# 14. Examples of measure-preserving transformations: the continued fraction map, toral endomorphisms

## §14.1 The continued fraction map

Recall that the continued fraction map $T : [0, 1) \to [0, 1)$ is defined by

$$T(x) = \begin{cases} 0 & \text{if } x = 0, \\ \left\{\frac{1}{x}\right\} = \frac{1}{x} \bmod 1 & \text{if } 0 < x < 1. \end{cases}$$

One can easily show that the continued fraction map does not preserve Lebesgue measure, i.e. there exists $B \in \mathcal{B}$ such that $T^{-1}B$ and $B$ have different measure. (Indeed, choose $B$ to be any interval.)

Although the continued fraction map does not preserve Lebesgue measure, it does preserve Gauss' measure $\mu$, defined by

$$\mu(B) = \frac{1}{\log 2} \int_B \frac{1}{1+x}\, dx.$$

**Remark.** Two measures are said to be *equivalent* if they have the same sets of measure zero. Gauss' measure and Lebesgue measure are equivalent. This means that any property that holds for $\mu$-almost every point also holds for Lebesgue almost every point. This remark will have applications later when we use Birkhoff's Ergodic Theorem to describe properties of the continued fraction expansion for typical (i.e. Lebesgue almost every) points.

**Proof.** Using the Kolmogorov Extension Theorem argument, we only have to check that $\mu(T^{-1}I) = \mu(I)$ for intervals. If $I = (a, b)$ then

$$T^{-1}(a, b) = \bigcup_{n=1}^{\infty} \left( \frac{1}{b+n}, \frac{1}{a+n} \right).$$

Thus

$$\begin{aligned} &\mu(T^{-1}(a, b)) \\ &= \frac{1}{\log 2} \sum_{n=1}^{\infty} \int_{\frac{1}{b+n}}^{\frac{1}{a+n}} \frac{1}{1+x}\, dx \\ &= \frac{1}{\log 2} \sum_{n=1}^{\infty} \left[ \log\left(1 + \frac{1}{a+n}\right) - \log\left(1 + \frac{1}{b+n}\right) \right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\log 2} \sum_{n=1}^{\infty} [\log(a+n+1) - \log(a+n) - \log(b+n+1) + \log(b+n)] \\
&= \lim_{N\to\infty} \frac{1}{\log 2} \sum_{n=1}^{N} [\log(a+n+1) - \log(a+n) - \log(b+n+1) + \log(b+n)] \\
&= \frac{1}{\log 2} \lim_{N\to\infty} [\log(a+N+1) - \log(a+1) - \log(b+N+1) + \log(b+1)] \\
&= \frac{1}{\log 2} \left( \log(b+1) - \log(a+1) + \lim_{N\to\infty} \log\left( \frac{a+N+1}{b+N+1} \right) \right) \\
&= \frac{1}{\log 2} (\log(b+1) - \log(a+1)) \\
&= \frac{1}{\log 2} \int_a^b \frac{1}{1+x} \, dx = \mu(a,b),
\end{aligned}
$$

as required. □

**Exercise 14.1**

*Define the map* $T : [0,1] \to [0,1]$ *by*

$$
T(x) = \begin{cases} \frac{x}{1-x} & \text{if } 0 \leq x \leq 1/2 \\ \frac{1-x}{x} & \text{if } 1/2 \leq x \leq 1. \end{cases}
$$

*Define the measure* $\mu$ *on* $[0,1]$ *by*

$$
\mu(B) = \int_B \frac{dx}{x}
$$

*(note that the measure* $\mu$ *is* not *a probability measure as* $\mu([0,1]) = \infty$*).*

*(i) Show that* $\mu([a,b]) = \log b - \log a$.

