other words

$$Y(t,\omega) = \log h_t(\omega) + \epsilon(t,\omega),$$

which we can discretize over a set of times $t_1, \ldots, t_I$ and frequencies $\omega_1, \ldots, \omega_J$ to get the nice linear model:

$$H : Y_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ij}.$$

in an obvious way.

Approximately $\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$ if t_i, ω_j spaced out enough.

Stationarity Tests (Priestley & Subba Rao, 1969, JRSS-B)

1. Use Priestley's (1965) 'double-window' estimator: $\hat{h}_t(\omega)$.
2. Define $Y(t, \omega) = \log \hat{h}_t(\omega)$.
3. Then, approximately, $\mathbb{E}\{Y(t,\omega)\} = \log h_t(\omega)$,
4. And, crucially, $\mathrm{var}\{Y(t,\omega)\} = \sigma^2$.

In other words

$$Y(t,\omega) = \log h_t(\omega) + \epsilon(t,\omega),$$

which we can discretize over a set of times $t_1, \ldots, t_I$ and frequencies $\omega_1, \ldots, \omega_J$ to get the nice linear model:

$$H : Y_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ij}.$$

in an obvious way.

Approximately $\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$ if t_i, ω_j spaced out enough.

1. Use Priestley's (1965) 'double-window' estimator: $\hat{h}_t(\omega)$.
2. Define $Y(t,\omega) = \log \hat{h}_t(\omega)$.
3. Then, approximately, $\mathbb{E}\{Y(t,\omega)\} = \log h_t(\omega)$,
4. And, crucially, $\text{var}\{Y(t,\omega)\} = \sigma^2$.

In other words

$$Y(t,\omega) = \log h_t(\omega) + \epsilon(t,\omega),$$

which we can discretize over a set of times $t_1, \ldots, t_I$ and frequencies $\omega_1, \ldots, \omega_J$ to get the nice linear model:

$$H : Y_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ij}.$$

in an obvious way.

Approximately $\epsilon_{ij} \overset{iid}{\sim} N(0,\sigma^2)$ if t_i, ω_j spaced out enough.

Stationarity Tests (Priestley & Subba Rao, 1969, JRSS-B)

1. Use Priestley's (1965) 'double-window' estimator: $\hat{h}_t(\omega)$.
2. Define $Y(t,\omega) = \log \hat{h}_t(\omega)$.
3. Then, approximately, $\mathbb{E}\{Y(t,\omega)\} = \log h_t(\omega)$,
4. And, crucially, $\text{var}\{Y(t,\omega)\} = \sigma^2$.

In other words

$$Y(t,\omega) = \log h_t(\omega) + \epsilon(t,\omega),$$

which we can discretize over a set of times $t_1, \ldots, t_I$ and frequencies $\omega_1, \ldots, \omega_J$ to get the nice linear model:

$$H : Y_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ij}.$$

in an obvious way.

Approximately $\epsilon_{ij} \overset{iid}{\sim} N(0,\sigma^2)$ if t_i, ω_j spaced out enough.

Stationarity Tests (Priestley & Subba Rao, 1969, JRSS-B)

1. Use Priestley's (1965) 'double-window' estimator: $\hat{h}_t(\omega)$.
2. Define $Y(t, \omega) = \log \hat{h}_t(\omega)$.
3. Then, approximately, $\mathbb{E}\{Y(t, \omega)\} = \log h_t(\omega)$,
4. And, crucially, $\text{var}\{Y(t, \omega)\} = \sigma^2$.

In other words

$$Y(t, \omega) = \log h_t(\omega) + \epsilon(t, \omega),$$

which we can discretize over a set of times $t_1, \ldots, t_I$ and frequencies $\omega_1, \ldots, \omega_J$ to get the nice linear model:

$$H : Y_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ij}.$$

in an obvious way.

Approximately $\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$ if t_i, ω_j spaced out enough.

Test Procedure and Modern Implementation

Test stationarity by inferring whether $\alpha_i = 0$ and $\gamma_{ij} = 0$, $\forall i, j$.

Implementation: `stationarity()` in `fractal` R package.

Uses improved multitaper estimate: reduces bias.

`fractal` posted in 2007. By Bill Constantine & Donald Percival of the Applied Physics Laboratory, U of Washington, USA.

Thirty-eight years after the Priestley and Subba Rao paper!

Way ahead of their time!

p-value for Cardinham data using `stationarity()` is 9×10^{-10}.

