25. Entropy as an Isomorphism Invariant. Generators and Sinai's theorem.

§25.1 Entropy as an isomorphism invariant

Recall the definition of what it means to say that two measure-preserving transformations are metrically isomorphic.

Definition. We say that two measure-preserving transformations $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, m, S)$ are *(measure theoretically) isomorphic* if there exist $M \in \mathcal{B}$ and $N \in \mathcal{C}$ such that

(i) $TM \subset M$, $SN \subset N$,

(ii) $\mu(M) = 1$, $m(N) = 1$,

and there exists a bijection $\phi : M \to N$ such that

(i) ϕ, ϕ^{-1} are measurable and measure-preserving (i.e. $\mu(\phi^{-1}A) = m(A)$ for all $A \in \mathcal{C}$),

(ii) $\phi \circ T = S \circ \phi$.

We prove that two metrically isomorphic measure-preserving transformations have the same entropy.

Theorem 25.1
Let $T : X \to X$ be a measure-preserving of $(X, \mathcal{B}, \mu)$ and let $S : Y \to Y$ be a measure-preserving transformation of $(Y, \mathcal{C}, m)$. If T and S are isomorphic then $h(T) = h(S)$.

Proof. Let $M \subset X$, $N \subset Y$ and $\phi : M \to N$ be as above. If α is a partition of Y then (changing it on a set of measure zero if necessary) it is also a partition of N. The inverse image $\phi^{-1}\alpha = \{\phi^{-1}A \mid A \in \alpha\}$ is a partition of M and hence of X. Furthermore,

$$\begin{aligned} H_\mu(\phi^{-1}\alpha) &= -\sum_{A\in\alpha} \mu(\phi^{-1}A) \log \mu(\phi^{-1}A) \\ &= -\sum_{A\in\alpha} m(A) \log m(A) \\ &= H_m(\alpha). \end{aligned}$$

More generally,

$$\begin{aligned} H_\mu\left(\bigvee_{j=0}^{n-1} T^{-j}(\phi^{-1}\alpha)\right) &= H_\mu\left(\phi^{-1}\left(\bigvee_{j=0}^{n-1} S^{-j}\alpha\right)\right) \\ &= H_m\left(\bigvee_{j=0}^{n-1} S^{-j}\alpha\right). \end{aligned}$$

Therefore, dividing by n and letting $n \to \infty$, we have

$$h(S,\alpha) = h(T,\phi^{-1}\alpha).$$

Thus

$$\begin{aligned} h(S) &= \sup\{h(S,\alpha) \mid \alpha \text{ partition of } Y, H_m(\alpha) < \infty\} \\ &= \sup\{h(T,\phi^{-1}\alpha) \mid \alpha \text{ partition of } Y, H_m(\alpha) < \infty\} \\ &\le \sup\{h(T,\beta) \mid \beta \text{ partition of } X, H_\mu(\beta) < \infty\} \\ &= h(T). \end{aligned}$$

By symmetry, we also have $h(T) \le h(S)$. Therefore $h(T) = h(S)$. □

Note that the converse to Theorem 25.1 is false in general: if two measure-preserving transformations have the same entropy then they are not necessarily metrically isomorphic.

§25.2 Calculating entropy

At first sight, the entropy of a measure-preserving transformation seems hard to calculate as it involves taking a supremum over all possible (finite entropy) partitions. However, some short cuts are possible.

§25.3 Generators and Sinai's theorem

A major complication in the definition of entropy is the need to take the supremum over all finite entropy partitions. Sinai's theorem guarantees that $h(T) = h(T,\alpha)$ for a partition α whose refinements generates the full σ-algebra.

We begin by proving the following result.