*(ii) Show that*

$$
T^{-1}[a,b] = \left( \frac{a}{1+a}, \frac{b}{1+b} \right) \cup \left( \frac{1}{1+a}, \frac{1}{1+b} \right).
$$

*(iii) Show that* $\mu$ *is* $T$*-invariant.*

*(iv) Define* $h : [0,1] \to [0,\infty]$ *by*

$$
h(x) = \frac{1}{x} - 1.
$$

*Define* $S = hTh^{-1} : [0,\infty] \to [0,\infty]$ *(so that* $S$ *and* $T$ *are topologically conjugate—i.e. they have the same dynamics). Show that we have*

$$
S(x) = \begin{cases} x - 1 & \text{if } 1 \leq x < \infty \\ 1/x & \text{if } 0 \leq x < 1. \end{cases}
$$

*Relate the map* $S$ *to continued fractions.*

## §14.2 Linear toral endomorphisms

Let $T : \mathbb{R}^k/\mathbb{Z}^k \to \mathbb{R}^k/\mathbb{Z}^k$ be a linear toral endomorphism. Recall that this means that $T$ is given as follows:

$$T(x_1, \ldots, x_k) = A(x_1, \ldots, x_k) \bmod 1$$

where $A = (a_{i,j})$ is a $k \times k$ matrix with entries in $\mathbb{Z}$ and with $\det A \neq 0$.

We shall show that $\mu$ is $T$-invariant by using Fourier series.

### §14.2.1 Fourier series in higher dimensions

Let $X = \mathbb{R}^k/\mathbb{Z}^k$ be the $k$-dimensional torus and let $\mu$ denote Lebesgue measure on $X$. Let $f \in L^1(X, \mathcal{B}, \mu)$ be an integrable function defined on the torus. For each $n = (n_1, \ldots, n_k) \in \mathbb{Z}^k$ define

$$c_n = \int f(x_1, \ldots, x_k) e^{-2\pi i \langle n, x\rangle} \, d\mu$$

where $\langle n, x\rangle = n_1x_1 + \cdots + n_kx_k$.

Then we can associate to $f$ the Fourier series:

$$\sum_{n \in \mathbb{Z}^k} c_n e^{2\pi i \langle n, x\rangle},$$

where $n = (n_1, \ldots, n_k)$, $x = (x_1, \ldots, x_k)$. Essentially the same convergence results hold as in the case $k = 1$, provided that we write

$$s_n(x) = \sum_{\ell_1=-n}^{n} \cdots \sum_{\ell_k=-n}^{n} c_\ell e^{2\pi i \langle \ell, x\rangle}.$$

As in the one-dimensional case, we have that

$$c_0 = \int f \, d\mu,$$

and

$$\int e^{2\pi i \langle n, x\rangle} \, d\mu = \begin{cases} 0 & \text{if } n \neq (0, \ldots, 0) \\ 1 & \text{if } n = (0, \ldots, 0). \end{cases}$$

### §14.2.2 Lebesgue measure is an invariant measure for a toral endomorphism

Let $\mu$ denote Lebesgue measure. To show that $\mu$ is $T$-invariant, it is sufficient to prove that for each continuous function $f \in C(X, \mathbb{R})$ we have

$$\int fT \, d\mu = \int f \, d\mu.$$

We associate to such an $f$ its Fourier series:

$$\sum_{n\in\mathbb{Z}^k} c_n e^{2\pi i\langle n,x\rangle}.$$

Then $f \circ T$ has Fourier series

$$\sum_{n\in\mathbb{Z}^k} c_n e^{2\pi i\langle n,Ax\rangle}.$$

Intuitively, we can write

$$\begin{aligned}
\int f\circ T\,d\mu &= \int \sum_{n\in\mathbb{Z}^k} c_n e^{2\pi i\langle n,Ax\rangle}\,d\mu \\
&= \int \sum_{n\in\mathbb{Z}^k} c_n e^{2\pi i\langle nA,x\rangle}\,d\mu \\
&= \sum_{n\in\mathbb{Z}^k} c_n \int e^{2\pi i\langle nA,x\rangle}\,d\mu.
\end{aligned}$$

Using the fact that $\det A \neq 0$, we see that $nA = 0$ if and only if $n = 0$, Hence, all of the integrals above are 0, except for the term corresponding to $n = 0$. Hence

$$\int f\circ T\,d\mu = c_0 = \int f\,d\mu.$$

(This argument can be made rigorous as in Lecture 14.)

Therefore, by Lemma 13.3, $\mu$ is $T$-invariant.

**Exercise 14.2**
*Fix $\alpha \in \mathbb{R}$ and define the map $T : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ by*

$$T(x,y) = (x+\alpha, x+y).$$

*By using Fourier Series, sketch a proof that Lebesgue measure is $T$-invariant.*