ARMA and GARCH Connections

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First, A Little History from 1970…

My first seminar invitation came via Maurice Priestly and was at UMIST on the 25th February 1970

The topic was

'Selective interaction of point processes"

This was based on a Chapter of my PhD thesis about stochastic point processes on the line

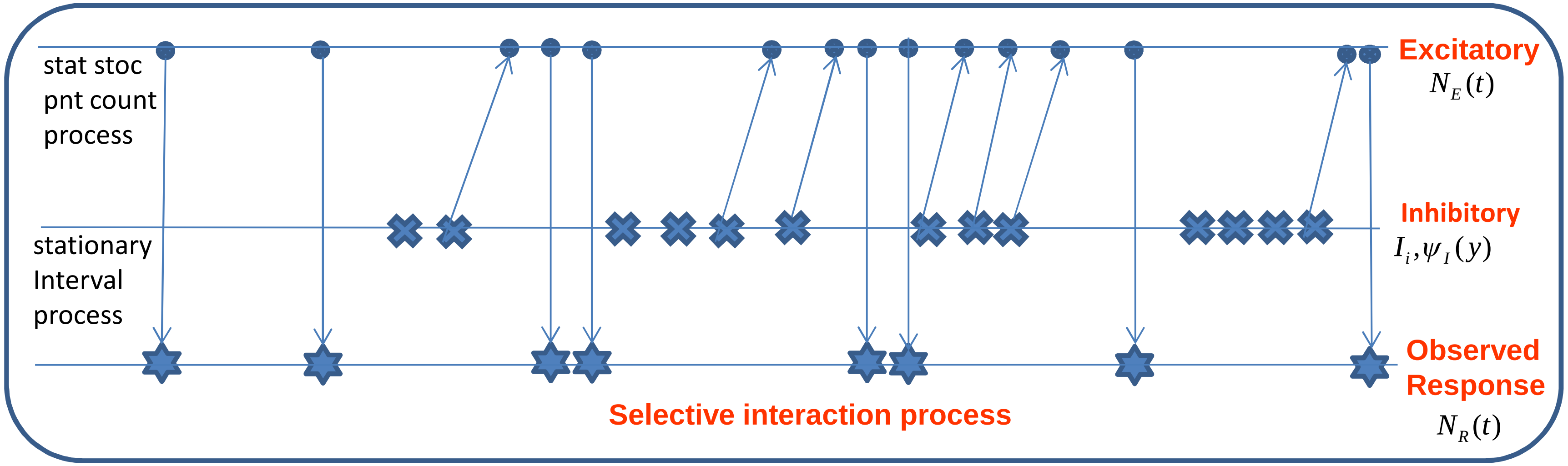
The selective interaction model was introduced by the neurophysiologists Ten Hoopen and Reuver (1965, 1967) to explain multi-modal inter-spike distributions for dark firing of lateral geniculate neurons observed by Bishop et al (1964)

I explored it as an applied probability model, but I should also have followed up on the statistical aspects, contacting the experimenters, analysing their data and doing simulations

The thesis was concerned with the general theory and results for particular point processes on the line – spatial point processes hardly touched on at the time

I focussed on superposition, interactions and branching behaviour, all effects which gave dependent 'arbitrary' intervals between events, and where stationary initial conditions were problematical

The Selective Interaction Neuron Firing Model



The model was motivated by a *multi-modal distribution of times between the responses*', in the 'spike trains' of observed neuron firings – convolutions of excitatory intervals