Theorem 25.2 (Abramov's theorem)
Suppose that $\alpha_1 \le \alpha_2 \le \cdots \uparrow \mathcal{B}$ are countable partitions such that $H(\alpha_n) < \infty$ for all $n \ge 1$. Then

$$h(T) = \lim_{n\to\infty} h(T,\alpha_n).$$

Proof. Choose any countable partition β such that $H(\beta) < \infty$. Fix $n > 0$. Then

$$\begin{aligned} H\left(\bigvee_{j=0}^{k-1} T^{-j}\beta\right) &\leq H\left(\bigvee_{j=0}^{k-1} T^{-j}\beta \vee \bigvee_{j=0}^{k-1} T^{-j}\alpha_n\right) \\ &\leq H\left(\bigvee_{j=0}^{k-1} T^{-j}\alpha_n\right) + H\left(\bigvee_{j=0}^{k-1} T^{-j}\beta \mid \bigvee_{j=0}^{k-1} T^{-j}\alpha_n\right), \end{aligned}$$

by the basic identity.

Observe that

$$\begin{aligned} & H\left(\bigvee_{j=0}^{k-1} T^{-j}\beta \mid \bigvee_{j=0}^{k-1} T^{-j}\alpha_n\right) \\ = \;& H\left(\beta \mid \bigvee_{j=0}^{k-1} T^{-j}\alpha_n\right) + H\left(\bigvee_{j=1}^{k-1} T^{-j}\beta \mid \beta \vee \bigvee_{j=0}^{k-1} T^{-j}\alpha_n\right) \\ \leq \;& H(\beta|\alpha_n) + H\left(\bigvee_{j=1}^{k-1} T^{-j}\beta \mid \bigvee_{j=1}^{k-1} T^{-j}\alpha_n\right) \\ = \;& H(\beta|\alpha_n) + H\left(\bigvee_{j=0}^{k-2} T^{-j}\beta \mid \bigvee_{j=0}^{k-2} T^{-j}\alpha_n\right). \end{aligned}$$

Continuing this inductively we see that

$$H\left(\bigvee_{j=0}^{k-1} T^{-j}\beta \mid \bigvee_{j=0}^{k-1} T^{-j}\alpha_n\right) \leq kH(\beta|\alpha_n).$$

Hence

$$\begin{aligned} h(T,\beta) &= \lim_{k\to\infty} \frac{1}{k} H\left(\bigvee_{j=0}^{k-1} T^{-j}\beta\right) \\ &\leq \lim_{k\to\infty} \frac{1}{k} H\left(\bigvee_{j=0}^{k-1} T^{-j}\alpha_n\right) + H(\alpha \mid \alpha_n) \\ &= h(T,\alpha_n) + H(\beta \mid \alpha_n). \end{aligned}$$

We now prove that $H(\beta \mid \alpha_n) \to 0$ as $n \to \infty$. To do this, it is sufficient to prove that $I(\beta \mid \alpha_n) \to 0$ in L^1 as $n \to \infty$. Recall that

$$I(\beta \mid \alpha_n)(x) = -\sum_{B\in\beta} \chi_B(x) \log \mu(B \mid \alpha_n)(x) = -\log \mu(B \mid \alpha_n)(x)$$

if $x \in B$, $B \in \beta$. By the Increasing Martingale Theorem, we know that

$$\mu(B \mid \alpha_n)(x) \to \chi_B \text{ a.e.}$$

Hence for $x \in B$

$$I(\beta \mid \alpha_n)(x) \to -\log \chi_B = 0.$$

Hence for any countable partition β with $H(\beta) < \infty$ we have that $h(T, \beta) \leq \lim_{n\to\infty} h(T, \alpha_n)$. The result follows by taking the supremum over all such β. □

Definition. We say that a countable partition α is a *generator* if T is invertible and

$$\bigvee_{j=-(n-1)}^{n-1} T^{-j}\alpha \to \mathcal{B}$$

as $n \to \infty$.

We say that a countable partition α is a *strong generator* if

$$\bigvee_{j=0}^{n-1} T^{-j}\alpha \to \mathcal{B}$$

as $n \to \infty$.

Remark. To check whether a partition α is a generator (respectively, a strong generator) it is sufficient to check that it separates almost every pair of points. That is, for almost every $x, y \in X$, there exists n such that x, y are in different elements of the partition $\bigvee_{j=-(n-1)}^{n-1} T^{-j}\alpha$ ($\bigvee_{j=0}^{n-1} T^{-j}\alpha$, respectively).

