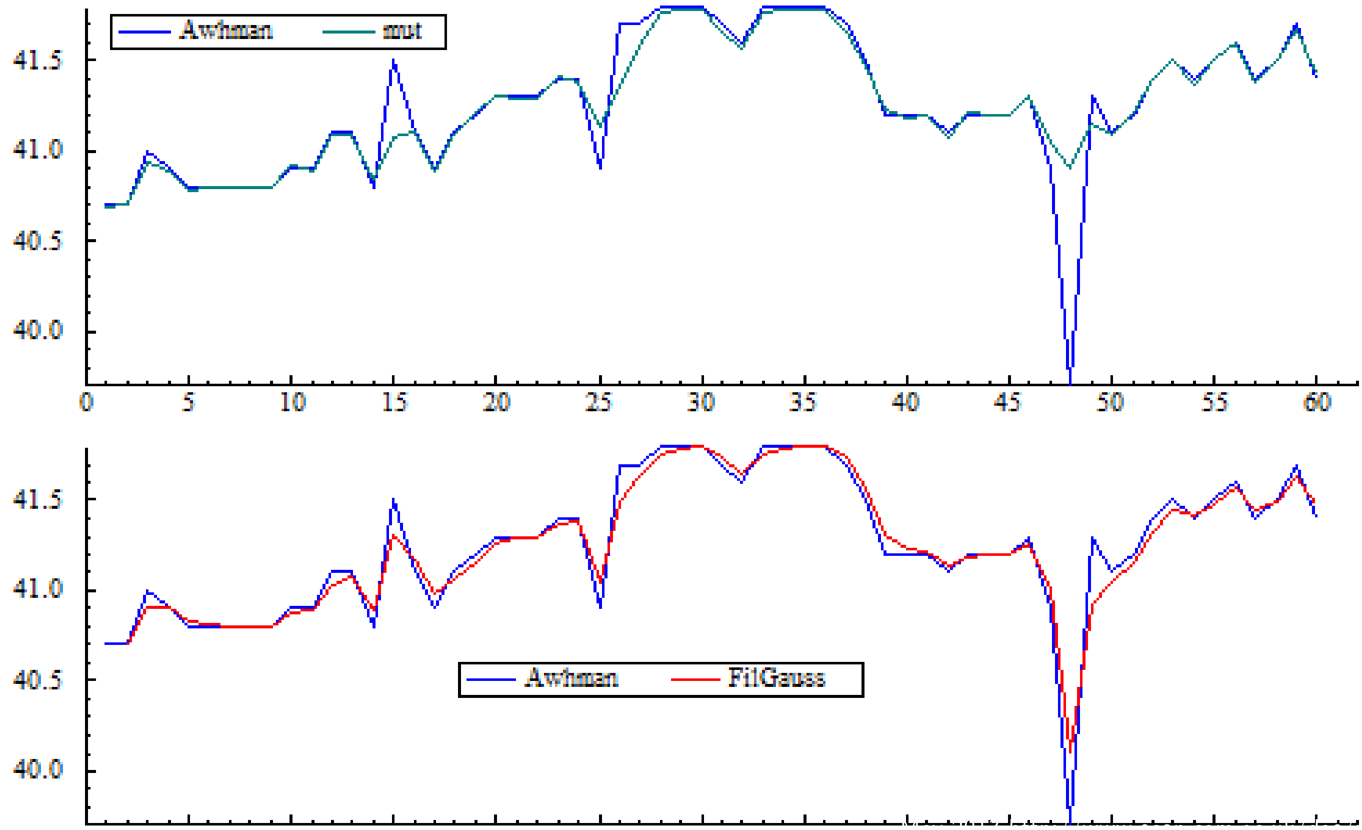
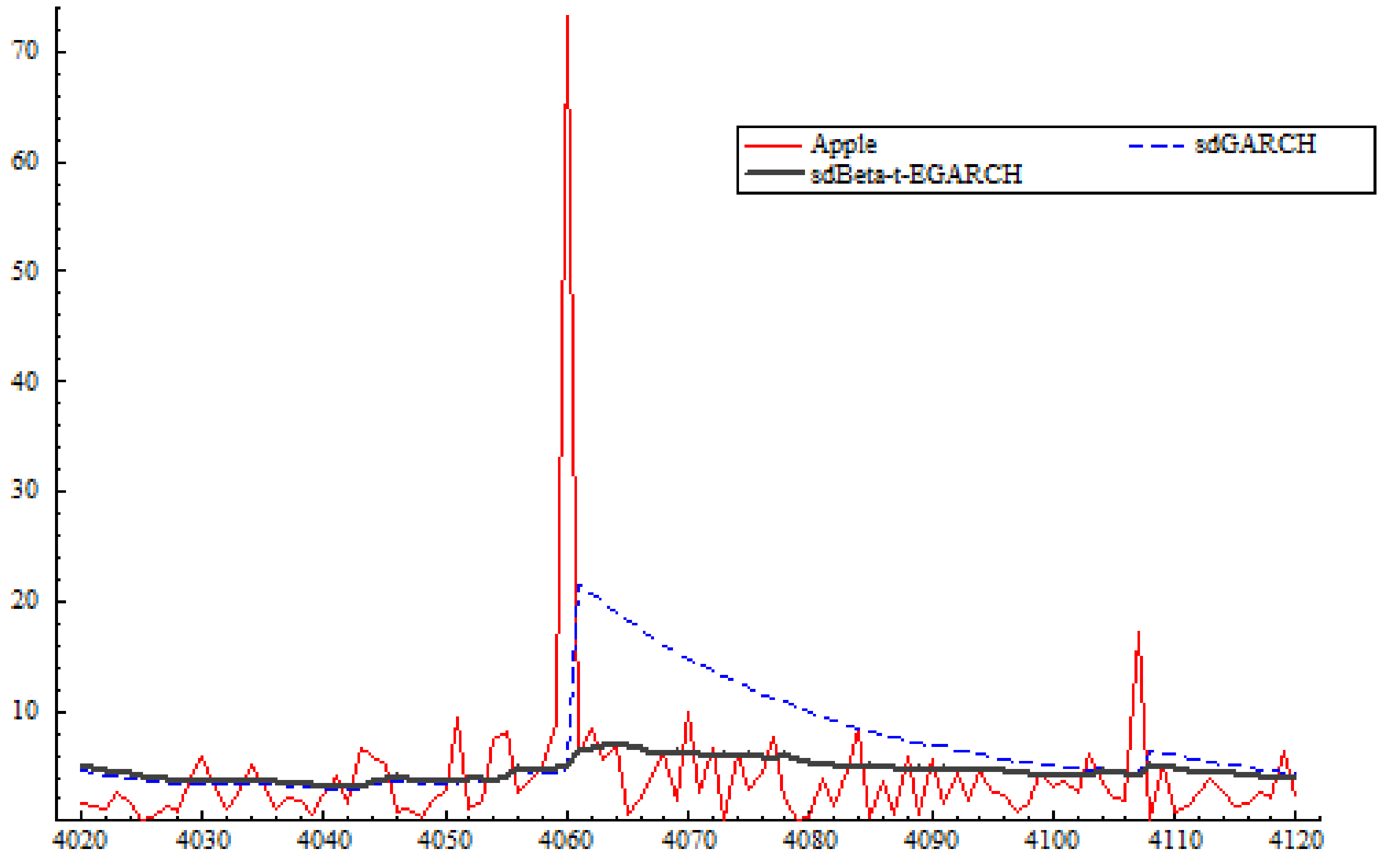
Robust Time Series Models

Andrew Harvey (ach34@cam.ac.uk)

Faculty of Economics, University of Cambridge
Based on joint papers with and Alessandra Luati (Bologna)
and Michele Caivano (Bank of Italy)

May 2013
http://www.econ.cam.ac.uk/dae/repec/cam/pdf/cwpe1255.pdf





Introduction to dynamic conditional score (DCS) models

A guiding principle is **signal extraction**. When combined with basic ideas of maximum likelihood estimation, the signal extraction approach leads to models which, in contrast to many in the literature, are relatively simple in their form and yield analytic expressions for their principal features.

For estimating location, DCS models are closely related to the unobserved components (UC) models described in Harvey (1989). Such models can be handled using state space methods and they are easily accessible using the STAMP package of Koopman et al (2008).

For estimating scale, the models are close to stochastic volatility (SV) models, where the variance is treated as an unobserved component.

Unobserved component models

A simple Gaussian signal plus noise model is

$$y_t = \mu_t + \varepsilon_t, \quad \varepsilon_t \sim NID\left(0, \sigma_\varepsilon^2\right), \quad t = 1, ..., T$$

$$\mu_{t+1} = \phi\mu_t + \eta_t, \quad \eta_t \sim NID(0, \sigma_\eta^2),$$

where the irregular and level disturbances, ε_t and η_t, are mutually independent. The AR parameter is ϕ, while the **signal-noise ratio**, $q = \sigma_\eta^2/\sigma_\varepsilon^2$, plays the key role in determining how observations should be weighted for prediction and signal extraction.
The reduced form (RF) is an ARMA(1,1) process

$$y_t = \phi y_{t-1} + \xi_t - \theta\xi_{t-1}, \quad \xi_t \sim NID\left(0, \sigma^2\right),$$

but with restrictions on θ. For example, when $\phi = 1$, $0 \leq \theta \leq 1$. The forecasts from the UC model and RF are the same.

Unobserved component models

The UC model is effectively in state space form (SSF) and, as such, it may be handled by the Kalman filter (KF). The parameters ϕ and q can be estimated by ML, with the likelihood function constructed from the one-step ahead prediction errors.
The KF can be expressed as a single equation. Writing this equation together with an equation for the one-step ahead prediction error, v_t, gives the innovations form (IF) of the KF:

$$\begin{aligned} y_t &= \mu_{t|t-1} + v_t \\ \mu_{t+1|t} &= \phi\mu_{t|t-1} + k_t v_t \end{aligned}$$

The Kalman gain, k_t, depends on ϕ and q.
In the steady-state, k_t is constant. Setting it equal to κ and re-arranging gives the **ARMA(1,1)** model with $\xi_t = v_t$ and $\phi - \kappa = \theta$.

Outliers

Suppose noise is from a heavy tailed distribution, such as Student's t. Outliers.
The RF is still an ARMA(1,1), but allowing the $\tilde{\zeta}_t' s$ to have a heavy-tailed distribution does not deal with the problem as a large observation becomes incorporated into the level and takes time to work through the system.
An ARMA models with a heavy-tailed distribution is designed to handle *innovations outliers*, as opposed to *additive outliers*. See the **robustness** literature.
But a *model-based approach* is not only simpler than the usual robust methods, but is also more amenable to diagnostic checking and generalization.
See Lange et al (JASA, 1989) for robustification with the t-distribution.

Unobserved component models for non-Gaussian noise

Simulation methods, such as MCMC, provide the basis for a direct attack on models that are nonlinear and/or non-Gaussian. The aim is to extend the Kalman filtering and smoothing algorithms that have proved so effective in handling linear Gaussian models. Considerable progress has been made in recent years; see Durbin and Koopman (2012).
But simulation-based estimation can be time-consuming and subject to a degree of uncertainty.
Also the statistical properties of the estimators are not easy to establish.

Observation driven model based on the score

The DCS approach begins by writing down the distribution of the $t-th$ observation, conditional on past observations. Time-varying parameters are then updated by a suitably defined filter. Such a model is *observation driven*, as opposed to a UC model which is *parameter driven*. In a *linear Gaussian UC* model, the KF is driven by the one step-ahead prediction error, v_t. The DCS filter replaces v_t in the KF equation by a variable, u_t, that is proportional to the score of the conditional distribution.
The innovations form becomes

$$\begin{aligned} y_t &= \mu_{t|t-1} + v_t, \qquad\qquad t = 1, ..., T \\ \mu_{t+1|t} &= \phi\mu_{t|t-1} + \kappa u_t \end{aligned}$$

where κ is an unknown parameter.

Dynamic location model

$$\begin{aligned} y_t &= \omega + \mu_{t|t-1} + v_t = \omega + \mu_{t|t-1} + \exp(\lambda)\varepsilon_t, \\ \mu_{t+1|t} &= \phi\mu_{t|t-1} + \kappa u_t, \end{aligned}$$

where ε_t is serially independent, standard t-variate and the conditional score is

$$u_t = \left(1 + \frac{(y_t - \mu_{t|t-1})^2}{\nu e^{2\lambda}}\right)^{-1} v_t,$$

where $v_t = y_t - \mu_{t|t-1}$ is the prediction error and $\varphi = \exp(\lambda)$ is the (time-invariant) scale.

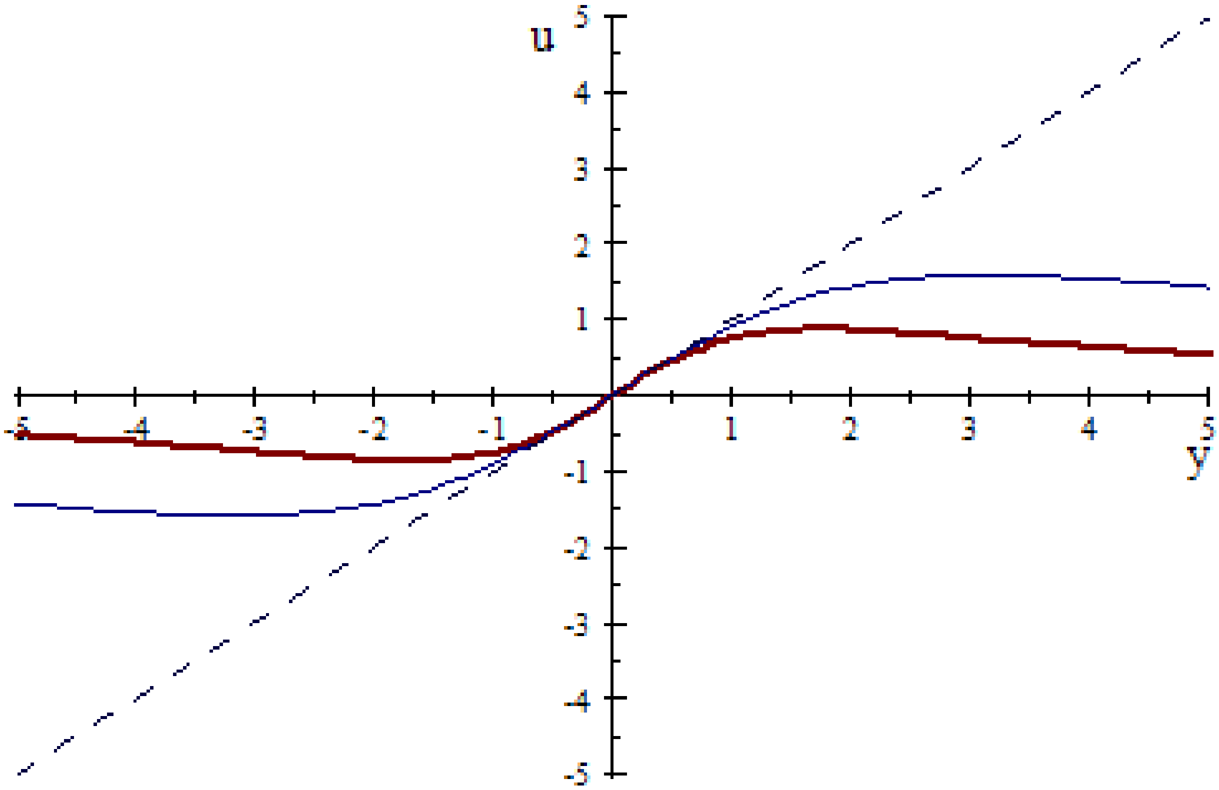


Figure: Impact of u_t for t_ν (with a scale of one) for $\nu = 3$ (thick), $\nu = 10$ (thin) and $\nu = \infty$ (dashed).

