

13. Examples of measure-preserving transformations: rotations of a torus, the doubling map

§13.1 Rotations of a torus, the doubling map

In this lecture we give two methods by which one can show that a given dynamical system preserves a given measure. We shall illustrate these two methods by proving that (i) a rotation of a torus, and (ii) the doubling map preserve Lebesgue measure. Let us first recall how these examples are defined.

§13.1.1 Rotations on tori

Take $X = \mathbb{R}^k / \mathbb{Z}^k$, the k -dimensional torus. Recall that Lebesgue measure μ is defined by first defining the measure of a k -dimensional cube $[a_1, b_1] \times \cdots \times [a_k, b_k]$ to be

$$\mu \left(\prod_{j=1}^k [a_j, b_j] \right) = \prod_{j=1}^k (b_j - a_j)$$

and then extending this to the Borel σ -algebra by using the Kolmogorov Extension Theorem.

Fix $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$ and define $T : X \rightarrow X$ by

$$T(x_1, \dots, x_k) = (x_1 + \alpha_1, \dots, x_k + \alpha_k) \mod 1.$$

(In multiplicative notation this becomes:

$$T(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_k}) = (e^{2\pi i (\theta_1 + \alpha_1)}, \dots, e^{2\pi i (\theta_k + \alpha_k)}).$$

This is the rotation of the k -dimensional torus $\mathbb{R}^k / \mathbb{Z}^k$ by the vector $(\alpha_1, \dots, \alpha_k)$.

In dimension $k = 1$ we get a rotation of a circle defined by

$$T : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z} : x \mapsto x + \alpha \mod 1.$$

§13.1.2 The doubling map

Let $X = \mathbb{R}/\mathbb{Z}$ denote the circle. The doubling map is defined to be

$$T : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z} : x \mapsto 2x \mod 1.$$

§13.2 Using the Kolmogorov Extension Theorem

Recall the Kolmogorov Extension Theorem:

Theorem 13.1 (Kolmogorov Extension Theorem)

Let \mathcal{A} be an algebra of subsets of X . Suppose that $\mu : \mathcal{A} \rightarrow \mathbb{R}^+ \cup \{\infty\}$ satisfies:

- (i) $\mu(\emptyset) = 0$;
- (ii) there exists finitely or countably many sets $X_n \in \mathcal{A}$ such that $X = \bigcup_n X_n$ and $\mu(X_n) < \infty$;
- (iii) if $E_n \in \mathcal{A}$, $n \geq 1$, are pairwise disjoint and if $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$ then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Then there is a unique measure $\mu : \mathcal{B}(\mathcal{A}) \rightarrow \mathbb{R}^+ \cup \{\infty\}$ which is an extension of $\mu : \mathcal{A} \rightarrow \mathbb{R}^+ \cup \{\infty\}$.

That is, if something looks like a measure on an algebra \mathcal{A} , then it extends *uniquely* to a measure defined on the σ -algebra $\mathcal{B}(\mathcal{A})$ generated by \mathcal{A} .

Corollary 13.2

Let \mathcal{A} be an algebra of subsets of X . Suppose that μ_1 and μ_2 are two measures on $\mathcal{B}(\mathcal{A})$ such that $\mu_1(E) = \mu_2(E)$ for all $E \in \mathcal{A}$. Then $\mu_1 = \mu_2$ on $\mathcal{B}(\mathcal{A})$.

To show that a dynamical system T preserves a probability measure μ we have to show that $T_*\mu = \mu$. By the above corollary, we see that it is sufficient to check that $T_*\mu = \mu$ on an algebra that generates the σ -algebra.

Recall that the collection of all finite unions of sub-intervals forms an algebra of subsets of both $[0, 1]$ and \mathbb{R}/\mathbb{Z} that generates the Borel σ -algebra. Similarly, the collection of all finite unions of k -dimensional sub-cubes of $\mathbb{R}^k/\mathbb{Z}^k$ forms an algebra of subsets of the k -dimensional torus $\mathbb{R}^k/\mathbb{Z}^k$ that generates the Borel σ -algebra.

