7. More on uniform distribution

§7.1 Generalisation to Higher Dimensions

We shall now look at the distribution of sequences in $\mathbb{R}^k$.

Definition. A sequence $x_n = (x_n^1, \ldots, x_n^k) \in \mathbb{R}^k$ is said to be *uniformly distributed mod 1* if, for each choice of k intervals $[a_1, b_1], \ldots, [a_k, b_k] \subset [0, 1)$, we have that

$$\frac{1}{n}\sum_{j=0}^{n-1}\prod_{i=1}^{k}\chi_{[a_i,b_i]}(\{x_j^i\}) \to \prod_{i=1}^{k}(b_i - a_i), \quad \text{as } n \to \infty.$$

We have the following criterion for uniform distribution.

Theorem 7.1 (Multi-dimensional Weyl's Criterion)
The sequence $x_n \in \mathbb{R}^k$ is uniformly distributed mod 1 if and only if

$$\frac{1}{n}\sum_{j=0}^{n-1} e^{2\pi i(\ell_1 x_j^1 + \cdots + \ell_k x_j^k)} \to 0, \quad \text{as } n \to \infty,$$

for all $\ell = (\ell_1, \ldots, \ell_k) \in \mathbb{Z}^k \setminus \{0\}$.

Remark. Here and throughout $0 \in \mathbb{Z}^k$ denotes the zero vector $(0, \ldots, 0)$.

Proof. The proof is essentially the same as in the case $k = 1$. □

We shall apply this result to the sequence $x_n = (n\alpha_1, \ldots, n\alpha_k)$, for real numbers $\alpha_1, \ldots, \alpha_k$.

Suppose first that the numbers $\alpha_1, \ldots, \alpha_k, 1$ are *rationally independent*. This means that if $r_1, \ldots, r_k, r$ are rational numbers such that

$$r_1\alpha_1 + \cdots + r_k\alpha_k + r = 0,$$

then $r_1 = \cdots = r_k = r = 0$. In particular, for $\ell = (\ell_1, \ldots, \ell_k) \in \mathbb{Z}^k \setminus \{0\}$ and $n \in \mathbb{N}$,

$$\ell_1 n\alpha_1 + \cdots + \ell_k n\alpha_k \notin \mathbb{Z},$$

so that

$$e^{2\pi i(\ell_1 n\alpha_1 + \cdots + \ell_k n\alpha_k)} \neq 1.$$

We therefore have that

$$\begin{aligned}\left|\frac{1}{n}\sum_{j=0}^{n-1} e^{2\pi i(\ell_1 j\alpha_1+\cdots+\ell_k j\alpha_k)}\right| &= \left|\frac{1}{n}\frac{e^{2\pi i n(\ell_1\alpha_1+\cdots+\ell_k\alpha_k)}-1}{e^{2\pi i(\ell_1\alpha_1+\cdots+\ell_k\alpha_k)}-1}\right| \\ &\leq \frac{1}{n}\frac{2}{|e^{2\pi i(\ell_1\alpha_1+\cdots+\ell_k\alpha_k)}-1|} \to 0, \quad \text{as } n\to\infty.\end{aligned}$$

Therefore, by Weyl's Criterion, $(n\alpha_1,\ldots,n\alpha_k)$ is uniformly distributed mod 1.

Now suppose that the numbers $\alpha_1,\ldots,\alpha_k,1$ are *rationally dependent*. Then there exists $\ell=(\ell_1,\ldots,\ell_k)\in\mathbb{Z}^k\setminus\{0\}$ such that

$$\ell_1\alpha_1+\cdots+\ell_k\alpha_k\in\mathbb{Z}.$$

Thus $e^{2\pi i(\ell_1 n\alpha_1+\cdots+\ell_k n\alpha_k)}=1$ for all $n\in\mathbb{N}$ and so

$$\frac{1}{n}\sum_{j=0}^{n-1} e^{2\pi i(\ell_1 j\alpha_1+\cdots+\ell_k j\alpha_k)} = 1 \not\to 0, \quad \text{as } n\to\infty.$$

Therefore, $(n\alpha_1,\ldots,n\alpha_k)$ is *not* uniformly distributed mod 1.

§7.2 Generalisation to polynomials

We shall now consider another generalisation of the sequence $n\alpha$. Write

$$p(n)=\alpha_k n^k+\alpha_{k-1}n^{k-1}+\cdots\alpha_1 n+\alpha_0.$$

Theorem 7.2 (Weyl)
If any one of $\alpha_1,\ldots,\alpha_k$ is irrational then $p(n)$ is uniformly distributed mod 1.

To prove this theorem we shall need the following technical result.

Lemma 7.3 (van der Corput's Inequality)
Let $z_0,\ldots,z_{n-1}\in\mathbb{C}$ and let $1\leq m\leq n-1$. Then

$$\begin{aligned}m^2\left|\sum_{j=0}^{n-1} z_j\right|^2 &\leq m(n+m)\sum_{j=0}^{n-1}|z_j|^2 \\ &\quad + 2(n+m)\,\mathrm{Re}\sum_{j=1}^{m-1}(m-j)\sum_{i=0}^{n-1-j} z_{i+j}\bar{z}_i.\end{aligned}$$

Let $x_n\in\mathbb{R}$. For each $m\geq 1$ define the sequence $x_n^{(m)}=x_{n+m}-x_n$ of m^{th} differences. The following lemma allows us to infer the uniform distribution of the sequence x_n if we know the uniform distribution of the each of the m^{th} differences of x_n.