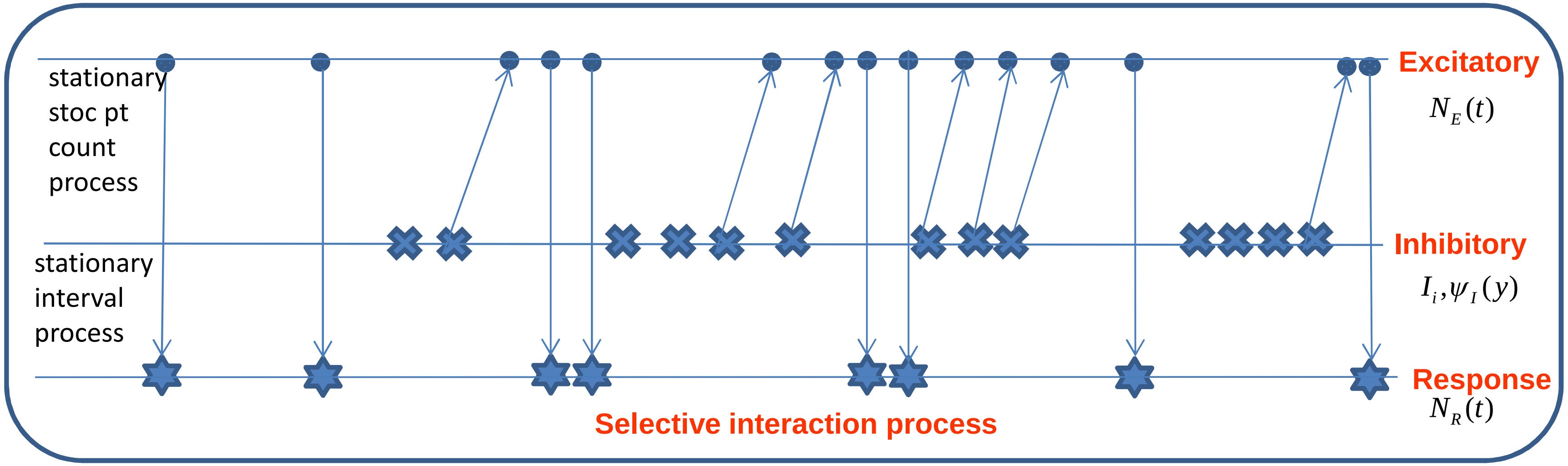
Poisson excitatory results by calculation – in my thesis

General results by appealing to the compound distribution structure of the observed response count, resulting in

$$N_R(t) = N_E(t) - \sum_{i=1}^{N_I(t)} \delta_i, \quad \delta_i = 1 \; with \; P\{N_E(I_i) \geq 1\}, = 0 \; otherwise$$

Continued, (more detailed Poisson excitatory results in 1970…)

$$N_R(t) = N_E(t) - \sum_{i=1}^{N_I(t)} \delta_i, \quad \delta_i = 1 \; with \; prob = P\{N_E(I_i) \geq 0\} = 1 - P\{N_E(I_i) = 0\}, = 0 \; otherwise$$



It follows

$$E\{N_R(t)\} = \left[\lambda_E - \lambda_I \int_{y=0}^{\infty} \Pr\{N_E(y) \geq 1\} \psi_I(y) dy\right] t$$

and approximately (?) via compound distribution results

$$\text{var}\{N_R(t)\} \sim [\sigma_E + \sigma_I E(\delta) + \lambda_I \text{ var}(\delta)]t \qquad \text{rates } \sigma_E, \sigma_I$$

Compounding the exciting process intervals using the inhibitory process to get the inter-response distribution is more difficult…

For more detailed results when the excitatory process is Poisson, see my 4 JAP papers in the 70's. No model fitting, no simulations – what a pity ! Last cited in 1996

and now for something different…

ARMA and GARCH Connections

Introductory Discussion

The connection between squared GARCH variables and ARMA variables has been noted in various places, starting with Bollerslev (1986) who reports the observation by Pantula and an anonymous referee that…

'the model structure of the squared-variable generated by a GARCH model is of ARMA form with uncorrelated innovations'

'model structure' is an important qualification here…

Other mentions include Tsay (2002), Fan & Yao (2003) and Lai & Xing (2008) who note that the innovations of the ARMA process is a martingale difference sequence – but not too useful to me…

A few other references seem (incorrectly) to imply that the equivalence is exact

Emphasis here is on:

Volatility structure of ARMA innovations
Use of ARMA structure in GARCH prediction
Extending ARMA models to be volatile

The ARCH(1) Model and its Equivalent AR(1) Model – so easy…

$$X_t = \sigma_t \varepsilon_{t,}\ \sigma_t = \sqrt{\alpha_0 + \alpha_1 X_{t-1}^2}\ ,\ \varepsilon_t \sim D(0,1)$$

Square the first equation and write as

$$X_t^2 = \sigma_t^2 + \sigma_t^2 (\varepsilon_t^2 - 1) \qquad *$$

Very easy but the main idea !