Thus to show that for a dynamical system T defined on \mathbb{R}/\mathbb{Z} preserves a measure μ we need only check that

$$T_*\mu(a, b) = \mu T^{-1}(a, b) = \mu(a, b)$$

for all subintervals (a, b) .

§13.2.1 Rotations of a circle

We claim that the rotation $T(x) = x + \alpha \bmod 1$ preserves Lebesgue measure μ . First note that

$$T^{-1}(a, b) = \{x \mid T(x) \in (a, b)\} = (a - \alpha, b - \alpha).$$

Hence

$$\begin{aligned} T_*\mu(a, b) &= \mu T^{-1}(a, b) \\ &= \mu(a - \alpha, b - \alpha) \\ &= (b - \alpha) - (a - \alpha) \\ &= b - a \\ &= \mu(a, b). \end{aligned}$$

Hence $T_*\mu = \mu$ on the algebra of finite unions of subintervals. As this algebra generates the Borel σ -algebra, by uniqueness in the Kolmogorov Extension Theorem we see that $T_*\mu = \mu$; i.e. Lebesgue measure is T -invariant.

§13.2.2 The doubling map

We claim that the doubling map $T(x) = 2x \bmod 1$ preserves Lebesgue measure μ . First note that

$$T^{-1}(a, b) = \{x \mid T(x) \in (a, b)\} = \left(\frac{a}{2}, \frac{b}{2}\right) \cup \left(\frac{a+1}{2}, \frac{b+1}{2}\right).$$

Hence

$$\begin{aligned} T_*\mu(a, b) &= \mu T^{-1}(a, b) \\ &= \mu\left(\frac{a}{2}, \frac{b}{2}\right) \cup \left(\frac{a+1}{2}, \frac{b+1}{2}\right) \\ &= \frac{b}{2} - \frac{a}{2} + \frac{(b+1)}{2} - \frac{(a+1)}{2} \\ &= b - a = \mu(a, b). \end{aligned}$$

Hence $T_*\mu = \mu$ on the algebra of finite unions of subintervals. As this semi-algebra generates the Borel σ -algebra, by uniqueness in the Kolmogorov Extension Theorem we see that $T_*\mu = \mu$; i.e. Lebesgue measure is T -invariant.

§13.3 Using Fourier series

Let \mathcal{B} denote the Borel σ -algebra on \mathbb{R}/\mathbb{Z} and let μ be Lebesgue measure. Given a Lebesgue integrable function $f \in L^1(\mathbb{R}/\mathbb{Z}, \mathcal{B}, \mu)$, we can associate to f the *Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nx + b_n \sin 2\pi nx),$$

where

$$a_n = 2 \int_0^1 f(x) \cos 2\pi n x \, d\mu, \quad b_n = 2 \int_0^1 f(x) \sin 2\pi n x \, d\mu.$$

(Notice that we are not claiming that the series converges—we are just formally associating the Fourier series to f .)

We shall find it more convenient to work with a complex form of the Fourier series:

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}, \quad (13.1)$$

where

$$c_n = \int_0^1 f(x) e^{-2\pi i n x} d\mu.$$

(In particular, $c_0 = \int_0^1 f \, d\mu$.)

We are still not making any assumption as to (i) whether the series (13.1) converges at all, or (ii) whether, if the series does converge, it converges to $f(x)$. In general, answering these questions relies on the class of function to which f belongs.

The weakest class of function is $f \in L^1(X, \mathcal{B}, \mu)$. In this case, we only know that the coefficients $c_n \rightarrow 0$ as $|n| \rightarrow \infty$. Although this condition is clearly necessary for (13.1) to converge, it is not sufficient, and there exist examples of functions $f \in L^1(X, \mathcal{B}, \mu)$ for which (13.1) does not converge to $f(x)$.