Basic properties

$$u_t = (1 - b_t)(y_t - \mu_{t|t-1}), \tag{1}$$

where

$$b_t = \frac{(y_t - \mu_{t|t-1})^2/\nu \exp(2\lambda)}{1 + (y_t - \mu_{t|t-1})^2/\nu \exp(2\lambda)}, \qquad 0 \leq b_t \leq 1, \quad 0 < \nu < \infty, \tag{2}$$

is distributed as $beta(1/2, \nu/2)$. The $u_t's$ are $IID(0, \sigma_u^2)$ and symmetrically distributed.

The fact that (2) has a beta distribution follows from the property of the t-distribution

Exponential DCS Volatility Models

$$y_t = \varepsilon_t \exp(\lambda_{t|t-1}/2), \quad t = 1, ..., T,$$

where the serially independent, zero mean variable ε_t has a $t_\nu-$distribution with degrees of freedom, $\nu > 0$, and the dynamic equation for the log of scale is

$$\lambda_{t|t-1} = \delta + \phi\lambda_{t-1|t-2} + \kappa u_{t-1}.$$

The conditional score is

$$u_t = \frac{(\nu+1)y_t^2}{\nu \exp(\lambda_{t|t-1}) + y_t^2} - 1, \quad -1 \leq u_t \leq \nu, \quad \nu > 0$$

NB The variance is equal to the square of the **scale**, that is $(\nu-2)\sigma^2_{t|t-1}/\nu$ for $\nu > 2$.

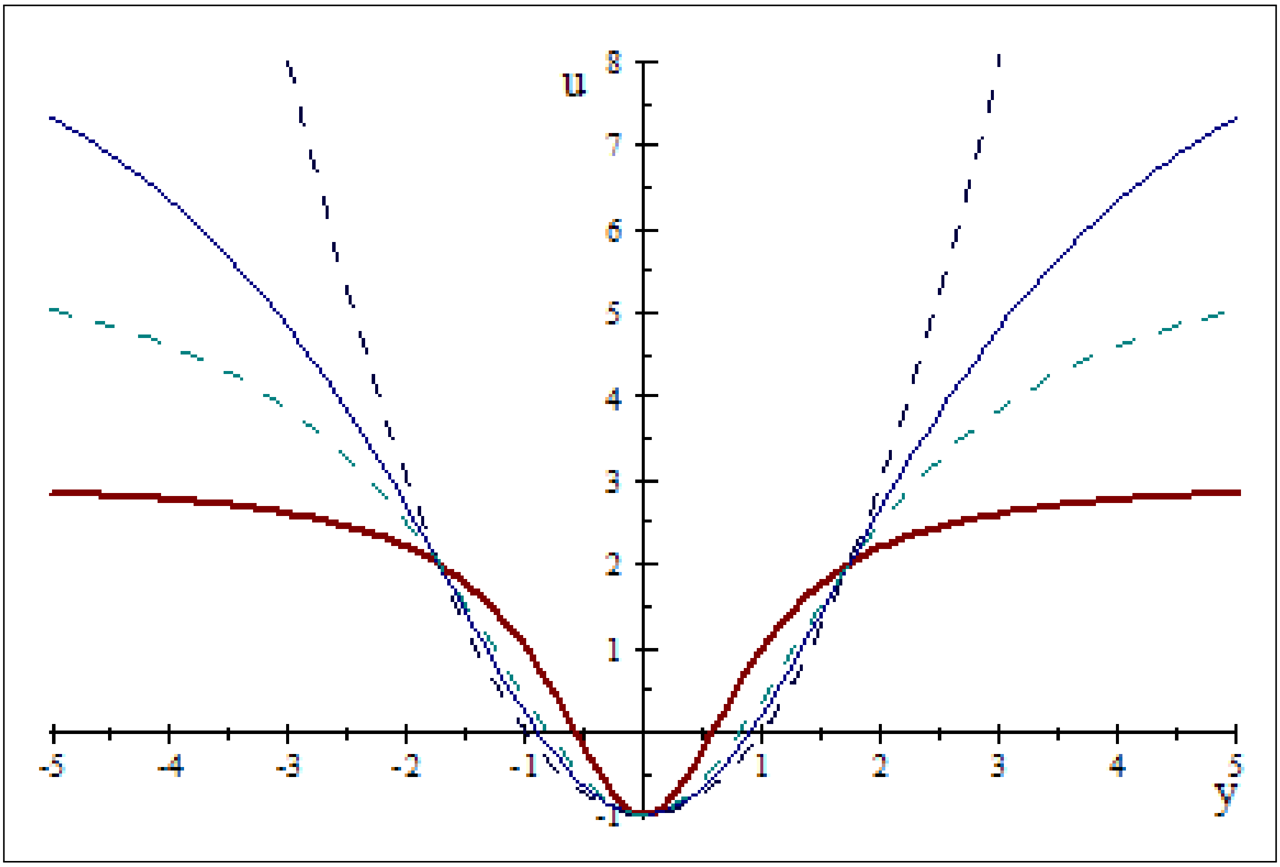
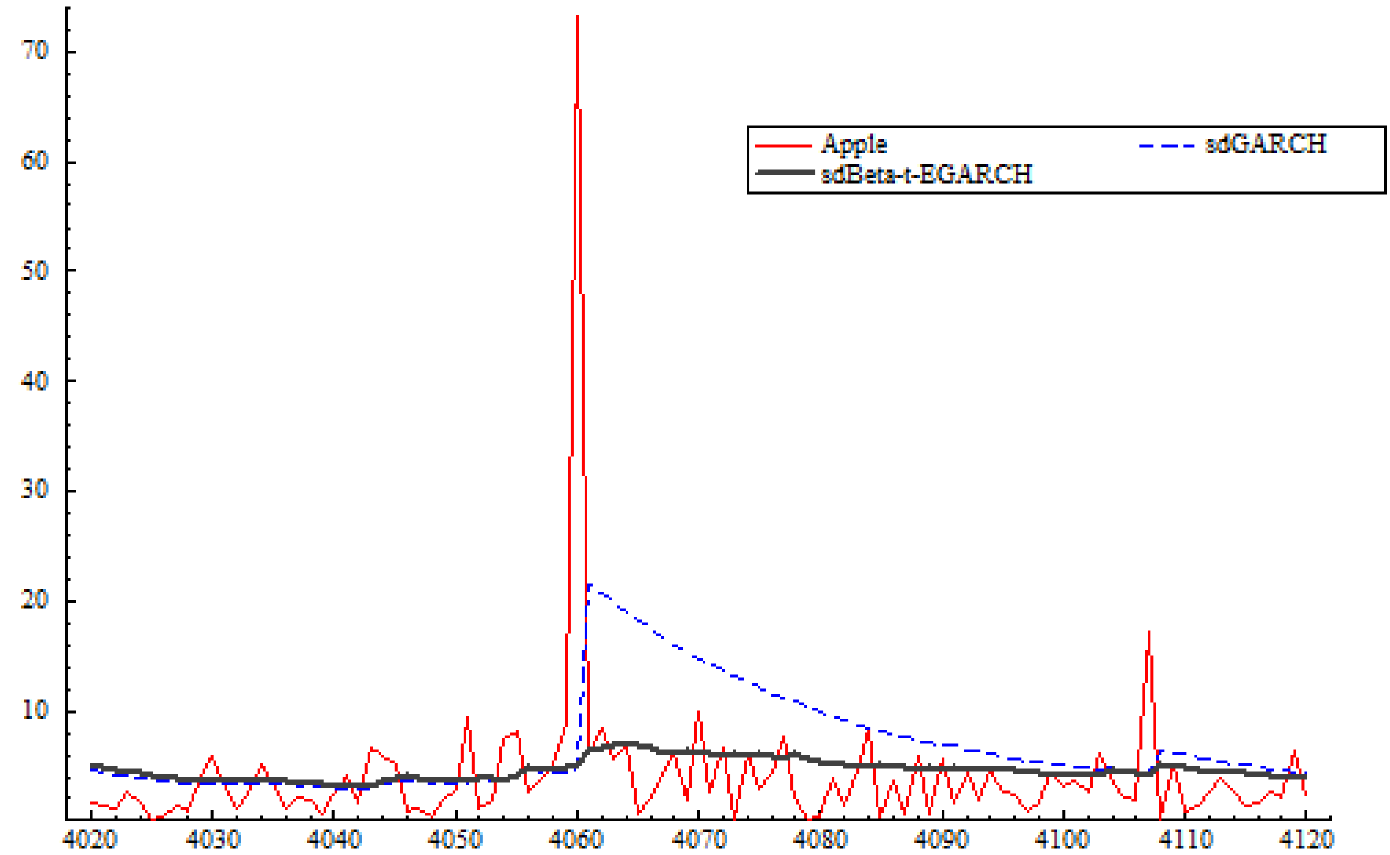


Figure: Impact of u_t for t_ν with $\nu = 3$ (thick), $\nu = 6$ (medium dashed) $\nu = 10$ (thin) and $\nu = \infty$ (dashed).



Beta-t-EGARCH

The variable u_t may be expressed as

$$u_t = (\nu + 1)b_t - 1,$$

where

$$b_t = \frac{y_t^2/\nu \exp(\lambda_{t|t-1})}{1 + y_t^2/\nu \exp(\lambda_{t|t-1})}, \qquad 0 \leq b_t \leq 1, \quad 0 < \nu < \infty,$$

is distributed as $Beta(1/2, \nu/2)$, a **Beta distribution**. Thus the $u_t's$ are IID.

Since $E(b_t) = 1/(\nu+1)$ and $Var(b_t) = 2\nu/\{(\nu+3)(\nu+1)^2\}$, u_t has zero mean and variance $2\nu/(\nu+3)$.

Beta-t-EGARCH

1) Moments exist and ACF of $|y_t|^c$, $c \geq 0$, can be derived.
2) Closed form expressions for moments of multi-step forecasts of volatility can be derived and full distribution easily simulated.
3) Asymptotic distribution of ML estimators with analytic expressions for standard errors.
4) Can handle time-varying trends (eg splines) and seasonals (eg time of day or day of week).

Location: basic properties

The filter may be generalized to:

$$\mu_{t+1|t} = \phi_1 \mu_{t|t-1} + ... + \phi_p \mu_{t-p+1|t-p} + \kappa_0 u_t + \kappa_1 u_{t-1} + ... + \kappa_r u_{t-r}.$$

Such a filter is denoted as $QARMA(p, r)$. The full model will be called $DCS - t - QARMA(p, r)$.

It corresponds to an unobserved component signal plus noise model in which the signal is $ARMA(p, r)$.

Basic properties

In the Gaussian case $u_t = v_t$. If q is defined as $max(p, r+1)$, we may write

$$y_t = \phi_1 y_{t-1} + ... + \phi_p y_{t-p} + v_t - (\phi_1 - \kappa_0)v_{t-1} - ... - (\phi_q - \kappa_q)v_{t-q},$$

which is an $ARMA(p, q)$ with MA coefficients $\theta_i = \phi_i - \kappa_{i-1}, i = 1, .., q$.

The invertibility conditions apply to $\theta_i = \phi_i - \kappa_{i-1}, i = 1, .., q$ rather than to $\kappa_i, i = 0, ...q$.

Maximum likelihood estimation

The log-likelihood function for the DCS-t model is

$$\begin{aligned} \ln L(\boldsymbol{\psi}, \nu) &= T \ln \Gamma\left((\nu+1)/2\right) - \frac{T}{2}\ln\pi - T\ln\Gamma\left(\nu/2\right) \\ &\quad -\frac{T}{2}\ln\nu - T\ln\varphi - \frac{(\nu+1)}{2}\sum_{t=1}^{T}\ln\left(1 + \frac{(y_t - \mu_{t|t-1})^2}{\nu\varphi^2}\right). \end{aligned}$$

Maximization of the log-likelihood function with respect to the unknown dynamic parameters in the vector $\boldsymbol{\psi}$ and the scale and shape parameters, λ and ν, can be carried out by numerical optimization.

Maximum likelihood estimation: information matrix

Let $y_t \mid Y_{t-1}$ have a t_ν-distribution with $\mu_{t|t-1}$ generated by the first-order model. Then, assuming that $|\phi| < 1$ and $b < 1$,

$$\mathbf{I}\begin{pmatrix} \boldsymbol{\psi} \\ \lambda \\ \nu \end{pmatrix} = \begin{bmatrix} \frac{\nu+1}{\nu+3}\exp(-2\lambda)\mathbf{D}(\boldsymbol{\psi}) & 0 & 0 \\ 0 & \frac{2\nu}{\nu+3} & \frac{1}{(\nu+3)(\nu+1)} \\ 0 & \frac{1}{(\nu+3)(\nu+1)} & h(\nu)/2 \end{bmatrix},$$

where $h(\nu)$ is a function of ν (involving trigamma functions) and

$$\mathbf{D}\begin{pmatrix} \kappa \\ \phi \\ \omega \end{pmatrix} = \frac{1}{1-b}\begin{bmatrix} \sigma_u^2 & \frac{a\kappa\sigma_u^2}{1-a\phi} & 0 \\ \frac{a\kappa\sigma_u^2}{1-a\phi} & \frac{\kappa^2\sigma_u^2(1+a\phi)}{(1-\phi^2)(1-a\phi)} & 0 \\ 0 & 0 & \frac{(1-\phi)^2(1+a)}{1-a} \end{bmatrix}$$

Maximum likelihood estimation: information matrix

$$a = \phi - \kappa\frac{\nu}{\nu+3},$$

$$b = \phi^2 - 2\phi\kappa\frac{\nu}{\nu+3} + \kappa^2\frac{\nu\left(\nu^3+10\nu^2+35\nu+38\right)}{(\nu+1)(\nu+3)(\nu+5)(\nu+7)},$$

Figure shows a plot of b against κ for $\phi = 0.9$ and $\nu = 6$. The admissible range is slightly bigger than in the Gaussian case where it is $-0.1 < \kappa < 1.9$.

