Cocycles in Hyperbolic Dynamics: Livšic Regularity Theorems and Applications to Stable Ergodicity

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Declaration

This thesis is, to the best of my knowledge, original except where stated otherwise. All three chapters have been submitted for publication.

Summary

This thesis is about cocycle equations over hyperbolic dynamical systems. The three chapters are inter-related but can be read independently. A more detailed description of our results can be found in the Introduction.

Chapters 1 and 2 are concerned with cocycles over hyperbolic flows. In Chapter 1 we discuss 'regularity theorems' for Hölder continuous cocycles taking values in either a compact Lie group or an abelian Lie group. In particular, we prove that any measurable solution to the cocycle equation (holding a.e. with respect to a Hölder equilibrium state) has a Hölder continuous version for which the cocycle equation holds everywhere. This generalises a well-known theorem due to Livšic.

In Chapter 2 we consider flows obtained from a hyperbolic flow by either a toral extension or a velocity change. A cocycle is associated to each flow so obtained. We link these cocycles to the first (Čech) cohomology group (realised as the Bruschlinsky group) of the underlying basic set. When the Bruschlinsky group has finite rank, we show how to use the results of Chapter 1 to construct open dense sets of mixing toral extensions and velocity changes. Similar results are proved for suspensions of shifts of finite type.

In Chapter 3 we consider Hölder continuous cocycles over hyperbolic diffeomorphisms. We discuss regularity theorems for cocycle equations that are 'twisted' with the action of an automorphism of the group. We shall mostly be interested in an automorphism of a compact connected Lie group with the following property: the eigenvalues of the derivative at the identity have modulus 1. (We say such an automorphism is 'unipotent'.) In this case, we prove that any measurable solution (again holding a.e. with respect to a Hölder equilibrium state) has a Hölder continuous version for which the cocycle equation holds everywhere. We also prove a C^r version. The unipotency hypothesis on the automorphism is optimal.

Introduction

Abstract

We give a brief summary of the results presented in this thesis together with some of the relevant background. We do not give complete definitions here of some of the required constructions in hyperbolic dynamics and Lie groups, but instead refer the reader either to the relevant chapter or to [KH]. Often, we will not state how smooth the various maps are assumed to be. Instead we just assume that everything is smooth enough to make sense.

§1 Hyperbolic dynamical systems

This thesis is primarily about hyperbolic dynamical systems. Here we give a brief summary of such systems and, in particular, relate them to Anosov systems and Smale's Axiom A.

Let M be a compact Riemannian manifold and let ϕ be a diffeomorphism of M. A closed ϕ-invariant subset $\Lambda \subset M$ is said to possess a *hyperbolic splitting* if the tangent bundle restricted to Λ, $T_\Lambda M$, has a continuous splitting into two sub-bundles E^s, E^u for which there exist constants $C > 0$, $\lambda \in (0,1)$ such that

$$\|d\phi^n|_{E^s}\| \leq C\lambda^n, \quad \|d\phi^{-n}|_{E^u}\| \leq C\lambda^n$$

for $n \geq 0$. If ϕ is a flow, we say a closed ϕ-invariant subset $\Lambda \subset M$ has a *hyperbolic splitting* if $T_\Lambda M = E^s \oplus E^u \oplus E^0$ where E^s, E^u satisfy the above conditions (replacing $n \in \mathbb{N}$ by $t \in \mathbb{R}^+$) and E^0 is the one-dimensional sub-bundle spanned by the vector field generating the flow.

If M is connected and has a hyperbolic splitting then we call ϕ an *Anosov diffeomorphism* or *Anosov flow*.

We call a closed ϕ-invariant set $\Lambda \subset M$ a *basic set* if:

1. Λ has a hyperbolic splitting;
2. the periodic orbits in Λ are dense in Λ (and Λ is not a single periodic orbit);
3. there is a dense orbit in Λ;
4. $\Lambda = \bigcap_{n=-\infty}^{\infty} \phi^n U$ for some open neighbourhood $\Lambda \subset U \subset M$ ($n \in \mathbb{Z}$ or $\mathbb{R}$, as appropriate);
5. (in the case of a flow) Λ contains no fixed points.