Lemma 13.3 (Riemann-Lebesgue Lemma)

If $f \in L^1(\mathbb{R}/\mathbb{Z}, \mathcal{B}, \mu)$ then $c_n \rightarrow 0$ as $|n| \rightarrow \infty$, i.e.:

$$\lim_{n \rightarrow \pm\infty} \int_0^1 f(x) e^{2\pi i n x} d\mu = 0.$$

It is of great interest and practical importance to know when and in what sense the Fourier series converges to the original function f . For convenience, we shall write the n th partial sum of (13.1) as

$$s_n(x) = \sum_{\ell=-n}^n c_\ell e^{2\pi i \ell x}$$

and the average of the first n partial sums as

$$\sigma_n(x) = \frac{1}{n} (s_0(x) + s_1(x) + \cdots + s_{n-1}(x)).$$

We define $L^2(X, \mathcal{B}, \mu)$ to be the set of all functions $f : X \rightarrow \mathbb{R}$ such that $\int |f|^2 d\mu < \infty$. Notice that $L^2 \subset L^1$.

Theorem 13.4

(i) If $f \in L^2(\mathbb{R}/\mathbb{Z}, \mathcal{B}, \mu)$ then s_n converges to f in L^2 , i.e.,

$$\int |s_n - f|^2 d\mu \rightarrow 0, \text{ as } n \rightarrow \infty.$$

(ii) If $f \in C(\mathbb{R}/\mathbb{Z})$ then σ_n converges uniformly to f as $n \rightarrow \infty$, i.e., $\|\sigma_n - f\|_\infty \rightarrow 0$, as $n \rightarrow \infty$.

In summary:

Class of function	Property of Fourier coefficients	Fourier series converges to function
L^1	$c_n \rightarrow 0$	Not in general
L^2	partial sums s_n converge	Yes, $s_n \rightarrow f$ (convergence in L^2 sense)
continuous	averages σ_n of partial sums converge	Yes, $\sigma_n \rightarrow f$ (uniform convergence)

§13.3.1 Rotations of a circle

Let $T(x) = x + \alpha \bmod 1$ be a circle rotation. We now give an alternative method of proving that μ is T -invariant using Fourier series. Recall Lemma 12.3: μ is T -invariant if and only if

$$\int f \circ T d\mu = \int f d\mu \text{ for all } f \in C(X, \mathbb{R}).$$

Heuristically, the argument is as follows. First note that

$$\int e^{2\pi i n x} d\mu = \begin{cases} 0, & \text{if } n \neq 0 \\ 1, & \text{if } n = 0. \end{cases}$$

If $f \in C(X, \mathbb{R})$ has Fourier series $\sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x}$ then $f \circ T$ has Fourier series $\sum_{n \in \mathbb{Z}} c_n e^{2\pi i n \alpha} e^{2\pi i n x}$. The underlying idea is the following:

$$\begin{aligned} \int f \circ T d\mu &= \int \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n \alpha} e^{2\pi i n x} d\mu \\ &= \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n \alpha} \int e^{2\pi i n x} d\mu \\ &= c_0 = \int f d\mu. \end{aligned}$$

Notice that the above involves saying the ‘the integral of an infinite sum is the infinite sum of the integrals’. This is not necessarily the case, so to make this argument rigorous we need to use Theorem 14.4(ii) to justify this step.