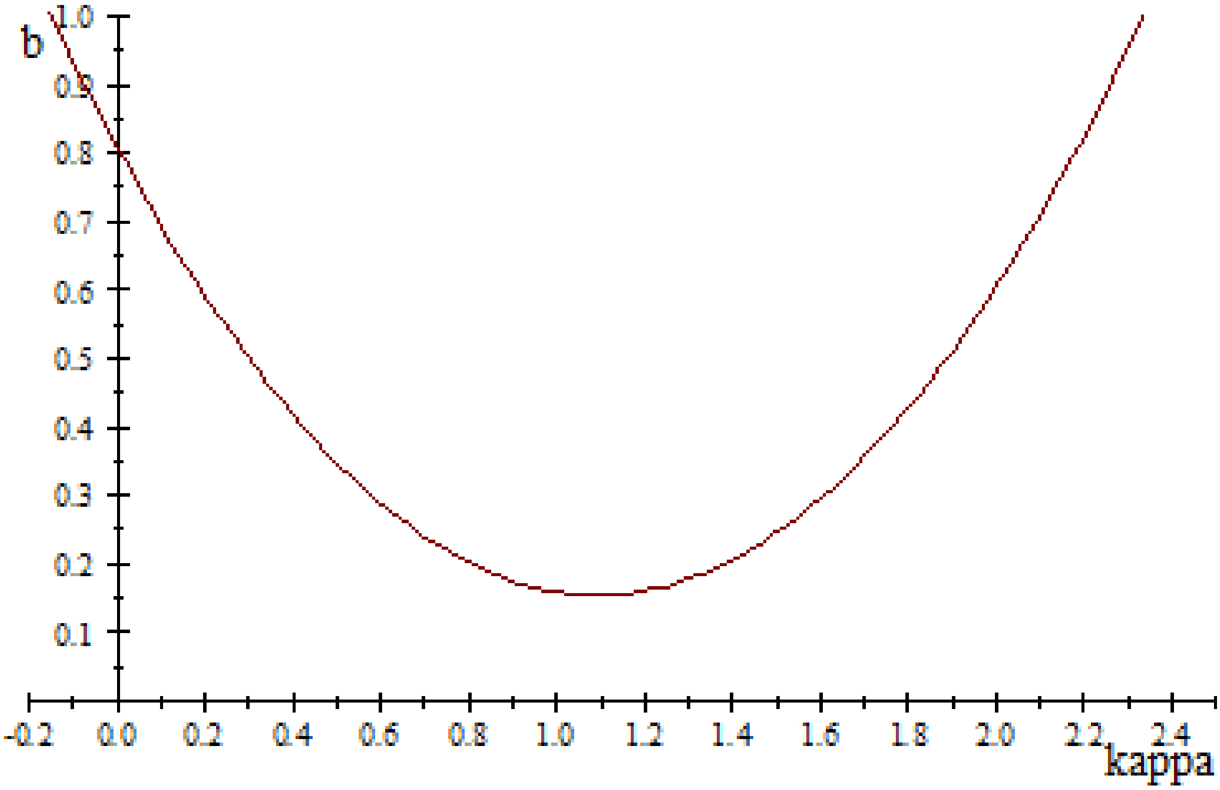


Figure: Plot of b against κ for $\phi = 0.9$ and $\nu = 6$

Maximum likelihood estimation: Gaussian model

For a Gaussian model, $b < 1$ provided that $\phi - 1 < \kappa < \phi + 1$.
The reduced form is the $ARMA(1,1)$ process

$$y_t = \phi y_{t-1} + v_t - \theta v_{t-1}.$$

The condition for strict invertibility in the ARMA(1,1) model is $|\theta| < 1$ and since $\theta = \phi - \kappa$, invertibility ensures that $b < 1$. The condition $\theta \neq \phi$ is needed for identifiability and this condition is equivalent to $\kappa \neq 0$.
When ϕ is known,

$$Var(\widetilde{\kappa}) = 1 - b = 1 - (\phi - \kappa)^2,$$

which is consistent with the standard $MA(1)$ result, $Var(\widetilde{\theta}) = 1 - \theta^2$.

Application to US GDP

A Gaussian AR(1) plus noise model with a constant, was fitted to the growth rate of US Real GDP, defined as the first difference of the logarithm, using the STAMP 8 package. The data were quarterly, from 1947(2) to 2012(1), and the parameter estimates were as follows:

$$\widetilde{\phi} = 0.501, \quad \widetilde{\sigma}_{\eta}^{2} = 7.62 \times 10^{-5}, \quad \widetilde{\sigma}_{\varepsilon}^{2} = 2.30 \times 10^{-5}, \quad \widetilde{\omega} = 0.0078.$$

There was little indication of residual serial correlation, but the Bowman-Shenton statistic is 30.04, which is clearly significant as the distribution under the null hypothesis of Gaussianity is χ_2^2. The non-normality clearly comes from excess kurtosis, which is 1.9, rather than from skewness.

Application to US GDP

DCS-location-t model. The estimated degrees of freedom of 6.3 means that the DCS filter is less responsive to more extreme observations, such as the fall of 2009(1).

Parameter	κ	ϕ	λ	ω	ν
Estimate	0.520	0.497	-4.878	0.0079	6.303
Num SE	0.098	0.102	0.073	0.0009	2.310
ASE	0.090	0.140	0.057	0.0009	1.807

Robust estimation and tail behaviour

The M-estimator, which features prominently in the robustness literature, has a Gaussian response until a certain point, b, whereupon it is constant; see Maronna *et al* (2006, p 25-31). This is known as Winsorizing as opposed to trimming where observations greater than b in absolute value are set to zero. In both cases setting b to a suitable value requires a (robust) estimate of scale to be pre-computed.
On the other hand, the t-score is like a redescending estimator - goes to zero as $|y| \to \infty$. eg Tukey biweight. Gradual form of trimming.

Is there a parametric distribution that gives some form of Winsorizing?

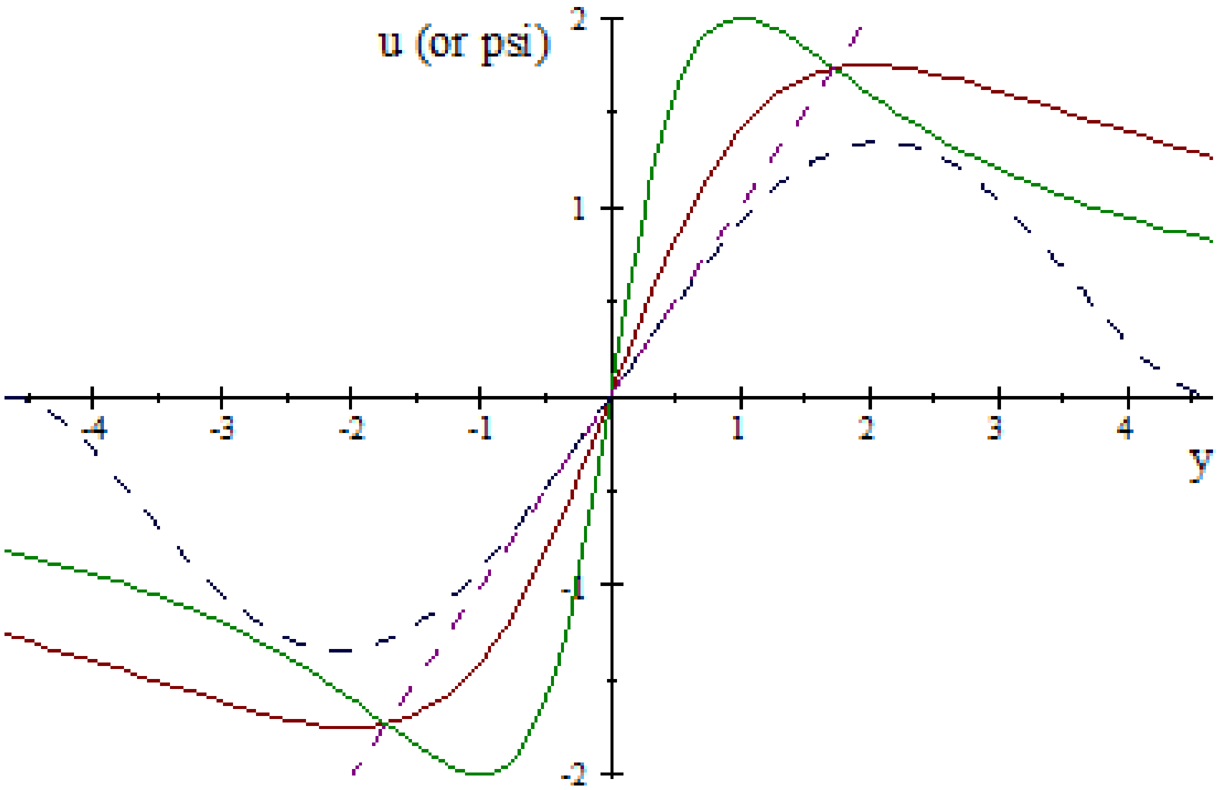


Figure: Score functions for t_3 (thin line) and t_6 (thick line) distributions, together with the linear score for the normal and Tukey's biweight function.

Robust estimation and tail behaviour

The Gaussian distribution has kurtosis of three and a distribution is said to exhibit *excess kurtosis* if its kurtosis is greater than three. Although many researchers take excess kurtosis as defining heavy tails, it is not, in itself, an ideal measure, particularly for asymmetric distributions.
A distribution is said to be *heavy-tailed* if

$$\lim_{y\to\infty} \exp(y/\beta)\overline{F}(y) = \infty \qquad \text{for all } \beta > 0, \tag{3}$$

where $\overline{F}(y) = \Pr(Y > y) = 1 - F(y)$ is the survival function.
When y has an exponential distribution, $\overline{F}(y) = \exp(-y/\alpha)$, so when $\beta = \alpha$, $\exp(y/\alpha)\overline{F}(y) = 1$ for all y. Thus the exponential distribution is not heavy-tailed.

Robust estimation and tail behaviour

A distribution is said to be *fat-tailed* if, for a fixed positive value of η,

$$\overline{F}(y) = cL(y)y^{-\eta}, \qquad \eta > 0, \tag{4}$$

where c is a non-negative constant and $L(y)$ is slowly varying, that is $\lim_{y\to\infty}(L(ky)/L(y)) = 1$.
The parameter η is the **tail index**. The implied PDF is a *power law* PDF

$$f(y) \sim cL(y)\eta y^{-\eta-1}, \qquad \eta > 0, \tag{5}$$

The m-th moment exists if $m < \eta$. The Pareto distribution is a simple case in which $\overline{F}(y) = y^{-\eta}$ for $y > 1$.
The complement to the Pareto distribution is the power function distribution- $\overline{F}(y) = y^{\overline{\eta}}$, $0 < y < 1$, $\overline{\eta} > 0$. More generally, $\overline{F}(y) = cL(y)y^{\overline{\eta}}$. Hence

$$f(y) \sim cL(y)\overline{\eta}y^{\overline{\eta}-1} \quad \textit{as} \quad y \to 0, \quad 0 < y < 1, \overline{\eta} > 0.$$

Robust estimation and tail behaviour

The above criteria are related to the behavior of the conditional score and whether or not it discounts large observations. This, in turn, connects to robustness. More specifically, consider a power law PDF with y divided by a scale parameter, φ, so that $f(y) \sim cL(y)\varphi^{-1}\eta(y/\varphi)^{-\eta-1}$.
Then

$$\partial \ln f / \partial \varphi \sim \eta / \varphi \quad \text{as} \quad y \to \infty$$

and so the score is bounded.
With the exponential link function, $\varphi = \exp(\lambda)$,

$$\partial \ln f / \partial \lambda \sim \eta \qquad \text{as} \qquad y \to \infty.$$

Similarly as $y \to 0$, $\partial \ln f / \partial \lambda \sim \overline{\eta}$.

Robust estimation and tail behaviour

The logarithm of a variable with a fat-tailed distribution has exponential tails.
Let x denote a variable with a fat-tailed distribution in which the scale is written as $\varphi = \exp(\mu)$ and let $y = \ln x$. Then for large y

$$f(y) \sim cL(e^y)\eta e^{-\eta(y-\mu)}, \qquad \eta > 0, \quad \text{as} \quad y \to \infty,$$

whereas as $y \to -\infty$, $f(y) \sim cL(e^y)\overline{\eta}e^{\overline{\eta}(y-\mu)}$, $\overline{\eta} > 0$.
Thus y is not heavy-tailed but it may exhibit excess kurtosis.
The score with respect to location, μ, is the same as the original score with respect to the logarithm of scale and so tends to η as $y \to \infty$.

EGB2 (work in progress with Michele Caivano)

The exponential generalized beta distribution of the second kind (EGB2) is obtained by taking the logarithm of a variable with a GB2 distribution. The PDF of a GB2 is

$$f(x) = \frac{\nu (x/\alpha)^{\nu\xi - 1}}{\alpha B(\xi, \varsigma) \left[(x/\alpha)^{\nu} + 1 \right]^{\xi+\varsigma}}, \qquad \alpha, \nu, \xi, \varsigma > 0,$$

where α is the scale parameter, ν, ξ and ς are shape parameters and $B(\xi, \varsigma)$ is the beta function; see Kleiber and Kotz (2003, ch6). The GB2 distribution contains many important distributions as special cases, including the Burr $(\xi = 1)$ and log-logistic $(\xi = 1, \varsigma = 1)$. GB2 distributions are fat tailed for finite ξ and ς with upper and lower tail indices of $\eta = \varsigma\nu$ and $\overline{\eta} = \xi\nu$ respectively. The absolute value of a t_f variate is GB2$(\varphi, 2, 1/2, f/2)$ with tail index is $\eta = \overline{\eta} = f$.

EGB2

Andres and Harvey (2012) study DCS models for time-varying scale.
The properties which EGB2 inherits from GB2 have important implications for the score function and hence for robustness to additive outliers.
The connection between the score for a t-distribution and redescending M-estimators is well-known. The fact that the EGB2 distribution gives a gentle form of Winsorizing is less well-known.

EGB2

If x is distributed as GB2$(\alpha, \nu, \xi, \varsigma)$ and $y = \ln x$, the PDF of the EGB2 variate y is

$$f(y; \mu, \nu, \xi, \varsigma) = \frac{\nu \exp\{\xi(y-\mu)\nu\}}{B(\xi,\varsigma)(1+\exp\{(y-\mu)\nu\})^{\xi+\varsigma}}. \quad (6)$$

What was the logarithm of scale in GB2 now becomes location in EGB, that is $\ln \alpha$ becomes μ. Furthermore ν is now a scale parameter, but ξ and ς are still shape parameters and they determine skewness and kurtosis. All moments exist.