Test Procedure and Modern Implementation

Test stationarity by inferring whether $\alpha_i = 0$ and $\gamma_{ij} = 0$, $\forall i, j$.

`Implementation:` `stationarity()` in `fractal` `R` package.

Uses improved multitaper estimate: reduces bias.

`fractal` posted in 2007. By Bill Constantine & Donald Percival of the Applied Physics Laboratory, U of Washington, USA.

Thirty-eight years after the Priestley and Subba Rao paper!

Way ahead of their time!

p-value for Cardinham data using `stationarity()` is 9×10^{-10}.

Test Procedure and Modern Implementation

Test stationarity by inferring whether $\alpha_i = 0$ and $\gamma_{ij} = 0$, $\forall i, j$.

`Implementation: stationarity()` in `fractal` `R` package.

Uses improved multitaper estimate: reduces bias.

`fractal` posted in 2007. By Bill Constantine & Donald Percival of the Applied Physics Laboratory, U of Washington, USA.

Thirty-eight years after the Priestley and Subba Rao paper!

Way ahead of their time!

p-value for Cardinham data using `stationarity()` is 9×10^{-10}.

Test Procedure and Modern Implementation

Test stationarity by inferring whether $\alpha_i = 0$ and $\gamma_{ij} = 0$, $\forall i, j$.

`Implementation:` `stationarity()` in `fractal` `R` package.

Uses improved multitaper estimate: reduces bias.

`fractal` posted in 2007. By Bill Constantine & Donald Percival of the Applied Physics Laboratory, U of Washington, USA.

Thirty-eight years after the Priestley and Subba Rao paper!

Way ahead of their time!

p-value for Cardinham data using `stationarity()` is 9×10^{-10}.

Test Procedure and Modern Implementation

Test stationarity by inferring whether $\alpha_i = 0$ and $\gamma_{ij} = 0$, $\forall i, j$.

`Implementation:` `stationarity()` in `fractal` `R` package.

Uses improved multitaper estimate: reduces bias.

`fractal` posted in 2007. By Bill Constantine & Donald Percival of the Applied Physics Laboratory, U of Washington, USA.

Thirty-eight years after the Priestley and Subba Rao paper!

Way ahead of their time!

p-value for Cardinham data using `stationarity()` is 9×10^{-10}.

Test Procedure and Modern Implementation

Test stationarity by inferring whether $\alpha_i = 0$ and $\gamma_{ij} = 0$, $\forall i, j$.

`Implementation: stationarity()` in `fractal` `R` package.

Uses improved multitaper estimate: reduces bias.

`fractal` posted in 2007. By Bill Constantine & Donald Percival of the Applied Physics Laboratory, U of Washington, USA.

Thirty-eight years after the Priestley and Subba Rao paper!

Way ahead of their time!

p-value for Cardinham data using `stationarity()` is 9×10^{-10}.

Test Procedure and Modern Implementation

Test stationarity by inferring whether $\alpha_i = 0$ and $\gamma_{ij} = 0$, $\forall i, j$.

`Implementation:` `stationarity()` in `fractal` `R` package.

Uses improved multitaper estimate: reduces bias.

`fractal` posted in 2007. By Bill Constantine & Donald Percival of the Applied Physics Laboratory, U of Washington, USA.

Thirty-eight years after the Priestley and Subba Rao paper!

Way ahead of their time!

p-value for Cardinham data using `stationarity()` is 9×10^{-10}.

STATISTICAL INFERENCE ON TIME SERIES BY HILBERT SPACE METHODS, I.

BY
EMANUEL PARZEN

TECHNICAL REPORT NO. 23
JANUARY 2, 1959

PREPARED UNDER CONTRACT Nonr-225(21)
(NR-042-993)
FOR
OFFICE OF NAVAL RESEARCH

APPLIED MATHEMATICS AND STATISTICS LABORATORY
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

Parzen and Priestley's Vision

"Parzen (1959) has pointed out that if there exists a representation $X(t) = \int \phi_t(\omega) dZ(\omega)$*, then there is a multitude of different representations of the process, each representation based on a different family of functions."*

"The situation is in some ways similar to the selection of a basis for a vector space."

"However, if the process is non-stationary this choice [complex exponential family] of family of functions is no longer valid."

Priestley (1981) p. 822.

