12. The space of invariant measures

§12.1 Existence of invariant measures

Given a continuous mapping $T : X \to X$ of a compact metric space, it is natural to ask whether invariant measures necessarily exist, i.e., whether $M(X,T) \neq \emptyset$. The next result shows that this is the case.

Theorem 12.1
Let $T : X \to X$ be a continuous mapping of a compact metric space. Then there exists at least one T-invariant probability measure.

Proof. Let $\sigma \in M(X)$ be a probability measure (for example, we could take σ to be a Dirac measure). Define the sequence $\mu_n \in M(X)$ by

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} T_*^j \sigma,$$

so that, for $B \in \mathcal{B}$,

$$\mu_n(B) = \frac{1}{n}(\sigma(B) + \sigma(T^{-1}B) + \cdots + \sigma(T^{-(n-1)}B)).$$

Since $M(X)$ is weak* compact, some subsequence μ_{n_k} converges, as $k \to \infty$, to a measure $\mu \in M(X)$. We shall show that $\mu \in M(X,T)$. By Lemma 12.3, this is equivalent to showing that

$$\int f\,d\mu = \int f \circ T\,d\mu \quad \forall f \in C(X).$$

To see this, note that

$$\begin{aligned}
\left|\int f \circ T\,d\mu - \int f\,d\mu\right| &= \lim_{k\to\infty}\left|\int f\circ T\,d\mu_{n_k} - \int f\,d\mu_{n_k}\right| \\
&= \lim_{k\to\infty}\left|\frac{1}{n_k}\int \sum_{j=0}^{n_k-1}(f\circ T^{j+1} - f\circ T^j)\,d\sigma\right| \\
&= \lim_{k\to\infty}\left|\frac{1}{n_k}\int (f\circ T^{n_k} - f)\,d\sigma\right| \\
&\leq \lim_{k\to\infty}\frac{2\|f\|_\infty}{n_k} = 0.
\end{aligned}$$

Therefore, $\mu \in M(X,T)$, as required. □

§12.2 Properties of $\mathbf{M(X, T)}$

We now know that $M(X,T) \neq \emptyset$. The next result gives us some basic information about its structure.

Theorem 12.2

(i) $M(X,T)$ *is convex: i.e.* $\mu_1, \mu_2 \in M(X,T) \Rightarrow \alpha\mu_1 + (1-\alpha)\mu_2 \in M(X,T)$, *for all* $0 \leq \alpha \leq 1$.

(ii) $M(X,T)$ *is weak* closed (and hence compact).*

Proof. (i) If $\mu_1, \mu_2 \in M(X,T)$ and $0 \leq \alpha \leq 1$ then

$$\begin{aligned}
&(\alpha\mu_1 + (1-\alpha)\mu_2)(T^{-1}B) \\
&= \alpha\mu_1(T^{-1}B) + (1-\alpha)\mu_2(T^{-1}B) \\
&= \alpha\mu_1(B) + (1-\alpha)\mu_2(B) = (\alpha\mu_1 + (1-\alpha)\mu_2)(B),
\end{aligned}$$

so $\alpha\mu_1 + (1-\alpha)\mu_2 \in M(X,T)$.

(ii) Let μ_n be a sequence in $M(X,T)$ and suppose that $\mu_n \rightharpoonup \mu \in M(X)$, as $n \to \infty$. For $f \in C(X)$,

$$\int f \, d\mu_n = \int f \circ T \, d\mu_n.$$

As $n \to \infty$, the left-hand side converges to $\int f \, d\mu$ and the right-hand side converges to $\int f \circ T \, d\mu$. Hence $\int f \, d\mu = \int f \circ T \, d\mu$ and so, by Lemma 13.3, $\mu \in M(X,T)$. This shows that $M(X,T)$ is closed. It is compact since it is a closed subset of the compact set $M(X)$. □