Standardize the variables $\tilde{X}_t^2 = X_t^2 - \mu_{X^2}$ and $\tilde{\varepsilon}_t^2 = \varepsilon_t^2 - 1$, then

$$\sigma_t^2 = \mu_{X^2} + \alpha_1 \tilde{X}_{t-1}^2$$

and define

$$\tilde{E}_t = (\mu_{X^2} + \alpha_1 \tilde{X}_{t-1}^2)\tilde{\varepsilon}_t^2$$

Combining above two results in * gives the autoregressive AR(*1*) structure

$$\tilde{X}_t^2 = \alpha_1 \tilde{X}_{t-1}^2 + \tilde{E}_t\ ,\ \tilde{E}_t = (\mu_{X^2} + \alpha_1 \tilde{X}_{t-1}^2)\tilde{\varepsilon}_t^2$$

Note the volatile dependent innovations – but uncorrelated as with linear AR(1) model

and extensions…

the ARCH(q) process

$$X_t = \sigma_t \varepsilon_t,\ \sigma_t = \left(\alpha_0 + \sum_{i=1}^{q} \alpha_i X_{t-i}^2 \right)^{1/2},\ \varepsilon_t \sim D(0,1)$$

has the AR(q) structure

$$\tilde{X}_t^2 = \sum_{i=1}^{q} \alpha_i \tilde{X}_{t-i}^2 + \tilde{E}_t,\ \tilde{E}_t = (\mu_{X^2} + \sum_{i=1}^{q} \alpha_i \tilde{X}_{t-i}^2)\tilde{\varepsilon}_t^2$$

The 'popular' GARCH(*1,1*) model…

$$X_t = \sigma_t \varepsilon_t,\ \sigma_t = \sqrt{\alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2},\ \varepsilon_t \sim D(0,1)$$

has the ARMA(*1,1*) structure

$$\tilde{X}_t^2 = (\alpha_1 + \beta_1)\tilde{X}_{t-1}^2 + \tilde{E}_t - \beta_1 \tilde{E}_{t-1},\ \ \tilde{E}_t = (\mu_{X^2} + \sum_{i=0}^{\infty} \alpha_1 \beta_1^i \tilde{X}_{t-1-i}^2)\tilde{\varepsilon}_t^2$$

'conditions apply'

or the AR(∞) structure

$$X_t^2 = \sigma_t^2 = \mu_{X^2} + \sum_{i=1}^{\infty} \alpha_1 \beta_1^{i-1} \tilde{X}_{t-i}^2 + \left\{ \mu_{X^2} + \sum_{i=0}^{\infty} \alpha_1 \beta_1^i \tilde{X}_{t-1-i}^2 \right\} \tilde{\varepsilon}_t^2$$

and the final generality…

The GARCH (*q*,*r*) Model

$$X_t^2 = \sigma_t^2 \varepsilon_t^2, \;\; \sigma_t^2 = \alpha_0 + \alpha_q(B) X_t^2 + \beta_r(B)\sigma_t^2$$

$$\alpha_q(B) = \sum_{i=1}^{q} \alpha_i B^i, \;\; \beta_r(B) = \sum_{j=1}^{r} \beta_j B^j$$

The ARMA structure of this model is

$$X_t^2 = \alpha_0 + \{\alpha_q(B) + \beta_r(B)\} X_t^2 + \{1 - \beta_r(B)\} E_t$$

$$E_t = \left(\alpha_0 \left(1 - \sum_{j=1}^{r} \beta_j \right)^{-1} + \alpha_q(B) \{1 - \beta_r(B)\}^{-1} X_t^2 \right) \tilde{\varepsilon}_t^2$$

'conditions apply'

which is ARMA(max(*q*,*r*), *r*)), and there is also an AR(∞) form

The I-GARCH(*q*,*r*) structure requires

$$1 - \alpha_q(B) - \beta_r(B) \equiv (1 - B)\phi_{q \wedge r-1}(B)$$

Remarks on Prediction for X_t^2

Cannot predict $X_{t+k} \mid X_t, X_{t-1}, \ldots$ for GARCH models – they have no linear component

Can predict $X_{t+k}^2 \mid X_t^2, X_{t-1}^2, \ldots$ and $\sigma_{t+k}^2 \mid \sigma_t^2, \sigma_{t-1}^2, \ldots$