Parzen and Priestley's Vision

"Parzen (1959) has pointed out that if there exists a representation $X(t) = \int \phi_t(\omega) dZ(\omega)$*, then there is a multitude of different representations of the process, each representation based on a different family of functions."*

"The situation is in some ways similar to the selection of a basis for a vector space."

"However, if the process is non-stationary this choice [complex exponential family] of family of functions is no longer valid."

Priestley (1981) p. 822.

Parzen and Priestley's Vision

"Parzen (1959) has pointed out that if there exists a representation $X(t) = \int \phi_t(\omega) dZ(\omega)$, then there is a multitude of different representations of the process, each representation based on a different family of functions."

"The situation is in some ways similar to the selection of a basis for a vector space."

"However, if the process is non-stationary this choice [complex exponential family] of family of functions is no longer valid."

Priestley (1981) p. 822.

Priestley's Oscillatory Functions

From p. 824 Priestley settles on $\theta(\omega) = \omega$ and

$$\phi_t(\omega) = A_t(\omega)e^{i\omega t}.$$

This captures uniformly modulated processes $A_t(\omega) = C(t)$.

Priestley's Oscillatory Functions

From p. 824 Priestley settles on $\theta(\omega) = \omega$ and

$$\phi_t(\omega) = A_t(\omega) e^{i\omega t}.$$

This captures uniformly modulated processes $A_t(\omega) = C(t)$.

Consider linear filter with frequency response function $\Gamma(\omega)$.

Stationary processes have useful property that $h^{(Y)}(\omega_1)$ is unaffected by $\omega \neq \omega_1$, i.e. $h^{(Y)}(\omega_1) = |\Gamma(\omega_1)|^2 h^{(X)}(\omega_1), \ldots$

Priestley mimics stationary case and approx. useful property.

He achieves this by $A_t(\omega)$ slowly evolving fn. of $t \implies$ semi-stationary processes.

Today, related to locally stationary processes; also has the advantage of permitting estimation.

Semi-Stationary Processes

Consider linear filter with frequency response function $\Gamma(\omega)$.

Stationary processes have useful property that $h^{(Y)}(\omega_1)$ is unaffected by $\omega \neq \omega_1$, i.e. $h^{(Y)}(\omega_1) = |\Gamma(\omega_1)|^2 h^{(X)}(\omega_1)$, . . .

Priestley mimics stationary case and approx. useful property.

He achieves this by $A_t(\omega)$ slowly evolving fn. of $t \implies$ semi-stationary processes.

Today, related to locally stationary processes; also has the advantage of permitting estimation.

Consider linear filter with frequency response function $\Gamma(\omega)$.

Stationary processes have useful property that $h^{(Y)}(\omega_1)$ is unaffected by $\omega \neq \omega_1$, i.e. $h^{(Y)}(\omega_1) = |\Gamma(\omega_1)|^2 h^{(X)}(\omega_1)$, ...

Priestley mimics stationary case and approx. useful property.

He achieves this by $A_t(\omega)$ slowly evolving fn. of $t \implies$ semi-stationary processes.

Today, related to locally stationary processes; also has the advantage of permitting estimation.

Semi-Stationary Processes

Consider linear filter with frequency response function $\Gamma(\omega)$.

Stationary processes have useful property that $h^{(Y)}(\omega_1)$ is unaffected by $\omega \neq \omega_1$, i.e. $h^{(Y)}(\omega_1) = |\Gamma(\omega_1)|^2 h^{(X)}(\omega_1)$, ...

Priestley mimics stationary case and approx. useful property.

He achieves this by $A_t(\omega)$ slowly evolving fn. of $t \implies$ semi-stationary processes.

Today, related to locally stationary processes; also has the advantage of permitting estimation.

Consider linear filter with frequency response function $\Gamma(\omega)$.

Stationary processes have useful property that $h^{(Y)}(\omega_1)$ is unaffected by $\omega \neq \omega_1$, i.e. $h^{(Y)}(\omega_1) = |\Gamma(\omega_1)|^2 h^{(X)}(\omega_1)$, ...

Priestley mimics stationary case and approx. useful property.

He achieves this by $A_t(\omega)$ slowly evolving fn. of $t \implies$ semi-stationary processes.

Today, related to locally stationary processes; also has the advantage of permitting estimation.

Heisenberg-Gabor uncertainty principle

"more accurately we estimate $h_t(\omega)$ as a function of time the less accurately we can determine it as a function of frequency, "

Priestley, 1981, p. 835 (Daniells, 1965 and Tjøstheim, 1976).