Let $f \in C(X, \mathbb{R})$. Then f has a Fourier series

$$\sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x}.$$

Let $s_n(x)$ denote the n th partial sum:

$$s_n(x) = \sum_{\ell=-n}^n c_\ell e^{2\pi i \ell x}.$$

Then

$$s_n(Tx) = \sum_{\ell=-n}^n c_\ell e^{2\pi i \ell \alpha} e^{2\pi i \ell x}$$

and this is the n th partial sum for the Fourier series of $f \circ T$. As $\int e^{2\pi i \ell x} d\mu = 0$ unless $\ell = 0$, it follows that

$$\int s_n d\mu = c_0 = \int s_n \circ T d\mu.$$

Consider

$$\sigma_n(x) = \frac{1}{n}(s_0 + \cdots + s_{n-1})(x).$$

Then $\sigma_n(x) \rightarrow f(x)$ uniformly. Moreover, $\sigma_n(Tx) \rightarrow f(Tx)$ uniformly. Hence

$$\int f d\mu = \lim_{n \rightarrow \infty} \int \sigma_n d\mu = c_0 = \lim_{n \rightarrow \infty} \int \sigma_n \circ T d\mu = \int f \circ T d\mu$$

and Lemma 13.3 implies that Lebesgue measure is invariant.

§13.4 The doubling map

Define $T : X \rightarrow X$ by

$$T(x) = 2x \bmod 1.$$

Heuristically, the argument is as follows: If f has Fourier series $\sum_n c_n e^{2\pi i n x}$ then $f \circ T$ has Fourier series $\sum_n c_n e^{2\pi i 2n x}$. Hence

$$\begin{aligned} \int f \circ T d\mu &= \int \sum_n c_n e^{2\pi i 2n x} d\mu \\ &= \sum_n c_n \int e^{2\pi i 2n x} d\mu \\ &= c_0 \\ &= \int f d\mu. \end{aligned}$$

Again, this needs to be made rigorous, and the argument is similar to that above.

§13.4.1 Higher dimensional Fourier series

Let $X = \mathbb{R}^k/\mathbb{Z}^k$ be the k -dimensional torus and let μ denote Lebesgue measure on X . Let $f \in L^1(X, \mathcal{B}, \mu)$ be an integrable function defined on the torus. For each $n = (n_1, \dots, n_k) \in \mathbb{Z}^k$ define

$$c_n = \int f(x) e^{-2\pi i \langle n, x \rangle} d\mu$$

where $\langle n, x \rangle = n_1 x_1 + \dots + n_k x_k$. Then we can associate to f the Fourier series:

$$\sum_{n \in \mathbb{Z}^k} c_n e^{2\pi i \langle n, x \rangle},$$

where $n = (n_1, \dots, n_k)$, $x = (x_1, \dots, x_k)$. Essentially the same convergence results hold as in the case $k = 1$, provided that we write

$$s_n(x) = \sum_{\ell_1=-n}^n \dots \sum_{\ell_k=-n}^n c_\ell e^{2\pi i \langle \ell, x \rangle}.$$

Exercise 13.1

For an integer $k \geq 2$ define $T : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ by $T(x) = kx \bmod 1$. Show that T preserves Lebesgue measure.

Exercise 13.2

Let $\beta > 1$ denote the golden ratio (so that $\beta^2 = \beta + 1$). Define $T : [0, 1] \rightarrow [0, 1]$ by $T(x) = \beta x \bmod 1$. Show that T does not preserve Lebesgue measure. Define the measure μ by $\mu(B) = \int_B k(x) dx$ where

$$k(x) = \begin{cases} \frac{1}{\frac{1}{\beta} + \frac{1}{\beta^3}} & \text{on } [0, 1/\beta) \\ \frac{1}{\beta(\frac{1}{\beta} + \frac{1}{\beta^3})} & \text{on } [1/\beta, 1). \end{cases}$$

By using the Kolmogorov Extension Theorem, show that T preserves μ .

Exercise 13.3

Define the logistic map $T : [0, 1] \rightarrow [0, 1]$ by $T(x) = 4x(1 - x)$. Define the measure μ by

$$\mu(B) = \frac{1}{\pi} \int_B \frac{1}{\sqrt{x(1-x)}} dx.$$

- (i) Check that μ is a probability measure.
- (ii) By using the Kolmogorov Extension Theorem, show that T preserves μ .