Although ν is a scale parameter, it is the inverse of what would be considered a more conventional measure of scale. Thus scale is better defined as $1/\nu$ or as the standard deviation

$$\sigma = \sqrt{\psi'(\xi) + \psi'(\varsigma)}/\nu = h(\xi,\varsigma)/\nu = h/\nu. \quad (7)$$

An exponential link function for the standard deviation yields a parameter $\lambda = \ln \sigma = -\ln \nu + \ln h$.

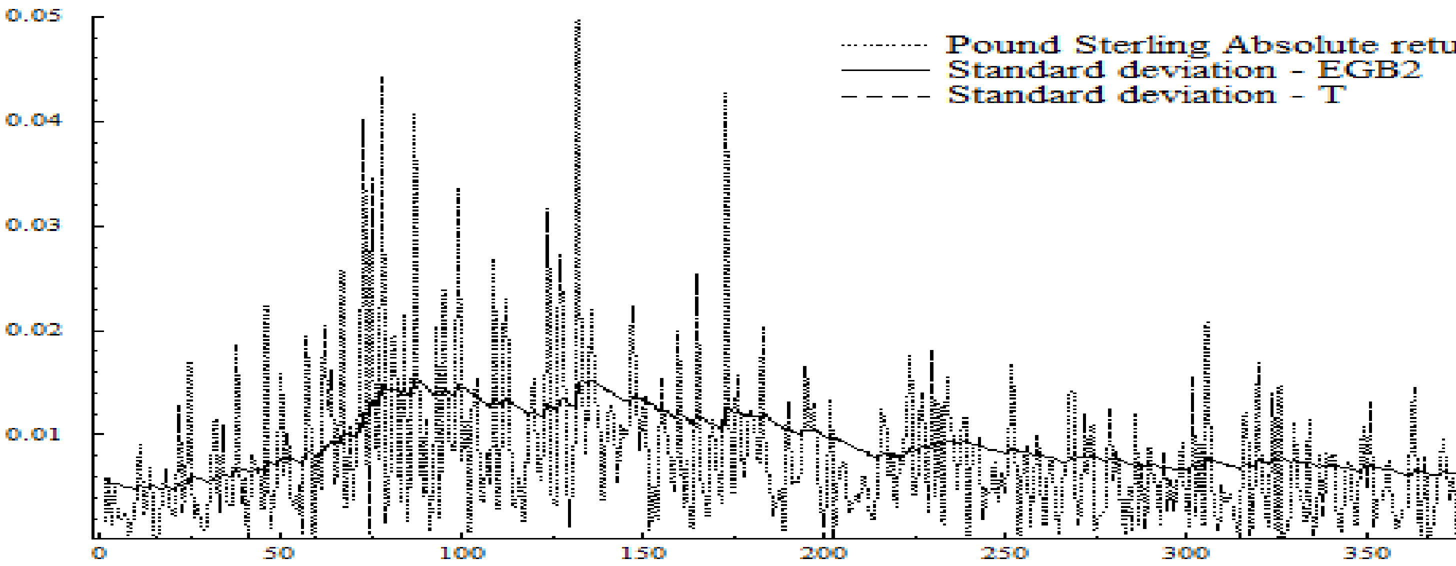
EGB2

When $\xi = \varsigma$, the distribution is symmetric; for $\xi = \varsigma = 1$ it is a logistic distribution and when $\xi = \varsigma \to \infty$ it tends to a normal distribution.

Proposition

When $\xi = \varsigma = 0$ in the EGB2, the distribution is double exponential or Laplace.

A plot of the (symmetric) EGB2, GED and Student's t with the same excess kurtosis shows them to be very similar. It is difficult to see the heavier tails of the t distribution from the graph, and the only discernible difference among the three distributions is in the peak, which is higher and more pointed for the GED. The EGB2 in turn is more peaked than the t. As the excess kurtosis increases, the differences between the peaks become more marked.



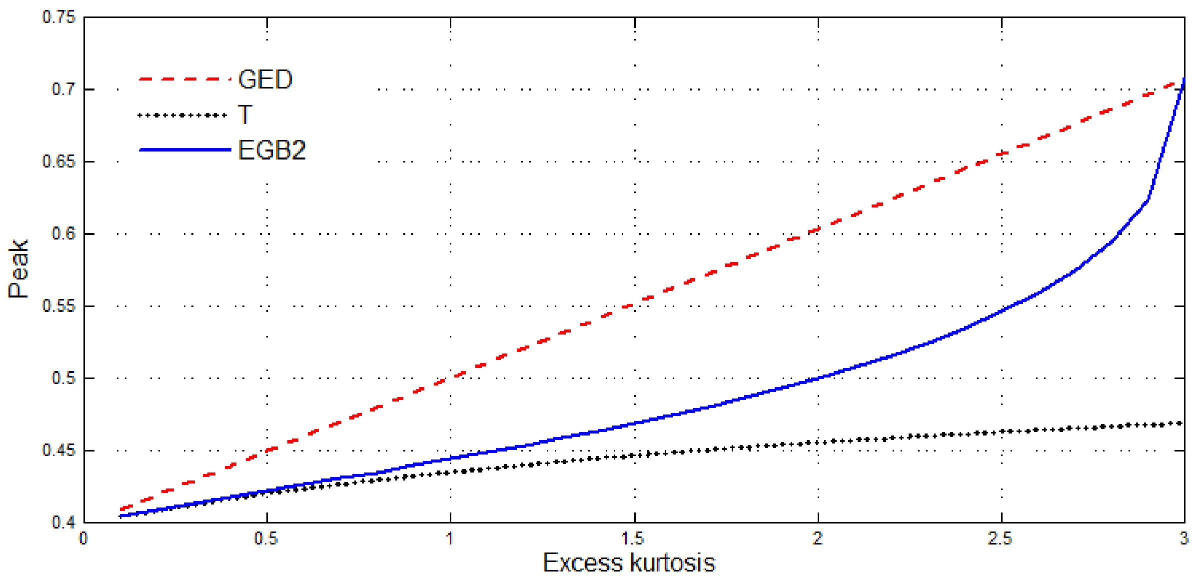


Figure: PDF at the mean for GED, EGB2 and t-distributions with the same kurtosis.

EGB2

The score function for the GB2 distribution with respect to $\ln \alpha$ is bounded, reflecting the fact that the distribution has a long tail (fat tail) and α is a scale parameter. The score function with respect to location in the EGB2 distribution is of exactly the same form and so inherits the boundedness property. Specifically,

$$\frac{\partial \ln f_t}{\partial \mu_{t|t-1}} = \nu(\xi + \varsigma)b_t(\xi, \varsigma) - \nu\xi, \qquad t = 1, ..., T,$$

where

$$b_t(\xi, \varsigma) = \frac{e^{(y_t - \mu_{t|t-1})\nu}}{e^{(y_t - \mu_{t|t-1})\nu} + 1}.$$

Because $0 \leq b_t(\xi, \varsigma) \leq 1$, it follows that as $y \to \infty$, the score approaches an upper bound of $\nu\varsigma$, whereas $y \to -\infty$ gives a lower bound of $\nu\xi$.

EGB2

It will prove more convenient to replace ν by h/σ and to define u_t as

$$u_t = \sigma^2 \frac{\partial \ln f_t}{\partial \mu_{t|t-1}} = \sigma h[(\xi + \varsigma)b_t(\xi, \varsigma) - \xi]. \qquad (8)$$

We note that the upper and lower bounds are $\sigma\sqrt{2}$ and $-\sigma\sqrt{2}$ respectively when $\varsigma = \xi = 0$. On the other hand, there is no upper (lower) bound for $\varsigma(or\ \xi) \to \infty$ because $h\varsigma \to \infty$ (as does $h\xi$). As $\varsigma = \xi \to \infty$, the distribution becomes normal and so for large ς and ξ, $u_t \simeq y_t - \mu_{t|t-1}$.
Andres and Harvey (2012) study time-varying scale in a GB2 DCS model parameterized with an exponential link function. Many of the results given there and in Harvey (2013) therefore apply to the EGB2 model with time-varying location.
The variable $b_t(\xi, \varsigma)$ is IID with a $beta(\xi, \varsigma)$ distribution at the true parameter values.

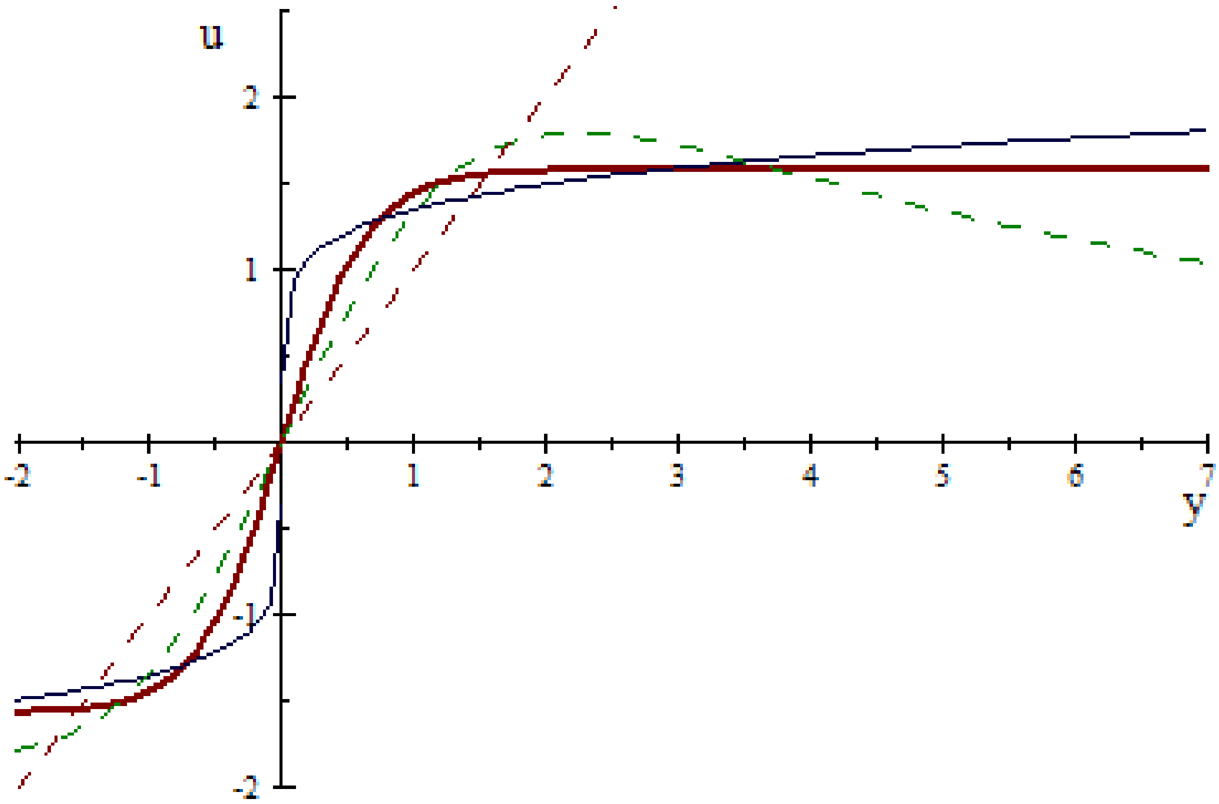


Figure: Score functions for EGB2 (thick line), GED (medium line) and t (thick dash), all with excess kurtosis of 2. Thin line shows normal score.

EGB2

Figure shows the score functions for standardized ($\sigma = 1$) EGB2, GED and t distributions, all with excess kurtosis of two. The shape parameters for the three distributions are $\xi = 0.5$, $\upsilon = 1.148$ and $\nu = 7$. Given the apparent similarity of the PDFs, the difference in the behaviour of the score functions is striking. The score for the t distribution is redescending, reflecting the fact that it has fat tails. There is no upper bound with GED, except when it becomes a Laplace distribution and the score is $\pm\sqrt{2}$ for $y \neq 0$.

Neither the EGB2 nor the GED distribution has heavy tails. However, the EGB2 distribution has exponential tails, whereas the GED distribution is super-exponential for $\upsilon > 1$. Hence the EGB2 score is bounded and what we get is a gentle form of Winsorizing.

Higher-order models and the state space form

The observation in the state space form is related to an $m \times 1$ state vector, $\boldsymbol{\alpha}_t$, through a measurement equation,

$$y_t = \mathbf{z}'\boldsymbol{\alpha}_t + \varepsilon_t, \quad t = 1, ..., T,$$

where $\mathbf{z}$ is an $m \times 1$ vector and ε_t is a serially uncorrelated disturbance with $E(\varepsilon_t) = \mathbf{0}$ and $Var(\varepsilon_t) = \sigma_\varepsilon^2$. The transition equation is

$$\boldsymbol{\alpha}_{t+1} = \mathbf{T}\boldsymbol{\alpha}_t + \boldsymbol{\eta}_t, \quad t = 1, ..., T.$$

The Kalman filter can be written as a single set of recursions going directly from $\boldsymbol{\alpha}_{t|t-1}$ to $\boldsymbol{\alpha}_{t+1|t}$, that is

$$\boldsymbol{\alpha}_{t+1|t} = \mathbf{T}\boldsymbol{\alpha}_{t|t-1} + \mathbf{k}_t v_t, \quad t = 1, ..., T,$$

where $v_t = y_t - \mathbf{z}_t'\boldsymbol{\alpha}_{t|t-1}$ is the innovation and $f_t = \mathbf{z}'\mathbf{P}_{t|t-1}\mathbf{z} + \sigma_\varepsilon^2$ is its variance. The gain vector, $\mathbf{k}_t$, is

$$\mathbf{k}_t = (1/f_t)\mathbf{T}\mathbf{P}_{t|t-1}\mathbf{z}, \quad t = 1, ..., T. \qquad (9)$$

Higher-order models and the state space form

Re-arranging the KF equations gives the innovations form

$$\begin{aligned} y_t &= \mathbf{z}'\boldsymbol{\alpha}_{t|t-1} + v_t, \quad t = 1, ..., T, \\ \boldsymbol{\alpha}_{t+1|t} &= \mathbf{T}\boldsymbol{\alpha}_{t|t-1} + \mathbf{k}_t v_t. \end{aligned} \qquad (10)$$

A general location DCS model may be set up in the same way as the innovations form of a Gaussian state space model. The model corresponding to the steady-state of (10) is

$$\begin{aligned} y_t &= \omega + \mathbf{z}'\boldsymbol{\alpha}_{t|t-1} + v_t, \quad t = 1, ..., T, \\ \boldsymbol{\alpha}_{t+1|t} &= \boldsymbol{\delta} + \mathbf{T}\boldsymbol{\alpha}_{t|t-1} + \boldsymbol{\kappa} u_t. \end{aligned} \qquad (11)$$

Higher-order models: asymptotic theory

The quantities a and b become

$$\mathbf{A}(\nu) = \mathbf{T} - \{\nu/(\nu+3)\}\boldsymbol{\kappa}\mathbf{z}',$$

and

$$\begin{aligned}\mathbf{B}(\nu) &= \mathbf{T}\otimes\mathbf{T} + \frac{\nu}{\nu+3}(\boldsymbol{\kappa}\mathbf{z}'\otimes\mathbf{T} + \mathbf{T}\otimes\boldsymbol{\kappa}\mathbf{z}') \\ &+ \frac{\nu\left(\nu^3+10\nu^2+35\nu+38\right)}{(\nu+1)(\nu+3)(\nu+5)(\nu+7)}\boldsymbol{\kappa}\mathbf{z}'\otimes\boldsymbol{\kappa}\mathbf{z}'\end{aligned}$$

The asymptotic theory requires that the roots of the $m^2 \times m^2$ matrix $\mathbf{B}(\nu)$ **have modulus less than one**.
For a Gaussian model this will be the case if the roots of $\mathbf{A}$ have modulus less than one because $\mathbf{B} = \mathbf{A}\otimes\mathbf{A}$.