To estimate time-varying behaviour, we will necessarily have to sacrifice some frequency resolution.

In some situations 'local Fourier' highly inappropriate

Heisenberg-Gabor uncertainty principle

"more accurately we estimate $h_t(\omega)$ as a function of time the less accurately we can determine it as a function of frequency, "

Priestley, 1981, p. 835 (Daniells, 1965 and Tjøstheim, 1976).

To estimate time-varying behaviour, we will necessarily have to sacrifice some frequency resolution.

In some situations 'local Fourier' highly inappropriate

Heisenberg-Gabor uncertainty principle

"more accurately we estimate $h_t(\omega)$ as a function of time the less accurately we can determine it as a function of frequency, "

Priestley, 1981, p. 835 (Daniells, 1965 and Tjøstheim, 1976).

To estimate time-varying behaviour, we will necessarily have to sacrifice some frequency resolution.

In some situations 'local Fourier' highly inappropriate

Other Possibilities

Single Bases:

- Fourier

Other Possibilities

Single Bases:

- Fourier (of course).

Other Possibilities

Single Bases:

- Fourier (of course).
  $T/2$ frequencies, T time steps

Other Possibilities

Single Bases:

- Fourier (of course).
  $T/2$ frequencies, T time steps
- Wavelets, $\mathcal{O}(N)$.

Other Possibilities

Single Bases:

- Fourier (of course).
  $T/2$ frequencies, T time steps
- Wavelets, $\mathcal{O}(N)$.
  $\log T$ frequency bands, T time steps

Other Possibilities

Single Bases:

- Fourier (of course).
  $T/2$ frequencies, T time steps
- Wavelets, $\mathcal{O}(N)$.
  $\log T$ frequency bands, T time steps
- Walsh series

Other Possibilities

Single Bases:

- Fourier (of course).
  $T/2$ frequencies, T time steps
- Wavelets, $\mathcal{O}(N)$.
  $\log T$ frequency bands, T time steps
- Walsh series

Libraries or Dictionaries of Bases (richer signal analysis)

Other Possibilities

Single Bases:

- Fourier (of course).
  $T/2$ frequencies, T time steps
- Wavelets, $\mathcal{O}(N)$.
  $\log T$ frequency bands, T time steps
- Walsh series

Libraries or Dictionaries of Bases (richer signal analysis)

- Local Cosine or Sine bases

Other Possibilities

Single Bases:

- Fourier (of course).
  $T/2$ frequencies, T time steps
- Wavelets, $\mathcal{O}(N)$.
  $\log T$ frequency bands, T time steps
- Walsh series

Libraries or Dictionaries of Bases (richer signal analysis)

- Local Cosine or Sine bases
- Wavelet packet library

Other Possibilities

Single Bases:

- Fourier (of course).
  $T/2$ frequencies, T time steps
- Wavelets, $\mathcal{O}(N)$.
  $\log T$ frequency bands, T time steps
- Walsh series

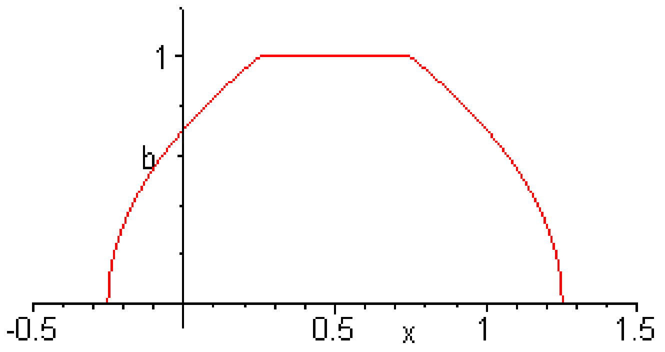
Libraries or Dictionaries of Bases (richer signal analysis)

- Local Cosine or Sine bases
- Wavelet packet library

"Adapted waveform analysis" of Coifman, fast $\mathcal{O}(N \log N)$ transforms.