We call a diffeomorphism or flow, when restricted to a basic set Λ, a *hyperbolic diffeomorphism* or *hyperbolic flow*. (Some authors, notably [KH], call Λ a *locally maximal hyperbolic set*, but we prefer the term basic set for its brevity.)

Hyperbolic dynamical systems are related to Axiom A systems via Smale's spectral decomposition. Recall that a dynamical system ϕ satisfies *Axiom A* if the non-wandering set $\Omega(\phi)$ has a hyperbolic splitting and the periodic points are dense in $\Omega(\phi)$. Spectral decomposition [Sm] allows us to write $\Omega(\phi) = \Lambda_1 \cup \cdots \cup \Lambda_n$ (in the case of a diffeomorphism) and $\Omega(\phi) = \Lambda_1 \cup \cdots \cup \Lambda_n \cup F$ (for a flow) where the Λ_i are disjoint ϕ-invariant basic sets and F is a finite collection of hyperbolic fixed points. Thus in defining a hyperbolic dynamical system, we are merely abstracting the properties of the basic sets appearing in Smale's spectral decomposition.

For a diffeomorphism, we can further decompose $\Omega(\phi)$ as follows. Each Λ_i can be written $\Lambda_i = \Lambda_{i,1} \cup \cdots \cup \Lambda_{i,k_i}$ (disjoint union) such that $\phi(\Lambda_{i,j}) = \Lambda_{i,j+1 \bmod k_i}$ and $\phi^{k_i}|_{\Lambda_{i,j}}$ is topologically mixing. Thus there is no great loss in assuming that hyperbolic diffeomorphisms are mixing. This is not the case for hyperbolic flows. Indeed, the constant suspension of a (mixing) hyperbolic diffeomorphism is a non-mixing hyperbolic flow.

For an Anosov flow ϕ, it is not necessarily true that $\Omega(\phi) = M$ [FW]. (Whether $\Omega(\phi) = M$ for an Anosov diffeomorphism is an important open problem.) When this is the case, we call ϕ a *(transitive) Anosov diffeomorphism* or *flow*. The spectral decomposition is then trivial and we see that any (transitive) Anosov diffeomorphism is mixing. Again, by taking a suspension, we see that Anosov flows need not be mixing.

Throughout this thesis, we will always restrict ϕ to a basic set Λ. It should be noted that Λ need not inherit any of the differential structure of the ambient manifold M. Indeed, Λ is often topologically very complicated. (It has been conjectured that, in the non-Anosov case, Λ is locally the product of a manifold and a Cantor set.) This is why, when considering functions defined on Λ, we only speak of Hölder continuous functions (using the metric induced from the Riemannian metric). When Λ is a manifold (the (transitive) Anosov case), we can speak of C^r functions for $1 \leq r \leq \infty, \omega$.

§2 Cocycles

Let M be a compact manifold and let ϕ^t be a flow on M restricted to a closed invariant subset $\Lambda \subset M$ (not necessarily basic). Let G be a Lie group. We shall usually assume G is compact or abelian. A continuous map $\mathcal{F} : \Lambda \times \mathbb{R} \to G$ is called a *cocycle* if

$$\mathcal{F}(x, t+s) = \mathcal{F}(\phi^t x, s)\mathcal{F}(x, t) \tag{1}$$

for all $x \in \Lambda$, $t, s \in \mathbb{R}$. We say that two cocycles $\mathcal{F}$, $\mathcal{G}$ are *cohomologous* if there exists a map $u : \Lambda \to G$ such that the *cocycle equation*

$$\mathcal{F}(x, t) = u(\phi^t x)\mathcal{G}(x, t)u(x)^{-1} \tag{2}$$

holds. We call u a *cobounding* or *transfer function*. It is often important to specify the class of maps to which u belongs. Indeed the aim of a large part of this thesis is to provide conditions for when a measurable solution u to an equation of the form (2) (holding a.e. with respect to a specific measure) has a Hölder continuous version. We call results of this form *Livšic regularity theorems.*

If $\mathcal{F}$ is cohomologous to the trivial cocycle $\mathcal{G}(x,t) \equiv e$, the group identity, then we call $\mathcal{F}$ a *coboundary*. The cocycle equation (2) becomes the *coboundary equation*

$$\mathcal{F}(x,t) = u(\phi^t x)u(x)^{-1}. \tag{3}$$

(Of course, if G is abelian then (2) reduces to (3).) If $\mathcal{F}$ is cohomologous to a cocycle $\mathcal{G}$ that is independent of the first co-ordinate (so that $t \mapsto \mathcal{G}(x,t)$ is a one-parameter subgroup of G) then we call $\mathcal{F}$ an *essential coboundary*.