Trend and seasonality

Stochastic trend and seasonal components may be introduced into UC models for location. These models, called structural time series models, are implemented in the STAMP package of Koopman *et al* (2009).
The Gaussian random walk plus noise or *local level* model is

$$\begin{aligned}y_t &= \mu_t + \varepsilon_t, & \varepsilon_t \sim NID(0,\sigma_\varepsilon^2), \\ \mu_t &= \mu_{t-1} + \eta_t, & \eta_t \sim NID(0,\sigma_\eta^2),\end{aligned}$$

where $E(\varepsilon_t\eta_s) = 0$ for all t and s. The signal noise ratio is $q = \sigma_\eta^2/\sigma_\varepsilon^2$.
The KF is an EWMA

$$\mu_{t+1|t} = (1-\kappa)\mu_{t|t-1} + \kappa y_t$$

Local level

For the DCS-t filter

$$\begin{aligned} y_t &= \mu_{t|t-1} + v_t, \\ \mu_{t+1|t} &= \mu_{t|t-1} + \kappa u_t. \end{aligned}$$

and the initial value, $\mu_{1|0}$,is treated as an unknown parameter that needs to estimated along with κ and ν.

Since $u_t = (1-b_t)(y_t - \mu_{t|t-1})$, re-arranging the dynamic equation gives

$$\mu_{t+1|t} = (1-\kappa(1-b_t))\mu_{t|t-1} + \kappa(1-b_t)y_t$$

A sufficient condition for the weights on current and past observations to be non-negative is that $\kappa(1-b_t) < 1$ and, because $0 \leq b_t \leq 1$, this is guaranteed by $0 < \kappa \leq 1$.

The restriction that $\kappa \leq 1$ is much stricter than is either necessary or desirable. Indeed the argument based on matching autocorrelations suggests an admissible range of $0 \leq \kappa \leq (\nu+1)/(\nu-2)$.

Local level

As regards asymptotic properties,

$$Var(\widetilde{\kappa}) = \left(2\kappa\frac{\nu}{\nu+3} - \kappa^2\frac{\nu\left(\nu^3+10\nu^2+35\nu+38\right)}{(\nu+1)(\nu+3)(\nu+5)(\nu+7)}\right)\left(\frac{\nu+3}{\nu}\right)^2.$$

In contrast to the case when $|\phi| < 1$, it is necessary that $\kappa > 0$. For finite degrees of freedom, the upper bound will be greater than the value of 2 for a Gaussian model.

Local level

Fitting a local level DCS model (initialized with $\mu_{2|1} = y_1$) to seasonally adjusted monthly data on U.S. Average Weekly Hours of Production and Nonsupervisory Employees: Manufacturing (AWHMAN) from February 1992 to May 2010 (220 observations) gave

$$\widetilde{\kappa} = 1.246 \qquad \widetilde{\lambda} = -3.625 \qquad \widetilde{\nu} = 6.35$$

with numerical (asymptotic) standard errors

$$SE(\widetilde{\kappa}) = 0.161(0.090) \qquad SE(\widetilde{\lambda}) = 0.120(0.062) \qquad SE(\widetilde{\nu}) = 1.630(1.991)$$

A drift term was initially included but it was statistically insignificant. The value of b is 0.151. Although $\widetilde{\kappa}$ is greater than one, the resulting filter is perfectly consistent with the properties of the series. Figure shows (part of) the series together with the contemporaneous filter, which for the random walk is $\mu_{t|t} = \mu_{t+1|t}$. Unusually large prediction errors result in a small value of $\kappa(1 - b_t)$ and most of weight in the filter is assigned to $\mu_{t|t-1}$.

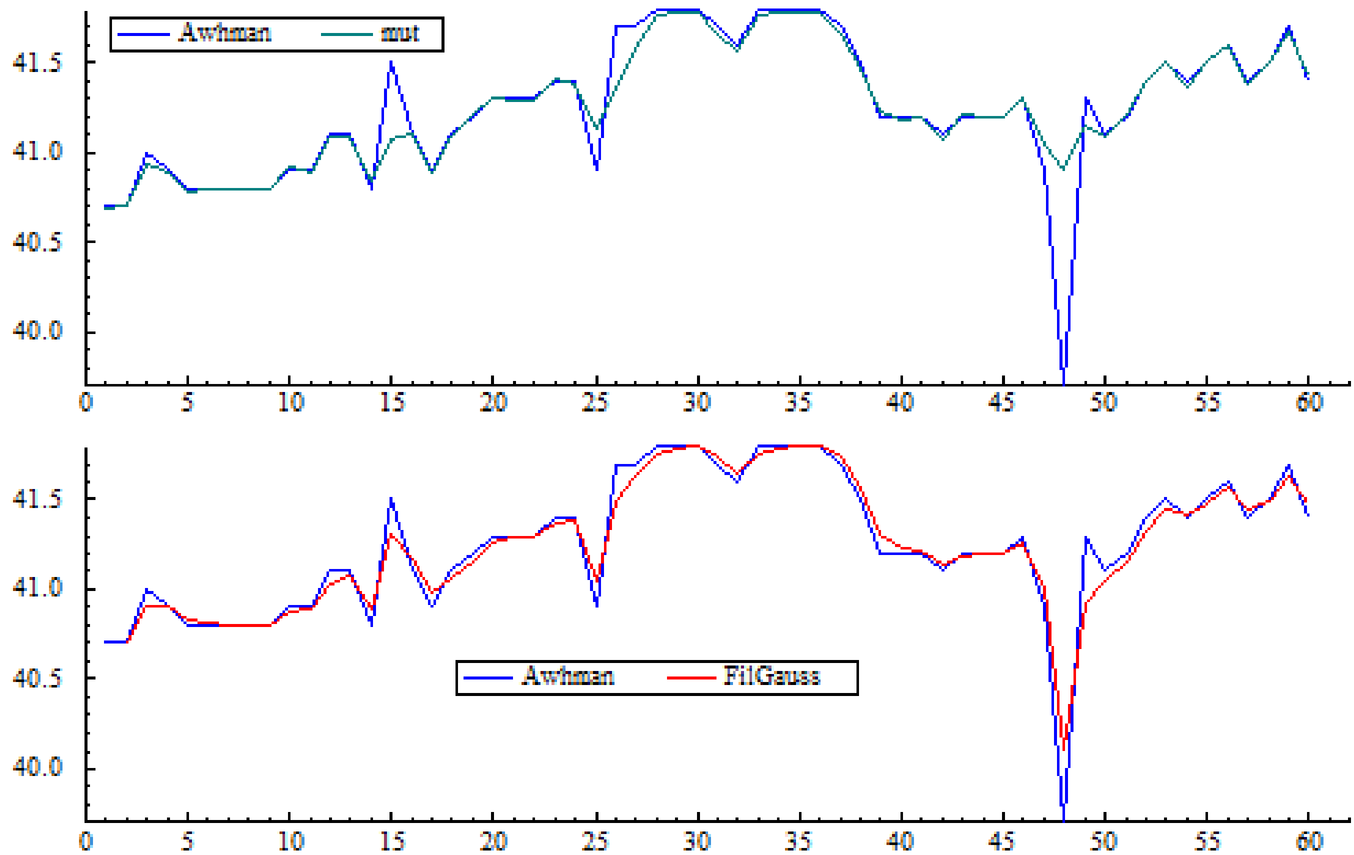


Figure: DCS and Gaussian (bottom panel) local level models fitted to US

Local linear trend

The DCS local linear trend filter is

$$\begin{aligned} y_t &= \mu_{t|t-1} + v_t, \quad t = 1, ..., T, \\ \mu_{t+1|t} &= \mu_{t|t-1} + \beta_{t|t-1} + \kappa_1 u_t \\ \beta_{t+1|t} &= \beta_{t|t-1} + \kappa_2 u_t. \end{aligned}$$

The initialization $\beta_{3|2} = y_2 - y_1$ and $\mu_{3|2} = y_2$ can be used, but, as in the local level model, initializing in this way is vulnerable to outliers at the beginning. *Estimating the fixed starting values, $\mu_{1|0}$ and $\beta_{1|0}$, is a better option*.

An IRW trend in the UC local linear trend model implies the contraint $\kappa_2 = \kappa_1^2/(2 - \kappa_1), \quad 0 < \kappa_1 < 1$, which may be found from Harvey (1989, p. 177). The restriction can be imposed on the DCS-t model by treating $\kappa_1 = \kappa$ as the unknown parameter, but without unity imposed as an upper bound.

Stochastic seasonal

A fixed seasonal pattern may be modeled as $\gamma_t = \sum_{j=1}^{s} \gamma_j z_{jt}$, where s is the number of seasons and the dummy variable z_{jt} is one in season j and zero otherwise. In order not to confound trend with seasonality, the coefficients, γ_j, $j = 1, ..., s$, are constrained to sum to zero.

The seasonal pattern may be allowed to change over time by letting the coefficients evolve as random walks. If γ_{jt} denotes the effect of season j at time t and $\boldsymbol{\gamma}_t = (\gamma_{1t}, ..., \gamma_{st})'$, then

$$\boldsymbol{\gamma}_t = \boldsymbol{\gamma}_{t-1} + \boldsymbol{\omega}_t, \qquad t = 1, ..., T,$$

where $\boldsymbol{\omega}_t$ is a normally distributed, zero-mean vector of disturbances.

Stochastic seasonal

Although all s seasonal components are continually evolving, only one affects the observations at any particular point in time, that is $\gamma_t = \gamma_{jt}$ when season j is prevailing at time t. The requirement that $\sum_{j=1}^{s} \gamma_{jt} = 0$, is enforced by the restriction that the disturbances sum to zero at each point in time.

This restriction is implemented by the correlation structure in

$$Var\left(\boldsymbol{\omega}_t\right) = \sigma_\omega^2 \left(\mathbf{I} - s^{-1}\mathbf{i}\mathbf{i}'\right)$$

where $\boldsymbol{\omega}_t = \left(\omega_{1t}, ..., \omega_{st}\right)'$ and $\mathbf{i}$ is a vector of ones, coupled with initial conditions requiring that the seasonals sum to zero at $t = 0$.

Stochastic seasonal

In the state space form, the transition matrix is just the identity matrix, but the $\mathbf{z}$ vector must change over time to accommodate the current season. Apart from replacing $\mathbf{z}$ by $\mathbf{z}_t$, the form of the KF remains unchanged. Adapting the innovations form to the DCS observation driven framework gives

$$y_t = \mathbf{z}_t'\boldsymbol{\alpha}_{t|t-1} + v_t, \qquad \boldsymbol{\alpha}_{t+1|t} = \boldsymbol{\alpha}_{t|t-1} + \boldsymbol{\kappa}_t u_t,$$

where $\mathbf{z}_t$ picks out the current season, $\gamma_{t|t-1}$, that is $\gamma_{t|t-1} = \mathbf{z}_t'\boldsymbol{\alpha}_{t|t-1}$. The only question is how to parameterize $\boldsymbol{\kappa}_t$.

Stochastic seasonal

The seasonal dummies in the UC model are constrained to sum to zero and the same is true of their filtered estimates. Thus $\mathbf{i}'\boldsymbol{\kappa}_t = 0$ in the Kalman filter and this property should carry across to the DCS filter.
If κ_{jt}, $j = 1, .., s$, denotes the $j - th$ element of $\boldsymbol{\kappa}_t$, then in season j we set $\kappa_{jt} = \kappa_s$, where κ_s is a non-negative unknown parameter, while

$$\kappa_{it} = -\kappa_s/(s-1) \qquad\qquad i \neq j.$$

The amounts by which the seasonal effects change therefore sum to zero.

Basic structural model

The seasonal recursions can be combined with the trend filtering equations to give a structure similar in form to that of the Kalman filter for the stochastic trend plus seasonal plus noise UC model, sometimes known as the 'basic structural model'. Thus

$$y_t = \mu_{t|t-1} + \gamma_{t|t-1} + v_t$$

The initial conditions at time $t = 0$ are estimated by treating them as parameters; there are $s - 1$ seasonal parameters because the remaining initial seasonal state is minus the sum of the others.