Autoregressive structure (not with MA component) provides a route to results for *predictive mean and predictive variance in squared prediction*

$$\chi_{t+k|t}^2 = E(\tilde{X}_{t+k}^2 \mid \tilde{X}_t, \tilde{X}_{t-1}, \ldots), \quad \upsilon_{t+k|t}^2 = \mathrm{var}\left\{ \tilde{X}_{t+k}^2 \middle| \tilde{X}_t, \tilde{X}_{t-1}, \ldots \right\}$$

For ARCH(1) case, using AR(1) for squares, and not surprisingly, the *k*-step predictor is

$$\chi_{t+k|t}^2 = \alpha_1^k \tilde{X}_t^2$$

and for predictive variances, *k=1, k=2* steps ahead, AR results lead to

$$\upsilon_{t+1|t}^2 = \left(\mu_{X^2} + \alpha_1 \tilde{X}_t^2 \right)^2 \mathrm{var}(\tilde{\varepsilon}_{t+1}^2)$$

$$\upsilon_{t+2|t}^2 = \alpha_1^2 \left(\mu_{X^2} + \alpha_1 \tilde{X}_t^2 \right)^2 \mathrm{var}(\tilde{\varepsilon}_{t+1}^2) + E_{|X_t} \left(\mu_{X^2} + \alpha_1 \tilde{X}_{t+1}^2 \right)^2 \left\{ \mathrm{var}(\tilde{\varepsilon}_{t+2}^2) \right\}$$

Note how ***prediction origin*** value X_t^2 affects ***the width of predictive intervals***

prediction, continued, GARCH(1,1)

$$X_t = \sigma_t \varepsilon_{t,}\ \sigma_t = \sqrt{\alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2}\ ,\ \varepsilon_t \sim D(0,1)$$

From ARMA(*1,1*), AR(∞) structures

$$\tilde{X}_t^2 = (\alpha_1 + \beta_1)\tilde{X}_{t-1}^2 + \tilde{E}_t - \beta_1 \tilde{E}_{t-1}, \qquad \tilde{X}_{t+1}^2 = \sum_{i=0}^{\infty} \alpha_1 \beta_1^i X_{t-i}^2 + \tilde{E}_{t+1}$$

From the AR(∞) form there are the one-step prediction results

$$\chi_{t+1|t}^2 = E(\tilde{X}_{t+1}^2 \mid \tilde{X}_t, \tilde{X}_{t-1}, \ldots) = \sum_{i=0}^{\infty} \alpha_1 \beta_1^i \tilde{X}_{t-i}^2$$

$$\upsilon_{t+1|t}^2 = \operatorname{var}\left\{ \tilde{X}_{t+1}^2 \middle| \tilde{X}_t, \tilde{X}_{t-1}, \ldots \right\} = \left(\mu_{X^2} + \sum_{i=0}^{\infty} \alpha_1 \beta_1^i \tilde{X}_{t-i}^2 \right)^2 \operatorname{var}(\tilde{\varepsilon}_{t+1}^2)$$

The predictor is noted as an exponential smooth of all past values with non-constant width prediction interval which is also depends on the exponential smooth

k-step predictor results similarly, prediction interval results more complicated

Suggestions for volatile ARMA families…

ARMA's with GARCH-like error structures – two possibilities

$$X_t = \phi_0 + \sum_{i=1}^{p} \phi_i X_{t-i} + E_t - \sum_{i=1}^{q} \psi_i E_{t-i}, \quad E_t = \left(\alpha_0 + \sum_{i=1}^{r} \alpha_i \left| X_{t-i} \right| \right) \varepsilon_t$$

Not investigated ?

$$X_t = \phi_0 + \sum_{i=1}^{p} \phi_i X_{t-1} + E_t - \sum_{i=1}^{q} \psi_i E_{t-i} \quad E_t = \left(\alpha_0 + \sum_{i=1}^{r} \alpha_i X_{t-i}^2 \right)^{1/2} \varepsilon_t$$

Generalization of Ling(2004)

An enquiry – are there any explicit results for the **full stationarity of GARCH processes**, apart from the GARCH(1,1) which include ARCH(1)

There is a numerical implementation of the general results of Bougerol & Picard (1992) in the book of Francq & Zakoian for ARCH(2), but little else

Thanks for sitting there

References…

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