Lemma 7.4
Let $x_n \in \mathbb{R}$ be a sequence. Suppose that for each $m \geq 1$ the sequence $x_n^{(m)}$ of m^{th} differences is uniformly distributed mod 1. Then x_n is uniformly distributed mod 1.

Proof. We shall apply Weyl's Criterion. We need to show that if $\ell \in \mathbb{Z}\backslash\{0\}$ then

$$\frac{1}{n}\sum_{j=0}^{n-1} e^{2\pi i \ell x_j} \to 0, \quad \text{as } n \to \infty.$$

Let $z_j = e^{2\pi i \ell x_j}$ for $j = 0, \ldots, n-1$. Note that $|z_j| = 1$. Let $1 < m < n$. By van der Corput's inequality,

$$\begin{aligned} \frac{m^2}{n^2}\left|\sum_{j=0}^{n-1} e^{2\pi i \ell x_j}\right|^2 &\leq \frac{m}{n^2}(n+m)n \\ &\quad + \frac{2(n+m)}{n}\,\mathrm{Re}\sum_{j=1}^{m-1}\frac{(m-j)}{n}\sum_{i=0}^{n-1-j} e^{2\pi i \ell (x_{i+j}-x_i)} \\ &= \frac{m}{n}(m+n) + \frac{2(n+m)}{n}\,\mathrm{Re}\sum_{j=1}^{m-1}(m-j)A_{n,j} \end{aligned}$$

where

$$A_{n,j} = \frac{1}{n}\sum_{i=0}^{n-1-j} e^{2\pi i \ell (x_{i+j}-x_i)} = \frac{1}{n}\sum_{i=0}^{n-1-j} e^{2\pi i \ell x_i^{(j)}}.$$

As the sequence $x_i^{(j)}$ of j^{th} differences is uniformly distributed mod 1, by Weyl's criterion we have that $A_{n,j} \to 0$ for each $j = 1, \ldots, m-1$. Hence for each $m \geq 1$

$$\limsup_{n\to\infty} \frac{m^2}{n^2}\left|\sum_{j=0}^{n-1} e^{2\pi i \ell x_j}\right|^2 \leq \limsup_{n\to\infty} m\frac{(n+m)}{n} = m.$$

Hence, for each $m > 1$ we have

$$\limsup_{n\to\infty} \frac{1}{n}\left|\sum_{j=0}^{n-1} e^{2\pi i \ell x_j}\right| \leq \frac{1}{\sqrt{m}}.$$

As $m > 1$ is arbitrary, the result follows. $\square$

Proof of Weyl's Theorem. We will only prove Weyl's theorem in the special case where the leading digit α_k of

$$p(n) = \alpha_k n^k + \cdots + \alpha_1 n + \alpha_0$$

is irrational. (The general case, where α_i is irrational for some $1 \leq i \leq k$ can be deduced very easily from this special case, but we will not go into this.)

We shall use induction on the degree of p. Let $\Delta(k)$ denote the statement 'for every polynomial q of degree $\leq k$, with irrational leading coefficient, the sequence $q(n)$ is uniformly distributed mod 1'. We know that $\Delta(1)$ is true.

Suppose that $\Delta(k-1)$ is true. Let $p(n) = \alpha_k n^k + \cdots + \alpha_1 n + \alpha_0$ be an *arbitrary* polynomial of degree k with α_k irrational. For each $m \in \mathbb{N}$, we have that

$$\begin{aligned}
&p(n+m) - p(n) \\
&\quad = \quad \alpha_k(n+m)^k + \alpha_{k-1}(n+m)^{k-1} + \cdots + \alpha_1(n+m) + \alpha_0 \\
&\qquad\quad - \alpha_k n^k - \alpha_{k-1}n^{k-1} - \cdots - \alpha_1 n - \alpha_0 \\
&\quad = \quad \alpha_k n^k + \alpha_k k n^{k-1} m + \cdots + \alpha_{k-1}n^{k-1} + \alpha_{k-1}(k-1)n^{k-2}h \\
&\qquad\quad + \cdots + \alpha_1 n + \alpha_1 m + \alpha_0 - \alpha_k n^k - \alpha_{k-1}n^{k-1} - \cdots - \alpha_1 n - \alpha_0.
\end{aligned}$$

After cancellation, we can see that, for each m, $p(n+m) - p(n)$ is a polynomial of degree $k-1$, with irrational leading coefficient $\alpha_k k m$. Therefore, by the inductive hypothesis, $p(n+m)-p(n)$ is uniformly distributed mod 1. We may now apply Lemma 7.4 to conclude that $p(n)$ is uniformly distributed mod 1 and so $\Delta(k)$ holds. This completes the induction. $\square$

Exercise 7.1

Let $p(n) = \alpha_k n^k + \alpha_{k-1}n^{k-1} + \cdots + \alpha_1 n + \alpha_0$, $q(n) = \beta_k n^k + \beta_{k-1}n^{k-1} + \cdots + \beta_1 n + \beta_0$. Show that $(p(n), q(n))$ is uniformly distributed mod 1 if at least one of $(\alpha_k, \beta_k, 1), \ldots, (\alpha_1, \beta_1, 1)$ is rationally independent.