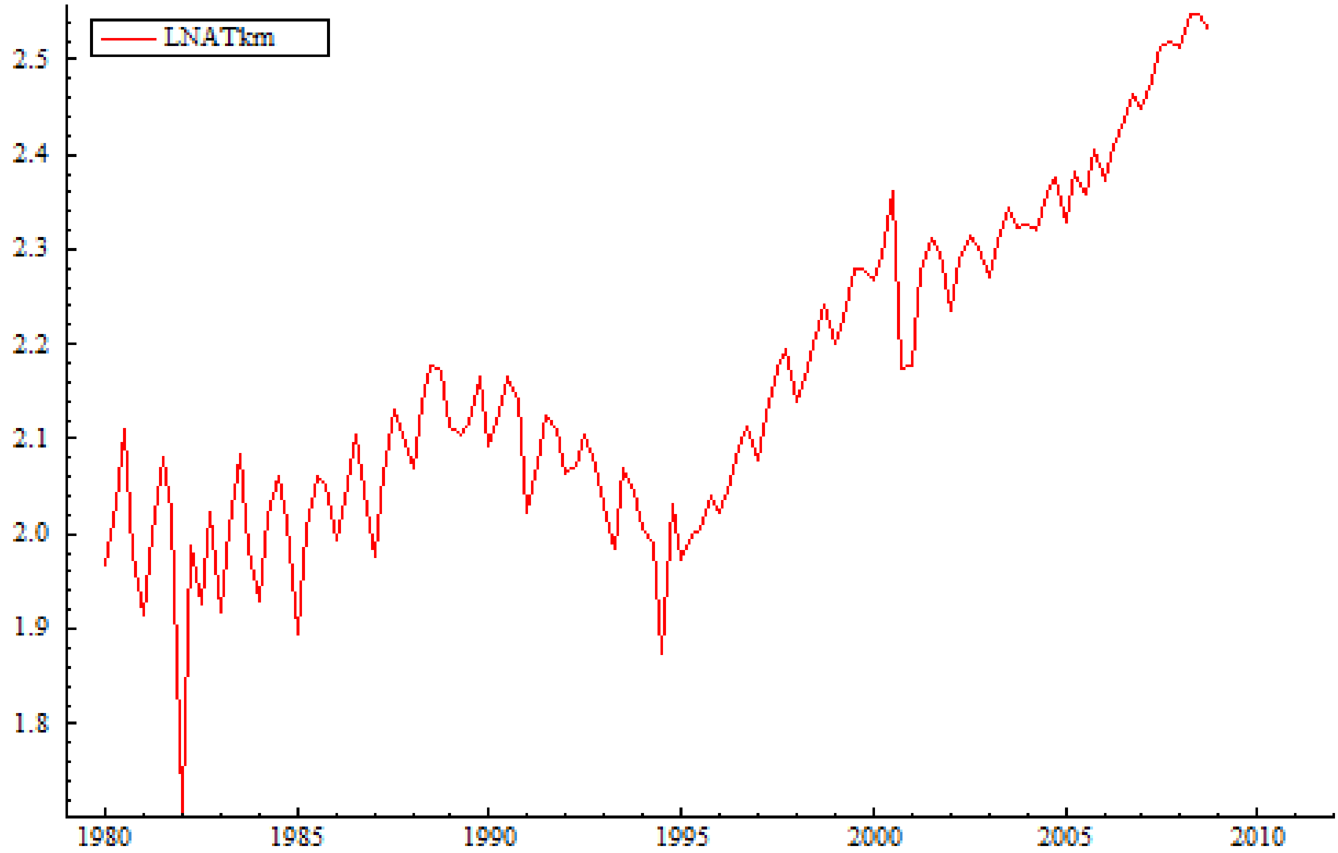


Figure: Logarithm of National Rail Travel in the UK (number of kilometres)

Application to rail travel

An unobserved components model was fitted to the rail series using the STAMP 8 package of Koopman et al (2009). Trend, seasonal and irregular components were included but the model was augmented with intervention variables to take out the effects of observations that are known to be unrepresentative.

The intervention dummies were:
(i) the train drivers strikes in 1982(1,3);
(ii) the Hatfield crash and its aftermath, 2000(4) and 2001(1); and
(iii) the signallers strike in 1994(3).

Application to rail travel

Fitting a DCS model with trend and seasonal avoids the need to deal explicitly with the outliers. The ML estimates for the parameters in a model with a random walk plus drift trend are

$$\widetilde{\kappa} = 1.421(0.161) \qquad \widetilde{\kappa}_s = 0.539\ (0.070) \qquad \widetilde{\lambda} = -3.787\ (0.053)$$
$$\widetilde{\nu} = 2.564\ (0.319) \qquad \widetilde{\beta} = 0.003\ (0.001)$$

with initial values $\widetilde{\mu} = 2.066(0.009)$, $\widetilde{\gamma}_1 = -0.094(0.007)$, $\widetilde{\gamma}_2 = -0.010(0.006)$ and $\widetilde{\gamma}_3 = 0.086(0.006)$.
The figures in parentheses are numerical standard errors. The last seasonal is $\widetilde{\gamma}_4 = 0.018$; it has no SE as it was constructed from the others.

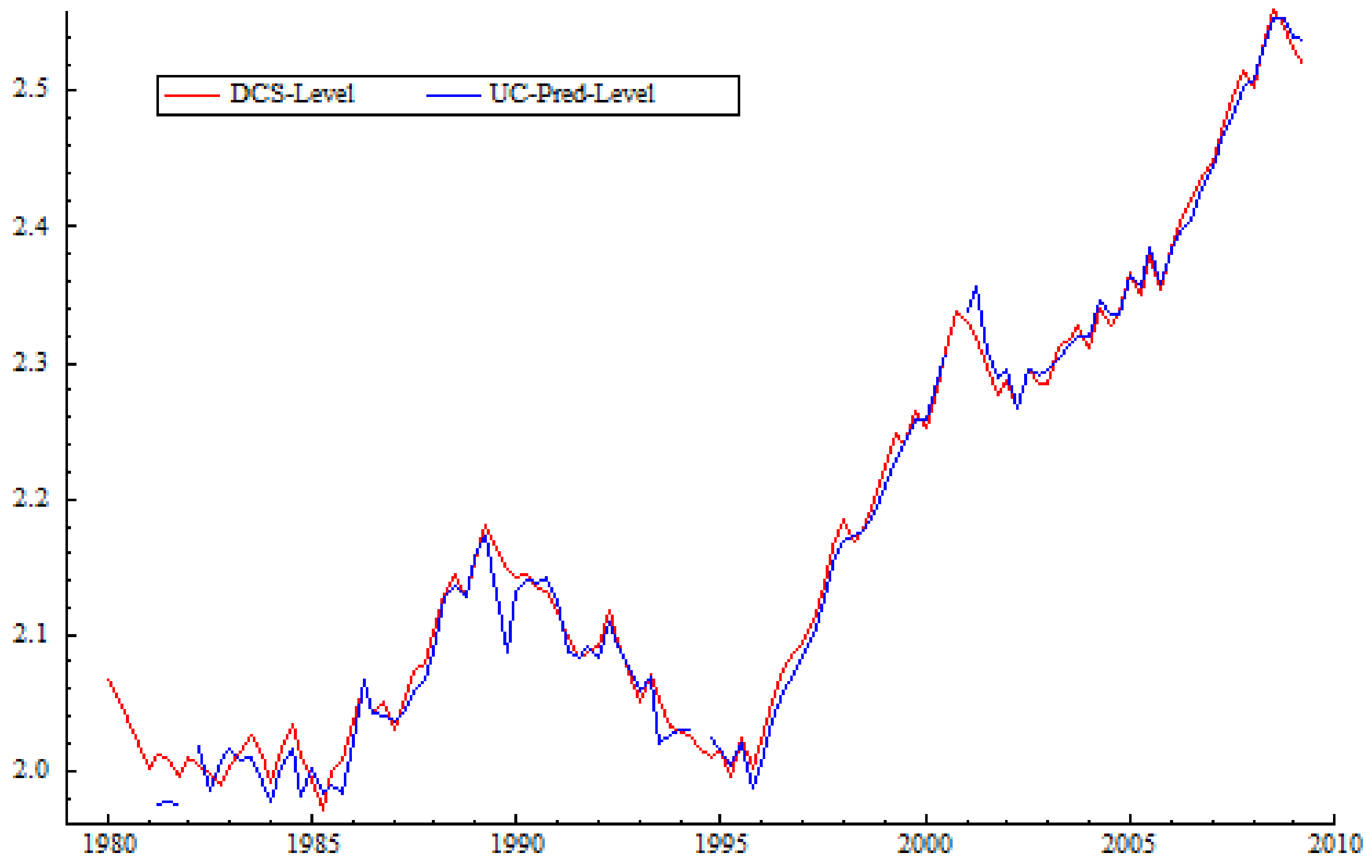


Figure: Trends in National Rail Travel from UC and DCS models

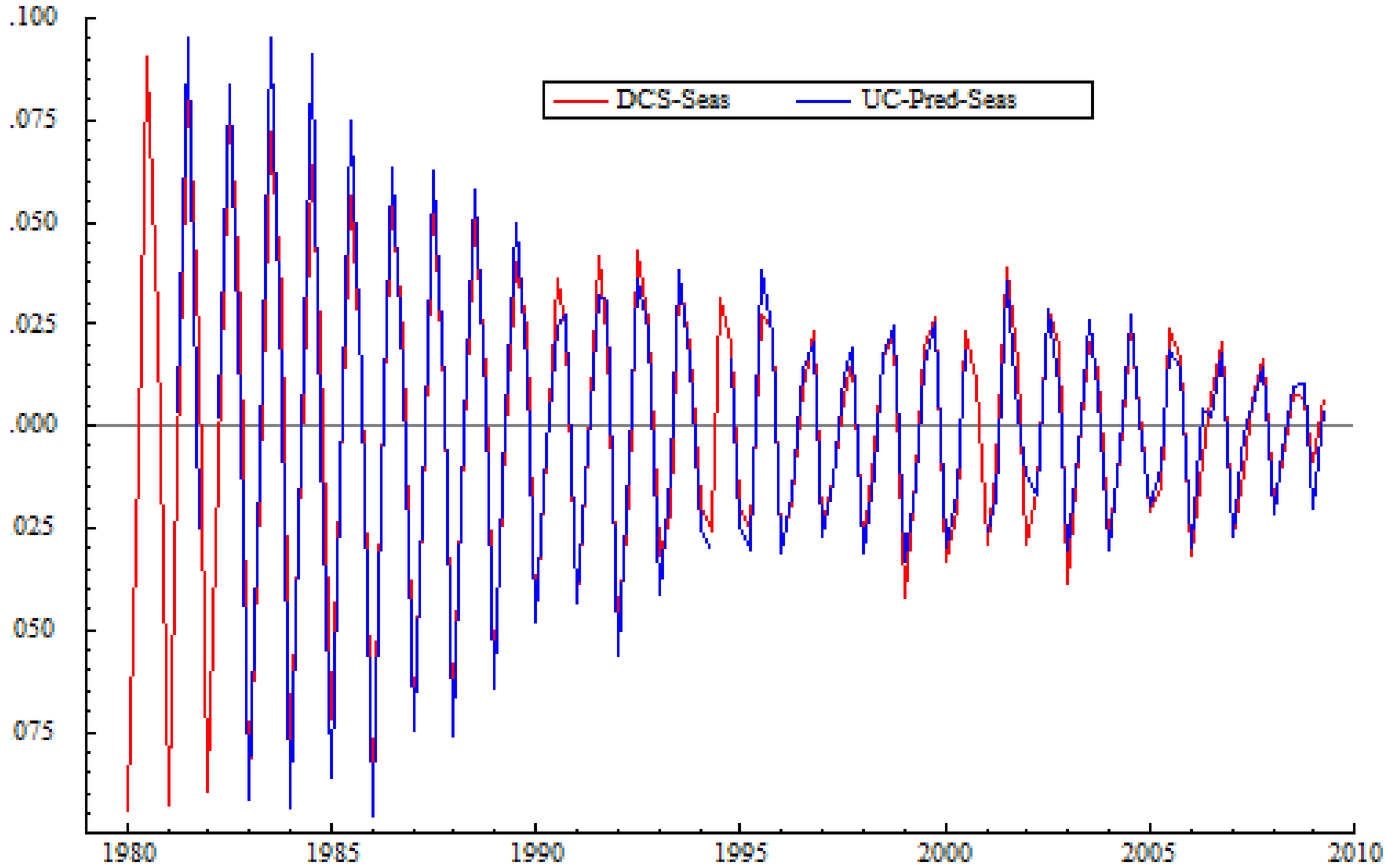


Figure: Seasonals in National Rail Travel from UC and DCS models

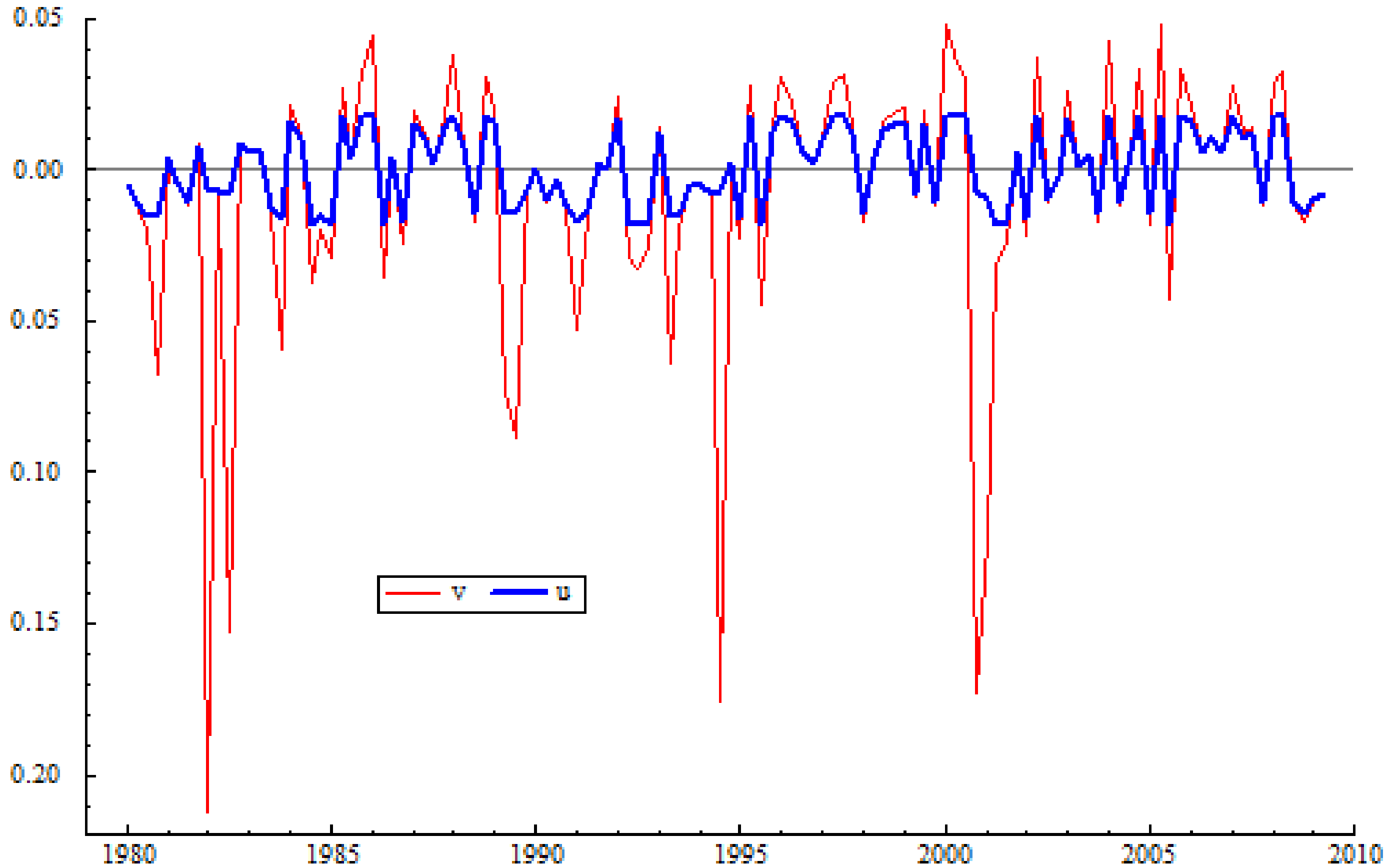


Figure: Residuals and scores from DCS-t model

Rail travel

Figure shows the residuals, that is the one-step ahead prediction errors, and score for the DCS model. The outliers, which were removed by dummies in the UC model, show up clearly in the residuals.
In the score series the outliers are downweighted and the autocorrelations for the score are slightly bigger than those of the residuals presumably because they are not weakened by aberrant values. The Box-Ljung $Q(12)$ statistic is 19.78 for the score and 12.40 for the residuals.
If it can be assumed that only the number of fitted dynamic parameters affects the distribution of the Box-Ljung statistic, its distribution under the null hypothesis of correct model specification is χ^2_{10}, which had a 5% critical value of 18.3. Thus the scores reject the null hypothesis, albeit only marginally, while the residuals do not. Having said that, the score autocorrelations do not exhibit any clear pattern and it is not clear how the dynamic specification might be improved.