Bell Function for Local Cosine Bases

Figure 7.4 A bell used for a local cosine basis



37

Local Cosine Basis

"Let $\{a_k\}$ *be a sequence of real numbers and* $\{\epsilon_k\}$ *of positive numbers such that* $a_{\pm k} \to \pm\infty$ *and*

$$a_k + \epsilon_k < a_{k+1} - \epsilon_{k+1};$$

let $b_k(x)$ *be the* $(\epsilon_k, \epsilon_{k+1})$ *bell over* $[a_k, a_{k+1}]$*; then* $\{u_{k,j}\}$ *where*

$$u_{k,j}(x) = \{2/(a_{k+1} - a_k)\}^{1/2}\, b_k(x) \cos\left\{\frac{(2j+1)\pi(x - a_k)}{2(a_{k+1} - a_k)}\right\},$$

$k \in \mathbb{Z}$, $j \in \mathbb{N}$ *is an orthonormal basis of* $L^2(\mathbb{R})$*."*

Walter and Shen (2001) Theorem 7.3
(Originally Coifman and Meyer (1991)).

Local Cosine Basis

"Let $\{a_k\}$ be a sequence of real numbers and $\{\epsilon_k\}$ of positive numbers such that $a_{\pm k} \to \pm\infty$ and

$$a_k + \epsilon_k < a_{k+1} - \epsilon_{k+1};$$

let $b_k(x)$ be the $(\epsilon_k, \epsilon_{k+1})$ bell over $[a_k, a_{k+1}]$; then $\{u_{k,j}\}$ where

$$u_{k,j}(x) = \{2/(a_{k+1} - a_k)\}^{1/2}\, b_k(x) \cos\left\{\frac{(2j+1)\pi(x - a_k)}{2(a_{k+1} - a_k)}\right\},$$

$k \in \mathbb{Z}$, $j \in \mathbb{N}$ is an orthonormal basis of $L^2(\mathbb{R})$."

Walter and Shen (2001) Theorem 7.3

Local Cosine Basis

"Let $\{a_k\}$ be a sequence of real numbers and $\{\epsilon_k\}$ of positive numbers such that $a_{\pm k} \to \pm\infty$ and

$$a_k + \epsilon_k < a_{k+1} - \epsilon_{k+1};$$

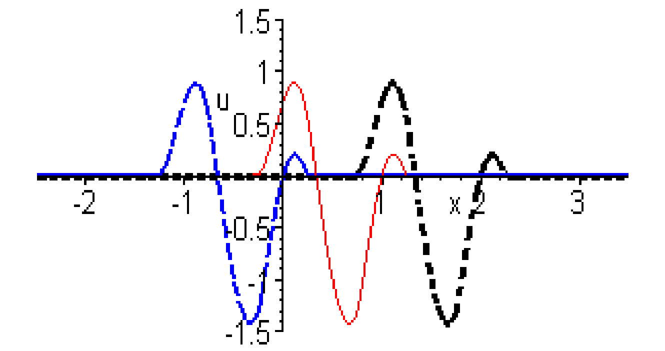
let $b_k(x)$ be the $(\epsilon_k, \epsilon_{k+1})$ bell over $[a_k, a_{k+1}]$; then $\{u_{k,j}\}$ where

$$u_{k,j}(x) = \{2/(a_{k+1} - a_k)\}^{1/2}\, b_k(x) \cos\left\{\frac{(2j+1)\pi(x - a_k)}{2(a_{k+1} - a_k)}\right\},$$

$k \in \mathbb{Z}$, $j \in \mathbb{N}$ is an orthonormal basis of $L^2(\mathbb{R})$."

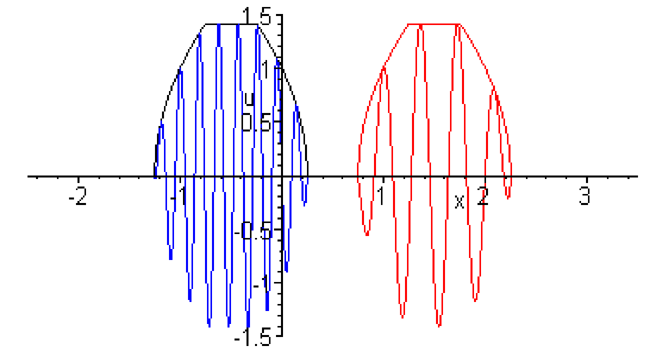
Walter and Shen (2001) Theorem 7.3

Figure 7.5 Three elements in the local cosine basis with bell of Figure 7.4



38

Figure 7.6 Two additional elements of the local cosine basis showing the bell



39

Different Families . . .

Back to Priestley:

> *"there is a multitude of different representations of the process, each representation based on a different family of functions."*

So, there are many things we might try.