We shall often assume that the cocycle satisfies some additional degree of regularity: Hölder or C^r for some $r = 1, \ldots, \infty, \omega$. We give the appropriate definitions in Chapter 1.

It is not yet clear that cocycles even exist. Let us give a method for constructing them, at least for compact or abelian Lie groups. Consider the direct product $\Lambda \times G$. There is a natural G-action on $\Lambda \times G$ given by $\gamma(x,y) = (x, y\gamma^{-1})$ for $\gamma \in G$. Let $\mathfrak{g}$ denote the Lie algebra of G and recall that an element $X \in \mathfrak{g}$ determines a right-invariant vector field by $y \mapsto dr_y X$ where dr_y denote the derivative of right multiplication by y. Let $f : \Lambda \to \mathfrak{g}$ be a map. Define a vector field on $\Lambda \times G$ by

$$(x,y) \mapsto (X_\phi(x), dr_y f(x)) \in T_x\Lambda \times T_y G$$

(here, X_ϕ is the vector field generating ϕ^t). This integrates to a flow ϕ_f^t on $\Lambda \times G$ and one can easily check that ϕ_f^t is G-equivariant: $\gamma\phi_f^t = \phi_f^t\gamma$ for all $\gamma \in G$. Hence we can write $\phi_f^t(x,y) = (\phi^t x, \mathcal{F}(x,t)y)$ for some $\mathcal{F}(x,t) \in G$. The condition that ϕ_f^t is a flow translates to the condition that $\mathcal{F}$ is a cocycle. Conversely, we may start with a cocycle $\mathcal{F}$ and construct a G-equivariant flow $\phi_{\mathcal{F}}^t$ by defining $\phi_{\mathcal{F}}^t(x,y) = (\phi^t x, \mathcal{F}(x,t)y)$. We call $\phi_{\mathcal{F}}^t$ a *G-extension* or *G-skew product* of ϕ^t.

For abelian Lie groups it is easy to obtain explicit formulæ for cocycles. If $f : \Lambda \to \mathbb{R}$ is continuous then we obtain an $\mathbb{R}$-valued and a circle-valued cocycle by defining

$$\mathcal{F}(x,t) = \int_0^t f(\phi^u x)\, du, \quad \mathcal{F}(x,t) = e^{2\pi i \int_0^t f(\phi^u x)\, du} \tag{4}$$

respectively.

We can use G-extensions to illustrate what it means for two cocycles to be cohomologous. If $\mathcal{F}$ and $\mathcal{G}$ are cohomologous cocycles with cobounding function u then the map $U(x,y) = (x, u(x)y)$ conjugates $\phi^t_{\mathcal{F}}$ and $\phi^t_{\mathcal{G}}$: $U\phi^t_{\mathcal{G}} = \phi^t_{\mathcal{F}}U$.

The situation is similar for a diffeomorphism ϕ. Let G be a Lie group. A map $\mathcal{F} : \Lambda \times \mathbb{Z} \to G$ is called a *cocycle* if

$$\mathcal{F}(x, n+m) = \mathcal{F}(\phi^n x, m)\mathcal{F}(x, n)$$

for all $x \in \Lambda$, $n, m \in \mathbb{Z}$. In this case, a cocycle is completely determined by $f(x) = \mathcal{F}(x, 1)$. Conversely, any map $f : \Lambda \to G$ determines a cocycle by setting

$$\begin{aligned} \mathcal{F}(x,n) &= f(\phi^{n-1}x)\cdots f(x) \\ \mathcal{F}(x,-n) &= f(\phi^{-n}x)^{-1}\cdots f(\phi^{-1}x)^{-1} \end{aligned}$$