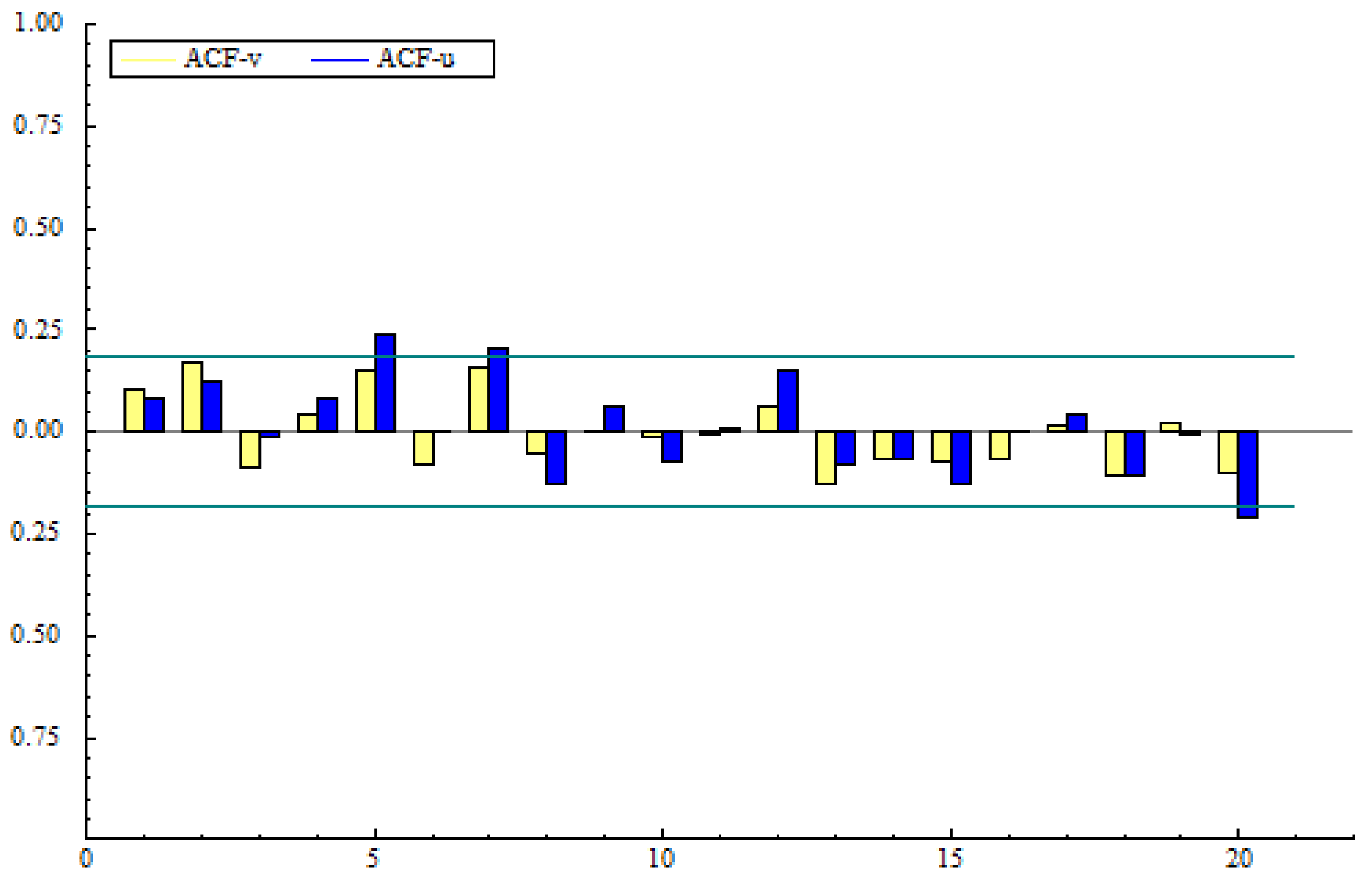


Figure: Residual correlograms for irregular and score residuals from DCS-t model

Explanatory variables

The location parameter may depend on a set of observable explanatory variables, denoted by the $k \times 1$ vector $\mathbf{w}_t$, as well as on its own past values and the score. The model can be set up as

$$y_t = \mu^{\dagger}_{t|t-1} + \mathbf{w}'_t\gamma + \varepsilon_t \exp(\lambda), \qquad t = 1, ..., T,$$

where $\mu^{\dagger}_{t|t-1}$ could be a stationary process or a stochastic trend.
The model may be augmented by a seasonal component.
If it is possible to make a sensible guess of initial values of the explanatory variable coefficients, the degrees of freedom parameter, ν, and the dynamic parameters, ϕ and κ for a stationary first-order model or β and κ for a random walk with drift, can be estimated by fitting a univariate model to the residuals, $y_t - \mathbf{w}'_t\widehat{\gamma}$, $t = 1, .., T$. These values are then used to start off numerical optimization with respect to all the parameters in the model.

Asymptotic theory

Consider a model with a stationary first-order component. Assume that the explanatory variables are weakly stationary with mean $\boldsymbol{\mu}_w$ and second moment Λ_w and are strictly exogenous in the sense that they are independent of the ε_t's and therefore of the $u'_t s$. Assuming that $b < 1$ and $\kappa \neq 0$, the limiting distribution of
$\sqrt{T}(\widetilde{\kappa} - \kappa, \widetilde{\phi} - \phi, \widetilde{\gamma}' - \gamma', \widetilde{\lambda} - \lambda, \widetilde{\nu} - \nu)'$ is multivariate normal with mean vector zero and covariance matrix given by the inverse of

$$\mathbf{I}\begin{pmatrix} \boldsymbol{\psi} \\ \lambda \\ \nu \end{pmatrix} = \begin{bmatrix} \frac{\nu+1}{\nu+3}\exp(-2\lambda)\mathbf{D}(\boldsymbol{\psi}) & 0 & 0 \\ 0 & \frac{2\nu}{\nu+3} & \frac{1}{(\nu+3)(\nu+1)} \\ 0 & \frac{1}{(\nu+3)(\nu+1)} & h(\nu)/2 \end{bmatrix},$$

but with $\boldsymbol{\psi}$ replaced by $(\kappa, \phi, \gamma')'$ and $\mathbf{D}(\boldsymbol{\psi})$ replaced by

Asymptotic theory

$$\mathbf{D}\begin{pmatrix}\kappa\\ \phi\\ \gamma\end{pmatrix}=\frac{1}{1-b}\begin{bmatrix}A & D & \mathbf{0}'\\ D & B & \mathbf{0}'\\ \mathbf{0} & \mathbf{0} & \mathbf{C}_w\end{bmatrix},$$

where

$$\mathbf{C}_w=(1+\phi^2)\Lambda_w-2\phi\Lambda_w(1)+2a(1-a)^{-1}(1-\phi)^2\boldsymbol{\mu}_w\boldsymbol{\mu}_w',$$

with $\Lambda_w(1)=E(\mathbf{w}_t\mathbf{w}_{t-1}')=E(\mathbf{w}_{t-1}\mathbf{w}_t')$.

Asymptotic theory

Corollary. When $\mu_{t|t-1}^{\dagger}$ is known to be a random walk with drift, β, and $\mu_{1|0}^{\dagger}$ is fixed and known,

$$\mathbf{D}\begin{pmatrix}\kappa\\ \gamma\\ \beta\end{pmatrix}=\frac{1}{1-b}\begin{bmatrix}\sigma_u^2 & \mathbf{0}' & \mathbf{0}'\\ \mathbf{0} & \mathbf{C}_{\Delta w} & \boldsymbol{\mu}_{\Delta w}\\ \mathbf{0} & \boldsymbol{\mu}_{\Delta w}' & 1\end{bmatrix},$$

where $\boldsymbol{\mu}_{\Delta w}=E(\Delta\mathbf{w}_t)$ and $\mathbf{C}_{\Delta w}=E(\Delta\mathbf{w}_t\Delta\mathbf{w}_t')$.
Assume $b<1$ and $\mathbf{C}_{\Delta w}$ is positive definite.

Asymptotic theory

The first differences of the explanatory variables must be weakly stationary but their levels may be nonstationary. Then the covariance matrix of the limiting distribution of $\sqrt{T}\widetilde{\gamma}$ is

$$Var(\widetilde{\gamma}) = \left(\frac{2\kappa\nu}{\nu+1} - \kappa^2 \frac{\nu\left(\nu^3+10\nu^2+35\nu+38\right)}{(\nu+1)^2(\nu+5)(\nu+7)} \right) e^{2\lambda}(\mathbf{C}_{\Delta w} - \boldsymbol{\mu}_{\Delta w}\boldsymbol{\mu}'_{\Delta w})^{-1}$$

In principle, the above Corollary may be extended to models where seasonals are included.

Rail travel

Potential explanatory variables for the rail travel series of Sub-section 6.5 are: (i) Real GDP (in £2003 prices), (ii) Real Fares, obtained by dividing total revenue by the number of kilometres travelled and the retail price index (RPI), and (iii) Petrol and Oil index (POI), divided by RPI. The fares series was smoothed by fitting a univariate UC model.

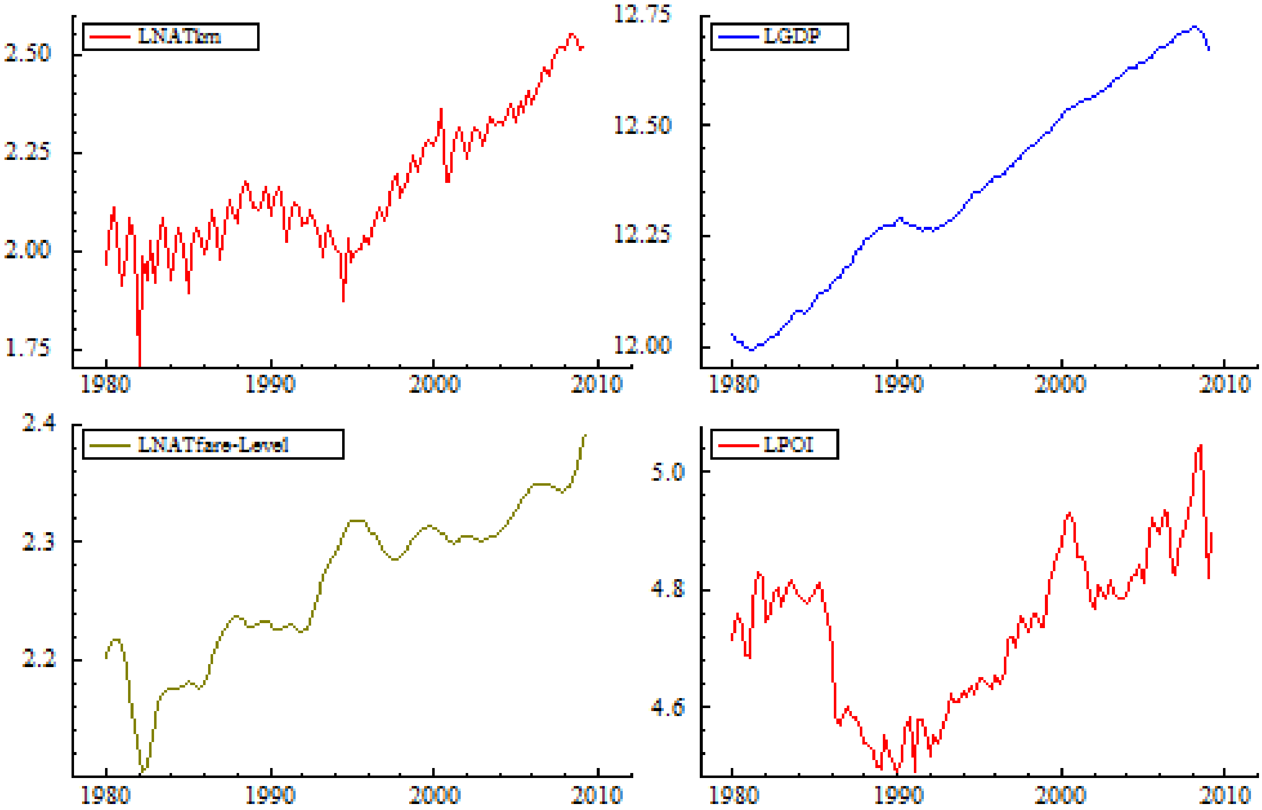


Figure: Rail travel in the UK and explanatory variables

Application to rail travel: unobserved components model

STAMP gave the following estimates for the coefficients of the logarithms of the explanatory variables:
GDP was 0.716 (0.267), fares was -0.416 (0.245) and POI was 0.050 (0.065).
All the estimates are all plausible. The coefficient of the petrol index is not statistically significant at any conventional level, but at least it has the right sign.
Failure to deal with outliers in a time series regression can lead to serious distortions and this is illustrated when the intervention variables are not included -
the fare estimate is *plus* 0.28 !

Rail travel: DCS-t without seasonal

When rail travel was seasonally adjusted by removing the seasonal component obtained from the univariate DCS-t model and LPOI was also seasonally adjusted, estimating the DCS-t model without a seasonal gave

$$\begin{aligned} \widetilde{\kappa} &= 1.346(0.151) \qquad \widetilde{\lambda} = -3.879\ (0.102) \\ \widetilde{\nu} &= 2.436\ (0.534) \qquad \widetilde{\beta} = 0.001\ (0.002), \end{aligned}$$

where the figures in parentheses are ASEs.
The ASEs calculated for the coefficients of LGDP, Lfare (level) and LPOI (seasonally adjusted) using $Var(\widetilde{\gamma})$ were 0.251, 0.246 and 0.050 respectively.