Not all of them are oscillatory functions, or we don't know

E.g. wavelets, Locally Stationary Wavelet Processes:

$$X_t = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} w_{j,k} \psi_{j,k-t} \xi_{j,k},$$

$w_{j,k}$ = amplitude, ψ = oscillation, ξ = randomness.

Different Families . . .

Back to Priestley:

> *"there is a multitude of different representations of the process, each representation based on a different family of functions."*

So, there are many things we might try.

Not all of them are oscillatory functions, or we don't know

E.g. wavelets, Locally Stationary Wavelet Processes:

$$X_t = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} w_{j,k} \psi_{j,k-t} \xi_{j,k},$$

$w_{j,k}$ = amplitude, ψ = oscillation, ξ = randomness.

Different Families ...

Back to Priestley:

> *"there is a multitude of different representations of the process, each representation based on a different family of functions."*

So, there are many things we might try.

Not all of them are oscillatory functions, or we don't know

E.g. wavelets, Locally Stationary Wavelet Processes:

$$X_t = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} w_{j,k} \psi_{j,k-t} \xi_{j,k},$$

$w_{j,k}$ = amplitude, ψ = oscillation, ξ = randomness.

Back to Priestley:

> *"there is a multitude of different representations of the process, each representation based on a different family of functions."*

So, there are many things we might try.

Not all of them are oscillatory functions, or we don't know

E.g. wavelets, Locally Stationary Wavelet Processes:

$$X_t = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} w_{j,k} \psi_{j,k-t} \xi_{j,k},$$

$w_{j,k}$ = amplitude, ψ = oscillation, ξ = randomness.

A Wavelet Based Test for Stationarity (Nason 2013)

Use raw wavelet periodogram, $I_{j,k} = d_{j,k}^2$

where $d_{j,k} = \sum_t X_t \psi_{j,k-t}$

Time-scale analogue of regular periodogram.

Define $\beta_j(z) = \mathbb{E} I_{j,k}$, where $z = k/T$

Under stationarity H_0 function $\beta_j(z)$ is constant.

In mind locally stationary wavelet process alternative, but not necessary

A Wavelet Based Test for Stationarity (Nason 2013)

Use raw wavelet periodogram, $I_{j,k} = d_{j,k}^2$

where $d_{j,k} = \sum_t X_t \psi_{j,k-t}$

Time-scale analogue of regular periodogram.

Define $\beta_j(z) = \mathbb{E} I_{j,k}$, where $z = k/T$

Under stationarity H_0 function $\beta_j(z)$ is constant.

In mind locally stationary wavelet process alternative, but not necessary

A Wavelet Based Test for Stationarity (Nason 2013)

Use raw wavelet periodogram, $I_{j,k} = d_{j,k}^2$

where $d_{j,k} = \sum_t X_t \psi_{j,k-t}$

Time-scale analogue of regular periodogram.

Define $\beta_j(z) = \mathbb{E} I_{j,k}$, where $z = k/T$

Under stationarity H_0 function $\beta_j(z)$ is constant.

In mind locally stationary wavelet process alternative, but not necessary

A Wavelet Based Test for Stationarity (Nason 2013)

Use raw wavelet periodogram, $I_{j,k} = d_{j,k}^2$

where $d_{j,k} = \sum_t X_t \psi_{j,k-t}$

Time-scale analogue of regular periodogram.

Define $\beta_j(z) = \mathbb{E} I_{j,k}$, where $z = k/T$

Under stationarity H_0 function $\beta_j(z)$ is constant.

In mind locally stationary wavelet process alternative, but not necessary

A Wavelet Based Test for Stationarity (Nason 2013)

Use raw wavelet periodogram, $I_{j,k} = d_{j,k}^2$

where $d_{j,k} = \sum_t X_t \psi_{j,k-t}$

Time-scale analogue of regular periodogram.

Define $\beta_j(z) = \mathbb{E} I_{j,k}$, where $z = k/T$

Under stationarity H_0 function $\beta_j(z)$ is constant.

In mind locally stationary wavelet process alternative, but not necessary

A Wavelet Based Test for Stationarity (Nason 2013)

Use raw wavelet periodogram, $I_{j,k} = d^2_{j,k}$

where $d_{j,k} = \sum_t X_t \psi_{j,k-t}$

Time-scale analogue of regular periodogram.

Define $\beta_j(z) = \mathbb{E} I_{j,k}$, where $z = k/T$

Under stationarity H_0 function $\beta_j(z)$ is constant.

