20. Recurrence and Unique Ergodicity

§20.1 Poincaré's Recurrence Theorem

We now go back to the general setting of a measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$. The following is the most basic result about the distribution of orbits.

Theorem 20.1 (Poincaré's Recurrence Theorem)
Let $T : X \to X$ be a measure-preserving transformation of $(X, \mathcal{B}, \mu)$ and let $A \in \mathcal{B}$ have $\mu(A) > 0$. Then for μ-a.e. $x \in A$, the orbit $\{T^n x\}_{n=0}^{\infty}$ returns to A infinitely often.

Proof. Let

$$E = \{x \in A \mid T^n x \in A \text{ for infinitely many } n\},$$

then we have to show that $\mu(A \backslash E) = 0$.

If we write

$$F = \{x \in A \mid T^n x \notin A \ \forall n \geq 1\}$$

then we have the identity

$$A \setminus E = \bigcup_{k=0}^{\infty} (T^{-k} F \cap A).$$

Thus we have the estimate

$$\begin{aligned} \mu(A \backslash E) &= \mu\left(\bigcup_{k=0}^{\infty} (T^{-k} F \cap A)\right) \\ &\leq \mu\left(\bigcup_{k=0}^{\infty} T^{-k} F\right) \\ &\leq \sum_{k=0}^{\infty} \mu(T^{-k} F). \end{aligned}$$

Since $\mu(T^{-k}F) = \mu(F) \ \forall k \geq 0$ (because the measure is preserved), it suffices to show that $\mu(F) = 0$.

First suppose that $n > m$ and that $T^{-m}F \cap T^{-n}F \neq \emptyset$. If y lies in this intersection then $T^m y \in F$ and $T^{n-m}(T^m y) = T^n y \in F \subset A$, which contradicts the definition of F. Thus $T^{-m}F$ and $T^{-n}F$ are disjoint.

Since $\{T^{-k}F\}_{n=0}^{\infty}$ is a disjoint family, we have

$$\sum_{k=0}^{\infty} \mu(T^{-k}F) = \mu\left(\bigcup_{k=0}^{\infty} T^{-k}F\right) \leq \mu(X) = 1.$$

Since the terms in the summation have the constant value $\mu(F)$, we must have $\mu(F) = 0$. □

Exercise 20.1

Construct an example to show that Poincaré's recurrence theorem does not hold on infinite measure spaces. (Recall that a measure space $(X, \mathcal{B}, \mu)$ *is infinite if* $\mu(X) = \infty$*.)*

§20.2 Unique Ergodicity

We finish this section by looking at the case where $T : X \to X$ has a *unique* invariant probability measure.

Definition. Let $(X, \mathcal{B})$ be a measurable space and let $T : X \to X$ be a measurable transformation. If there is a *unique* T-invariant probability measure then we say that T is *uniquely ergodic*.

Remark. You might wonder why we don't just call such T 'uniquely invariant'. Recall from Lecture 19 that the extremal points of $M(X, T)$ are precisely the ergodic measures. If $M(X, T)$ consists of just one measure then that measure is extremal, and so must be ergodic.

Unique ergodicity (for continuous maps) implies the following strong convergence result.

Theorem 20.2

Let X *be a compact metric space and let* $T : X \to X$ *be a continuous transformation. The following are equivalent:*

- *(i)* T *is uniquely ergodic;*
- *(ii) for each* $f \in C(X)$ *there exists a constant* $c(f)$ *such that*

$$\frac{1}{n}\sum_{j=0}^{n-1} f(T^j x) \to c(f),$$

uniformly for $x \in X$*, as* $n \to \infty$*.*

Proof. (ii) ⇒ (i): Suppose that μ, ν are T-invariant probability measures; we shall show that $\mu = \nu$.

Integrating the expression in (ii), we obtain

$$\begin{aligned}\int f\,d\mu &= \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\int f\circ T^j\,d\mu \\ &= \int \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} f\circ T^j\,d\mu \\ &= \int c(f)\,d\mu = c(f),\end{aligned}$$

and, by the same argument

$$\int f\,d\nu = c(f).$$

Therefore

$$\int f\,d\mu = \int f\,d\nu \quad \forall f\in C(X)$$

and so $\mu=\nu$ (by the Riesz Representation Theorem).

(i) ⇒ (ii): Let $M(X,T)=\{\mu\}$. If (ii) is true, then, by the Dominated Convergence Theorem, we must necessarily have $c(f)=\int f\,d\mu$. Suppose that (ii) is false. Then we can find $f\in C(X)$ and sequences $n_k\in\mathbb{N}$ and $x_k\in X$ such that

$$\lim_{k\to\infty}\frac{1}{n_k}\sum_{j=0}^{n_k-1} f(T^j x_k) \neq \int f\,d\mu.$$

For each $k\geq 1$, define a measure $\nu_k\in M(X)$ by

$$\nu_k = \frac{1}{n_k}\sum_{j=0}^{n_k-1} T_*^j\delta_{x_k},$$

so that

$$\int f\,d\nu_k = \frac{1}{n_k}\sum_{j=0}^{n_k-1} f(T^j x_k).$$

By the proof of Theorem 13.1, ν_k has a subsequence which converges weak* to a measure $\nu\in M(X,T)$. In particular, we have

$$\int f\,d\nu = \lim_{k\to\infty}\int f\,d\nu_k \neq \int f\,d\mu.$$

Therefore, $\nu\neq\mu$, contradicting unique ergodicity. □

§20.3 Example: The Irrational Rotation

Let $X = \mathbb{R}/\mathbb{Z}$, $T : X \to X : x \mapsto x + \alpha \bmod 1$, α irrational. Then T is uniquely ergodic (and $\mu =$ Lebesgue measure is the unique invariant probability measure).

Proof. Let m be an arbitrary T-invariant probability measure; we shall show that $m = \mu$.

Write $e_k(x) = e^{2\pi i k x}$. Then

$$\begin{aligned} \int e_k(x)\,dm &= \int e_k(Tx)\,dm \\ &= \int e_k(x+\alpha)\,dm \\ &= e^{2\pi i k\alpha} \int e_k(x)\,dm. \end{aligned}$$

Since α is irrational, if $k \neq 0$ then $e^{2\pi i k\alpha} \neq 1$ and so

$$\int e_k\,dm = 0. \tag{20.1}$$

Let $f \in C(X)$ have Fourier series $\sum_{k=-\infty}^{\infty} a_k e_k$, so that $a_0 = \int f\,d\mu$. For $n \geq 1$, we let σ_n denote the average of the first n partial sums. Then $\sigma_n \to f$ uniformly as $n \to \infty$. Hence

$$\lim_{n\to\infty} \int \sigma_n\,dm = \int f\,dm.$$

However using (20.1), we may calculate that

$$\int \sigma_n\,dm = a_0 = \int f\,d\mu.$$

Thus we have that $\int f\,dm = \int f\,d\mu$, for every $f \in C(X)$, and so $m = \mu$. $\square$