Rail travel: DCS-t with seasonal

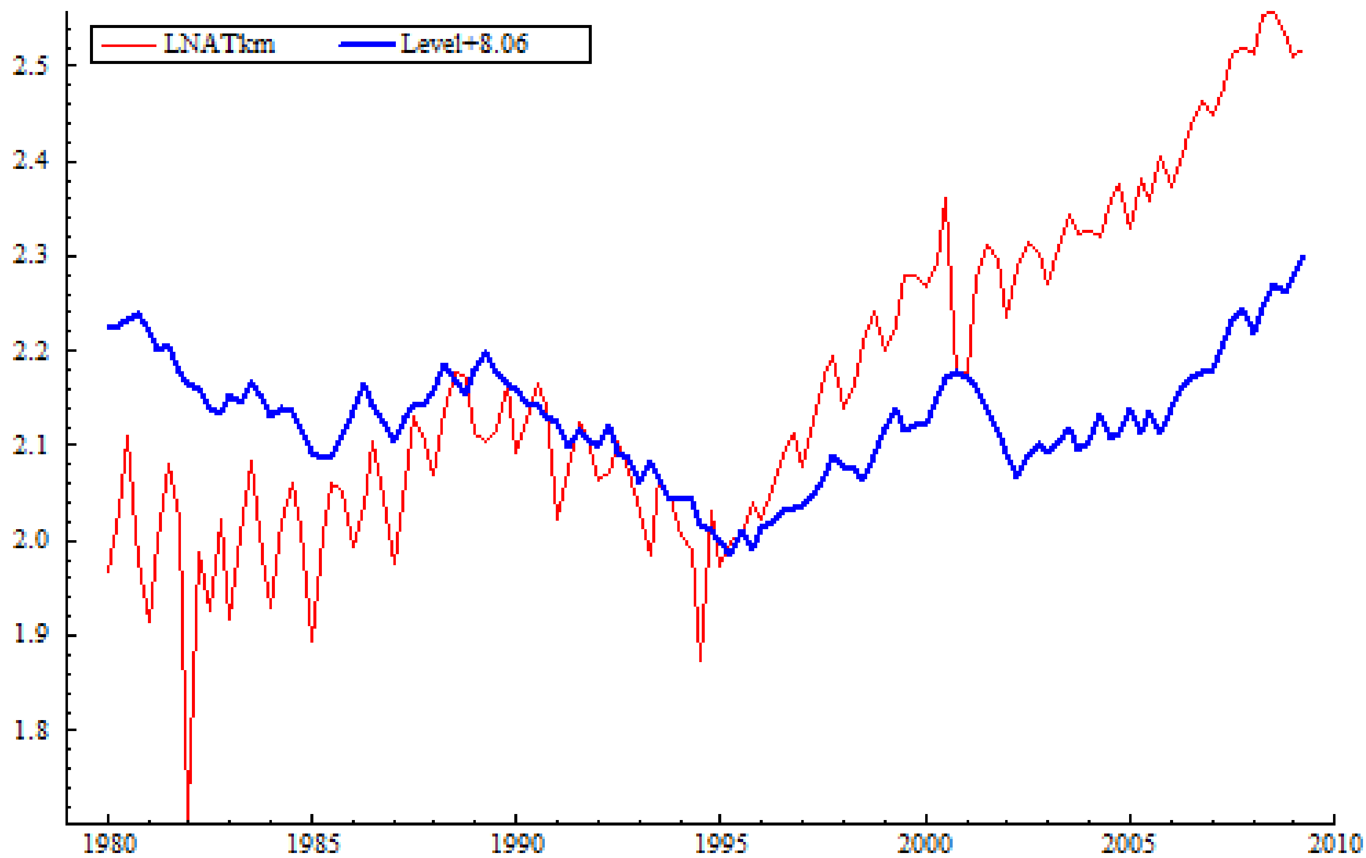
Fitting a DCS-t model with seasonal gave

$$\begin{aligned} \widetilde{\kappa} &= 2.212 \qquad \widetilde{\kappa}_s = 0.771 \qquad \widetilde{\lambda} = -4.059 \\ \widetilde{\nu} &= 2.070 \qquad \widetilde{\beta} = 0.0004 \end{aligned}$$

with initial values $\widetilde{\mu} = -6.162$, $\widetilde{\gamma}_1 = -0.084$, $\widetilde{\gamma}_2 = -0.007$ and $\widetilde{\gamma}_3 = 0.070$.
The coefficients of the explanatory variables were:

$$LGDP = 0.734 \qquad Lfare = -0.427 \quad LPOI = 0.056$$

A good deal, but by no means all, of the growth in rail travel from the mid-nineties is due to the increase in GDP. The continued fall after the economy had moved out of the recession of the early nineties is partly explained by the fact that fares increased sharply in 1993 in anticipation of rail privatisation and continued to increase till 1995.
Nevertheless, as is apparent from the Figure, there remain long-term movements in rail travel that cannot be accounted for by the exogenous variables.



Andrew Harvey (ach34@cam.ac.uk) (Faculty Robust Time Series Models / 82

Stochastic location and stochastic scale

The Student t model for time-varying location may be combined with Beta-t-EGARCH. In other words $y_t \mid Y_{t-1}$ has a t_ν distribution with mean $\mu_{t|t-1}$ and scale $\exp(\lambda_{t|t-1})$, that is

$$y_t = \mu_{t|t-1} + \varepsilon_t \exp(\lambda_{t|t-1}), \qquad t = 1, ..., T.$$

The structure of the information matrix in the static model is such that the form of the dynamic equations for $\mu_{t|t-1}$ and $\lambda_{t|t-1}$ is unchanged. The Beta-t-EGARCH score is

$$u_t = \frac{(\nu+1)(y_t - \mu_{t|t-1})^2}{\nu \exp(2\lambda_{t|t-1}) + (y_t - \mu_{t|t-1})^2} - 1$$

Estimation by ML is straightforward - asymptotics is not.

Example - seasonally adjusted rate of inflation in the US.

The rate of inflation is often a random walk plus noise. Thus for the DCS-t model

$$\mu_{t+1|t} = \mu_{t|t-1} + \kappa^{\dagger} u_t,$$

where u_t is the level score and the dagger serves to differentiate $\kappa^{\dagger}$ from the κ parameter in the dynamic scale equation.
For the Gaussian unobserved components model, u_t is the prediction error and $\kappa^{\dagger}$ is the Kalman gain.

Fitting such a model using the STAMP package gave an estimate of 0.579 for $\kappa^{\dagger}$. The plot of the filtered level, $\mu_{t+1|t}$, shows it to be sensitive to extreme values, while the ACF of the absolute values of the residuals provides strong evidence of serial correlation in variance.

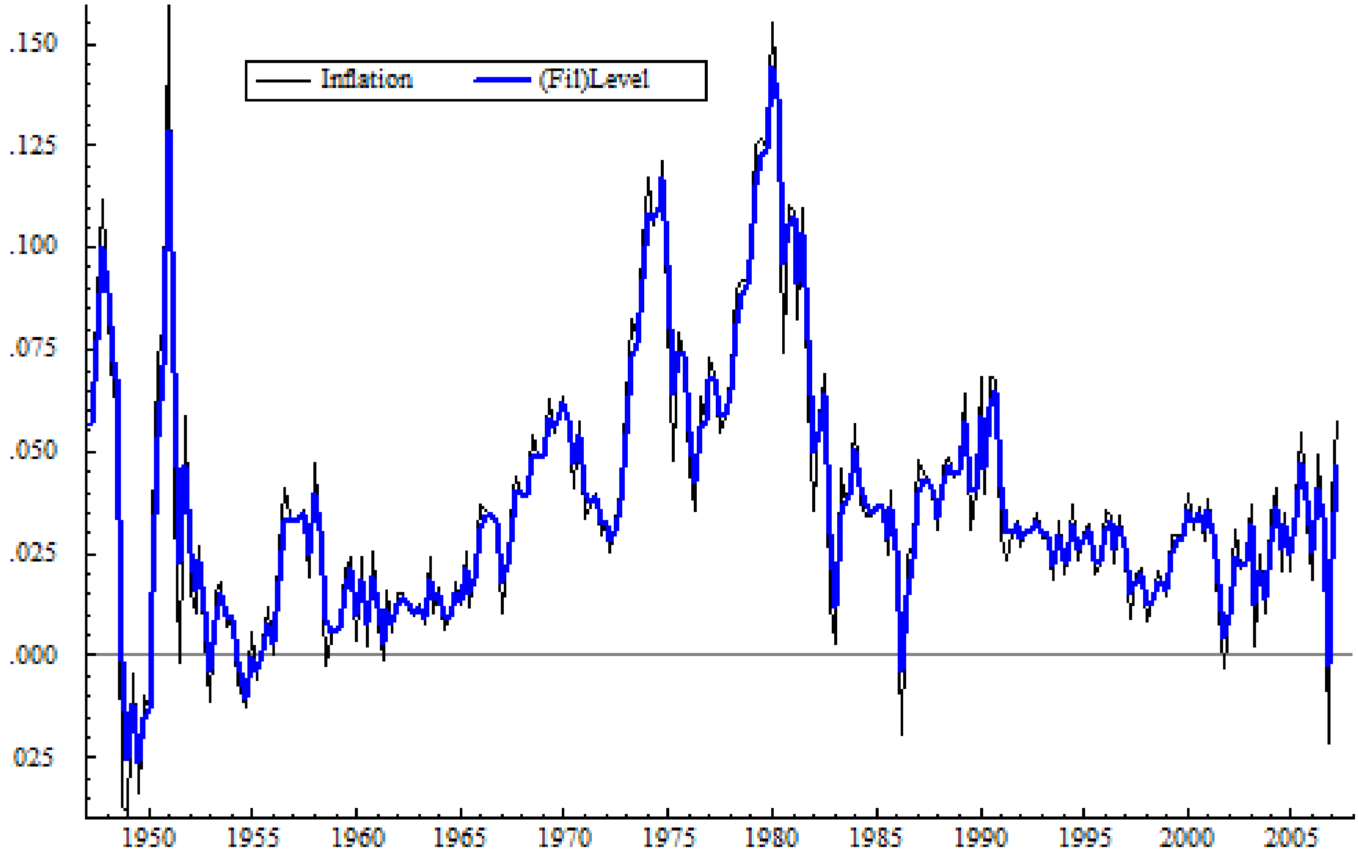


Figure: Filtered level from a Gaussian model.

Estimating a model in which filtered location is a random walk and scale evolves as a first-order Beta-t-EGARCH process gives the following ML estimates:
for location, $\widetilde{\kappa}^{\dagger} = 0.699(0.097)$,

and for scale, $\widetilde{\delta} = -0.370(0.214)$, $\widetilde{\phi} = 0.912(0.051)$ and $\widetilde{\kappa} = 0.$
$118(0.041)$, with $\widetilde{\nu} = 11.71(4.58)$.

The filtered DCS estimates respond less to extreme values than those from the homoscedastic Gaussian model.

Andrew Harvey (ach34@cam.ac.uk) (Faculty Robust Time Series Models / 82

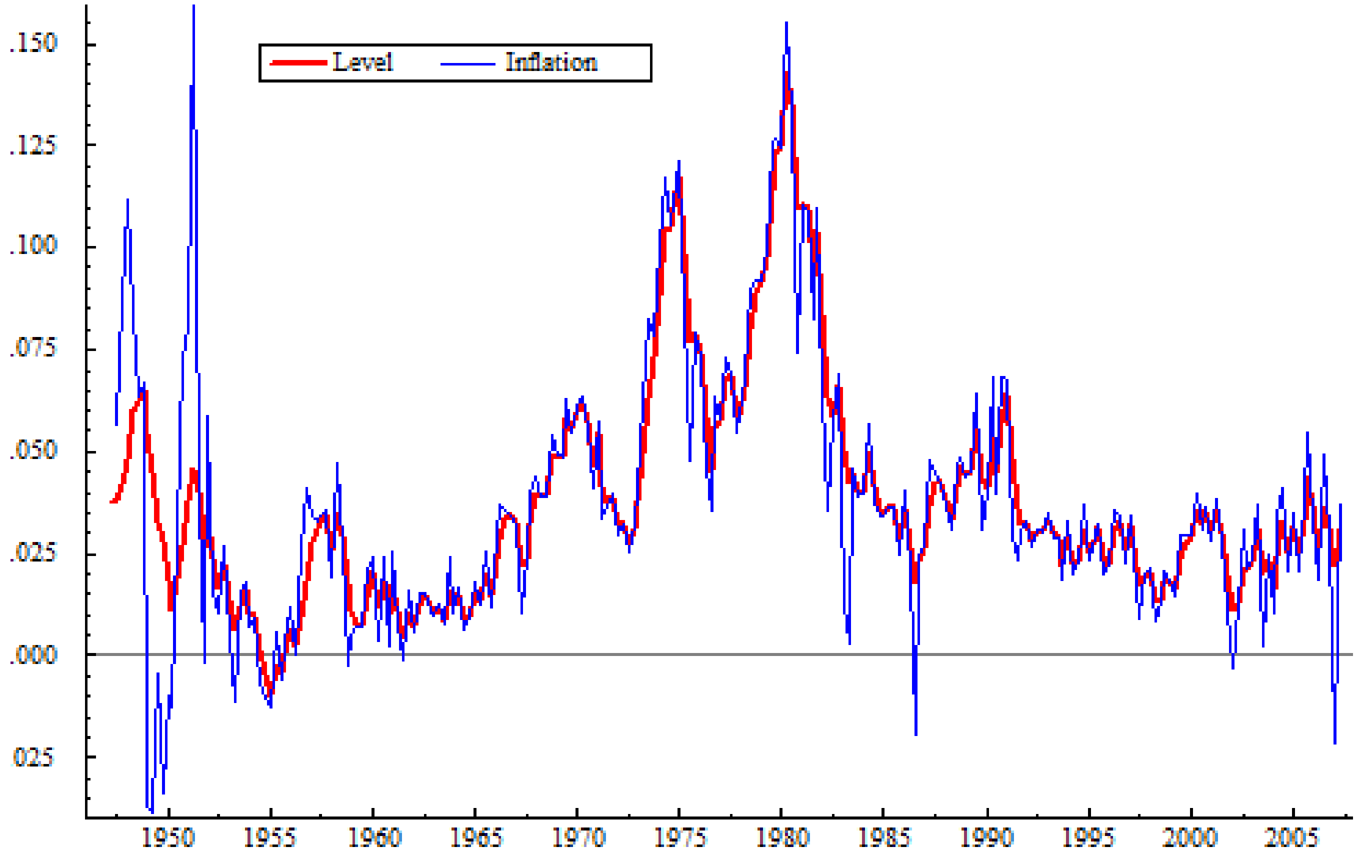


Figure: Filtered DCS level with Beta-t-GARCH.

Conclusions

DCS filter enables a time series model to be handled robustly.
Model-based approach, based on a t-distribution, is relatively simple, both conceptually and computationally, and is amenable to diagnostic checking and generalization.
Consider stationary models and then move on to include trend and seasonal components.
The same techniques could be applied to robustify ARIMA and seasonal ARIMA models.
Optimal forecasts can be computed recursively, either as in an ARMA model or by using the SSF, and multi-step conditional distributions can be easily constructed by simulation.
Explanatory variables.

Other generalizations are possible. eg a skewed-t model may be adopted using the method used by Harvey and Sucarrat (2012) for a volatility model.