In mind locally stationary wavelet process alternative, but not necessary

A Wavelet Based Test for Stationarity, 2

Use Neumann and von Sachs (2000) to test constancy of $\beta_j(z)$

Uses Haar wavelet coefficients of $I_{j,k}$ as fn. of k, which are $\hat{v}_{\ell,m}$

Test $H_0 : v_{\ell,m} = 0$ for all ℓ, m, asymptotic Gaussian theory

Use multiple test control, Bonferroni, FDR

R package `locits` contains the software

Use Neumann and von Sachs (2000) to test constancy of $\beta_j(z)$

Uses Haar wavelet coefficients of $I_{j,k}$ as fn. of k, which are $\hat{v}_{\ell,m}$

Test $H_0 : v_{\ell,m} = 0$ for all ℓ, m, asymptotic Gaussian theory

Use multiple test control, Bonferroni, FDR

R package `locits` contains the software

A Wavelet Based Test for Stationarity, 2

Use Neumann and von Sachs (2000) to test constancy of $\beta_j(z)$

Uses Haar wavelet coefficients of $I_{j,k}$ as fn. of k, which are $\hat{v}_{\ell,m}$

Test $H_0 : v_{\ell,m} = 0$ for all ℓ, m, asymptotic Gaussian theory

Use multiple test control, Bonferroni, FDR

R package `locits` contains the software

A Wavelet Based Test for Stationarity, 2

Use Neumann and von Sachs (2000) to test constancy of $\beta_j(z)$

Uses Haar wavelet coefficients of $I_{j,k}$ as fn. of k, which are $\hat{v}_{\ell,m}$

Test $H_0 : v_{\ell,m} = 0$ for all ℓ, m, asymptotic Gaussian theory

Use multiple test control, Bonferroni, FDR

R package `locits` contains the software

A Wavelet Based Test for Stationarity, 2

Use Neumann and von Sachs (2000) to test constancy of $\beta_j(z)$

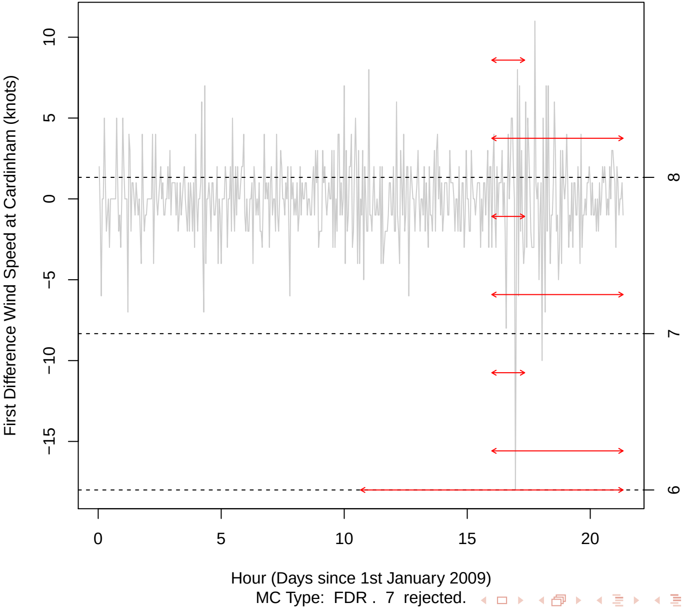
Uses Haar wavelet coefficients of $I_{j,k}$ as fn. of k, which are $\hat{v}_{\ell,m}$

Test $H_0 : v_{\ell,m} = 0$ for all ℓ, m, asymptotic Gaussian theory

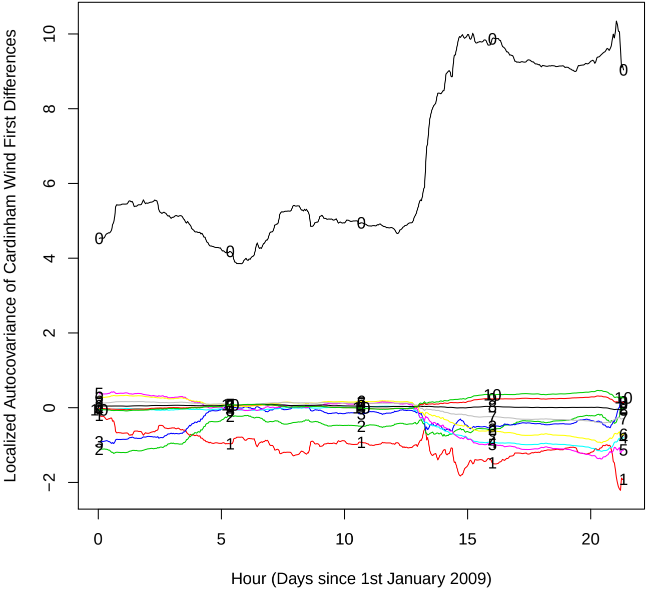
Use multiple test control, Bonferroni, FDR