The following important theorem will be the main tool in calculating entropy.

Theorem 25.3 (Sinai's theorem)
Suppose α is a strong generator or that T is invertible and α is a generator. If $H(\alpha) < \infty$ then

$$h(T) = h(T, \alpha).$$

Proof. The proofs of the two cases are similar, we prove the case when T is invertible and α is a generator of finite entropy.

Let $n \geq 1$. Then

$$\begin{aligned} & h(T, \bigvee_{j=-n}^{n} T^{-j}\alpha) \\ = {} & \lim_{k\to\infty} \frac{1}{k} H(T^n\alpha \vee \cdots \vee T^{-n}\alpha \vee T^{-(n-1)}\alpha \vee \cdots \vee T^{-(n+k-1)}\alpha) \\ = {} & \lim_{k\to\infty} \frac{1}{k} H(\alpha \vee \cdots \vee T^{-(2n+k-1)}\alpha) \\ = {} & h(T, \alpha) \end{aligned}$$

for each n. As α is a generator, we have that

$$\bigvee_{j=-n}^{n} T^{-j}\alpha \to \mathcal{B}.$$

By Abramov's theorem, $h(T, \alpha) = h(T)$. □

§25.4 Entropy of a power

Observe that if T preserves the measure μ then so does T^k. The following result relates the entropy of T and T^k.

Theorem 25.4

(i) For $k \geq 0$ we have that $h(T^k) = kh(T)$.

(ii) If T is invertible then $h(T) = h(T^{-1})$.

Proof. We prove (i), leaving the case $k = 0$ as an exercise. Choose a countable partition α with $H(\alpha) < \infty$. Then

$$\begin{aligned} h\left(T^k, \bigvee_{j=0}^{k-1} T^{-j}\alpha\right) &= \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{j=0}^{nk-1} T^{-j}\alpha\right) \\ &= k \lim_{n\to\infty} \frac{1}{nk} H\left(\bigvee_{j=0}^{nk-1} T^{-j}\alpha\right) = kh(T, \alpha). \end{aligned}$$

Thus,

$$\begin{aligned} kh(T) &= \sup_{H(\alpha)<\infty} kh(T, \alpha) \\ &= \sup_{H(\alpha)<\infty} h\left(T^k, \bigvee_{j=0}^{k-1} T^{-j}\alpha\right) \\ &\leq \sup_{H(\alpha)<\infty} h(T^k, \alpha) = h(T^k). \end{aligned}$$

On the other hand,

$$\begin{aligned} h(T^k, \alpha) &= \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} T^{-jk}\alpha\right) \\ &\leq \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{j=0}^{nk-1} T^{-j}\alpha\right) \quad \text{by Corollary 25.3} \\ &= k \lim_{n\to\infty} \frac{1}{nk} H\left(\bigvee_{j=0}^{nk-1} T^{-j}\alpha\right) = kh(T, \alpha), \end{aligned}$$

and so $h(T^k) \leq kh(T)$, completing the proof.

We prove (ii). We have

$$\begin{aligned} H\left(\bigvee_{j=0}^{n-1} T^{-j}\alpha\right) &= H\left(T^{n-1}\bigvee_{j=0}^{n-1} T^{-j}\alpha\right) \\ &= H\left(\bigvee_{j=0}^{n-1} T^{j}\alpha\right). \end{aligned}$$

Therefore

$$\begin{aligned} h(T,\alpha) &= \lim_{n\to\infty}\frac{1}{n}H\left(\bigvee_{j=0}^{n-1} T^{-j}\alpha\right) \\ &= \lim_{n\to\infty}\frac{1}{n}H\left(\bigvee_{j=0}^{n-1} T^{j}\alpha\right) = h(T^{-1},\alpha). \end{aligned}$$

Taking the supremum over α gives $h(T) = h(T^{-1})$. $\square$

Exercise 25.1
Prove that the entropy of the identity map is zero.