2. Some examples of dynamical systems

§2.1 The circle

Several of the key examples in the course take place on the circle. There are two different—although equivalent—ways of thinking about the circle.

We can think of the circle as the quotient group

$$\mathbb{R}/\mathbb{Z} = \{x + \mathbb{Z} \mid x \in \mathbb{R}\}$$

which is easily seen to be equivalent to $[0,1) \bmod 1$. We refer to this as *additive notation*.

Alternatively, we can regard the circle as

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\} = \{\exp 2\pi i\theta \mid \theta \in [0,1)\}.$$

We refer to this as *multiplicative notation*.

The two viewpoints are obviously equivalent, and we shall use whichever is most convenient given the circumstances.

We will also be interested in maps of the k-dimensional torus. In additive notation this is given by

$$\mathbb{R}^k/\mathbb{Z}^k = \{x + \mathbb{Z}^k \mid x \in \mathbb{R}^k\} = [0,1)^k \bmod 1$$

(in additive notation) and

$$S^1 \times \cdots \times S^1 (k\text{-times}) = \{(\exp 2\pi i\theta_1, \ldots, \exp 2\pi i\theta_k) \mid \theta_1, \ldots, \theta_k \in [0,1)\}$$

(in multiplicative notation).

§2.2 Rotations on a circle

Fix $\alpha \in [0,1)$ and define the map

$$T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} : x \mapsto x + \alpha \bmod 1.$$

(In multiplicative notation this is: $\exp 2\pi i\theta \mapsto \exp 2\pi i(\theta + \alpha)$.) This map acts on the circle by rotating it by angle α.

Suppose that $\alpha = p/q$ is rational (here, $p, q \in \mathbb{Z}$, $q \neq 0$). Then

$$T^q(x) = x + qp/q \bmod 1 = x + p \bmod 1 = x.$$

Hence *every* point of $\mathbb{R}/\mathbb{Z}$ is periodic.

When α is irrational, one can show that every point $x \in \mathbb{R}/\mathbb{Z}$ has a dense orbit. This will follow from a more general result later in the course, but it can also be proved directly.

Exercise 2.1
Prove that, for an irrational rotation of the circle, every orbit is dense. (Recall that the orbit of x is dense if: for all $y \in \mathbb{R}/\mathbb{Z}$ and for all $\varepsilon > 0$, there exists $n > 0$ such that $d(T^n(x), y) < \varepsilon$.)

(Hints: (1) First show that $T^n(x) = T^n(0) + x$ and conclude that it's sufficient to prove that the orbit of 0 is dense. (2) Prove that $T^n(x) \neq T^m(x)$ for $n \neq m$. (3) Show that for each $\varepsilon > 0$ there exists $n > 0$ such that $0 < n\alpha \bmod 1 < \varepsilon$ (you will need to remember that the circle is sequentially compact). (4) Now show that the orbit of 0 is dense.)

§2.3 The doubling map

We have already seen the doubling map

$$T : \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z} : x \mapsto 2x \bmod 1.$$

(In multiplicative notation this is

$$T(\exp 2\pi i\theta) = \exp 2\pi i(2\theta).)$$

Proposition 2.1
Let T be the doubling map.

(i) There are $2^n - 1$ points of period n.

(ii) The periodic points are dense.

(iii) There exists a dense orbit.

Proof. We prove (i). Notice that

$$T^n(x) = 2^n x = x \bmod 1$$

if there exists an integer $p > 0$ such that

$$2^n x = x + p.$$

Hence

$$x = \frac{p}{2^n - 1}.$$

We get distinct values of $x \in [0, 1)$ for $p = 0, 1, \ldots, 2^n - 2$. Hence there are $2^n - 1$ periodic points.

We leave (ii) as an exercise.

Exercise 2.2
Prove (ii).

We sketch the proof of (iii). Let us denote the interval $[0, 1/2)$ by the symbol 0 and denote the interval $[1/2, 1)$ by 1. Let $x \in [0, 1)$. For each $n \geq 0$ let x_n denote the symbol corresponding to the interval in which $T^n(x)$ lies. Thus to each $x \in [0, 1)$ we associate a sequence $(x_0, x_1, \ldots)$ of 0s and 1s. It is easy to see that

$$x = \sum_{n=0}^{\infty} \frac{x_n}{2^{n+1}}$$

so that the sequence $(x_0, x_1, \ldots)$ corresponds to the base 2 expansion of x.

Notice that if x has coding $(x_0, x_1, \ldots)$ then

$$T(x) = 2x \bmod 1 = \sum_{n=0}^{\infty} \frac{2x_n}{2^{n+1}} \bmod 1 = x_0 + \sum_{n=0}^{\infty} \frac{x_{n+1}}{2^{n+1}} \bmod 1 = \sum_{n=0}^{\infty} \frac{x_{n+1}}{2^{n+1}}$$

so that $T(x)$ has expansion $(x_1, x_2, \ldots)$, i.e. T can be thought of as acting on the coding of x be shifting the associated sequence one place to the left.

For each n-tuple $x_0, x_1, \ldots, x_{n-1}$ let

$$I(x_0, \ldots, x_{n-1}) = \{x \in [0, 1) \mid T^k(x) \text{ lies in interval } x_k \text{ for } k = 0, 1, \ldots, n-1\}.$$

That is, $I(x_0, \ldots, x_{n-1})$ corresponds to the set of all $x \in [0, 1)$ whose base 2 expansion starts $x_0, \ldots, x_{n-1}$. We call $I(x_0, \ldots, x_{n-1})$ a cylinder of rank n.

Exercise 2.3
Draw all cylinders of length ≤ 4.

One can show:

(i) la cylinder of rank n is an interval of length 2^{-n}.

(ii) for each $x \in [0, 1)$ with base 2 expansion $x_0, x_1, \ldots$, the intervals $I(x_0, \ldots, x_n)$ 'converge' as $n \to \infty$ (in an appropriate sense) to x.

From these observations it is easy to see that, in order to construct a dense orbit, it is sufficient to construct a point x such that for every cylinder I there exists $n = n(I)$ such that $T^n(x) \in I$. To do this, firstly write down all possible cylinders (there are countably many):

$$0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, 0000, 0001, \ldots.$$

Now take x to be the point with base 2 expansion

$$0100011011000001010011100101110111000000001\ldots$$

(that is, just adjoin all the symbolic representations of all cylinders in some order). One can easily check that such a point x has a dense orbit. $\square$

Exercise 2.4
(Assuming that you know what a metric space is.) Write down the proof of Proposition 2.1(iii), adding in complete details.

Remark. This technique of coding the orbits of a given dynamical system by partitioning the space X and forming an itinerary map is a very powerful technique that can be used to study many different classes of dynamical system.