R package `locits` contains the software

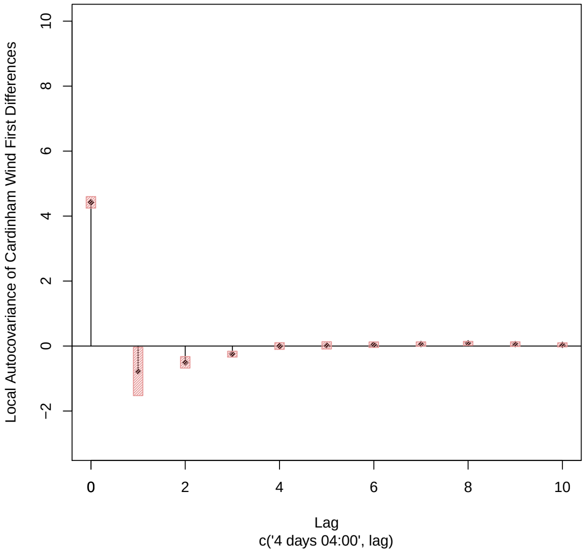
Wavelet Test of Stationarity on Cardinham 1st Diffs



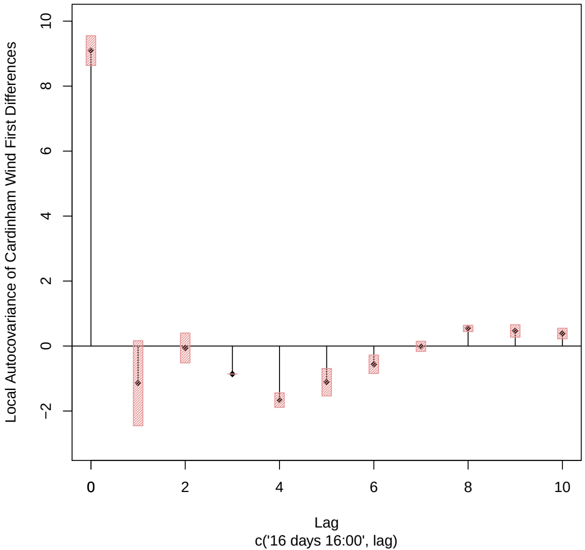
Cardinham Localized Autocovariance



Localized Autocovariance for Cardinham: 4 days 0400



Localized Autocovariance for Cardinham: 16 days 1600



- Nonstationary time series models
- Oscillatory & Semi-Stationary processes
- Multitude of representations, which one?
- Wavelets, Computational Harmonic Analysis
- Essential for picking up alternatives.
- Priestley: major contributions to statistics and time series.

Summary

- Nonstationary time series models
- Oscillatory & Semi-Stationary processes
- Multitude of representations, which one?
- Wavelets, Computational Harmonic Analysis
- Essential for picking up alternatives.
- Priestley: major contributions to statistics and time series.

Summary

- Nonstationary time series models
- Oscillatory & Semi-Stationary processes
- Multitude of representations, which one?
- Wavelets, Computational Harmonic Analysis
- Essential for picking up alternatives.
- Priestley: major contributions to statistics and time series.

Summary

- Nonstationary time series models
- Oscillatory & Semi-Stationary processes
- Multitude of representations, which one?
- Wavelets, Computational Harmonic Analysis
- Essential for picking up alternatives.
- Priestley: major contributions to statistics and time series.

Summary

- Nonstationary time series models
- Oscillatory & Semi-Stationary processes
- Multitude of representations, which one?
- Wavelets, Computational Harmonic Analysis
- Essential for picking up alternatives.
- Priestley: major contributions to statistics and time series.

- Nonstationary time series models
- Oscillatory & Semi-Stationary processes
- Multitude of representations, which one?
- Wavelets, Computational Harmonic Analysis
- Essential for picking up alternatives.
- Priestley: major contributions to statistics and time series.