for $n > 0$ and $f^0(x) \equiv e$. Hence we can identify $\mathcal{F}$ with f and we often write $f^n(x)$ for $\mathcal{F}(x,n)$. Two cocycles f, g are *cohomologous* if

$$f = u\phi \cdot g \cdot u^{-1} \tag{5}$$

for some $u : \Lambda \to G$. We call f a *coboundary* if

$$f = u\phi \cdot u^{-1}. \tag{6}$$

As with flows, cocycles generate skew products of ϕ by defining $\phi_f(x,y) = (\phi x, f(x)y)$. Clearly $\phi_f\gamma = \gamma\phi_f$ for all $\gamma \in G$.

More generally, we could consider skew products of ϕ that also include an automorphism of G. Let $\alpha \in \mathrm{Aut}(G)$ and define $\phi_{f,\alpha}(x,y) = (\phi x, f(x)\alpha(y))$. Then $\alpha(\gamma)\phi_{f,\alpha} = \phi_{f,\alpha}\gamma$. Such skew products generate *twisted cocycles* or *α-cocycles* by defining

$$\begin{aligned} f^n(x) &= f(\phi^{n-1}x)\alpha f(\phi^{n-2}x)\cdots\alpha^{n-1}f(x) \\ f^{-n}(x) &= \alpha^{-1}f(\phi^{-n}x)^{-1}\alpha^{-2}f(\phi^{-n+1}x)^{-1}\cdots\alpha^{-n}f(\phi^{-1}x)^{-1} \end{aligned}$$

for $n > 0$ and $f^0(x) \equiv e$. Then $f^{n+m}(x) = f^n(\phi^m x)\alpha^n f^m(x)$. We say f and g are *α-cohomologous* if the *twisted cocycle equation*

$$f = u\phi \cdot g \cdot \alpha u^{-1} \tag{7}$$

has a solution u. Again, f and g are α-cohomologous precisely when $(x, u(x)y)$ conjugates $\phi_{f,\alpha}$ and $\phi_{g,\alpha}$. If $f = u\phi \cdot \alpha u^{-1}$ then we call f an *α-coboundary*.

A lot of our work had its origins in [PP1, PP2]. In these papers, compact Lie group valued cocycles over hyperbolic diffeomorphisms were considered. Thus this thesis can be viewed as a generalisation of [PP1, PP2] in two different directions: to continuous time and to α-cocycles. We now describe our main results.

§3 Livšic theorems for hyperbolic dynamics

There are some obvious questions that one can ask about cocycles. When is a cocycle a coboundary? When are two cocycles cohomologous? What is the degree of regularity of a cobounding function? In [Li1, Li2], Livšic[1] considered such questions in the case when ϕ is a hyperbolic dynamical system. Most of his work concerns theorems of the following kind.

Theorem (Livšic's periodic data criterion)
Let ϕ be a hyperbolic diffeomorphism or flow and let G be a Lie group with a left-right invariant metric.[2] Let $\mathcal{F}$ be a Hölder cocycle of exponent θ. If $\mathcal{F}(x,t) = e$ whenever $\phi^t x = x$ ($t \in \mathbb{Z}$ or $\mathbb{R}$, as appropriate) then $\mathcal{F}(x,t) = u(\phi^t x)u(x)^{-1}$ for some Hölder u of exponent θ.

[1] A note on pronunciation. In Cyrillic, Livšic is spelled 'Лившиц' and so a better transliteration would be 'Livshits' or 'Livschitz'. However, in the English translations of Livšic's papers [Li1, Li2], the spelling 'Livšic' is used and we continue to use it here.

[2] This is unnecessarily restrictive. When such a metric does exist, the proof is particularly easy [KH, §19.2]. However, Livšic extended the periodic data criterion to all finite dimensional Lie groups and (if $\mathcal{F}$ only takes values in a small neighbourhood of the identity) to some infinite dimensional groups.

Theorem (Livšic's regularity theorem)
Let ϕ be a $C^{1+\varepsilon}$ (transitive) Anosov diffeomorphism or flow and let m be an invariant measure, equivalent to the Riemannian volume. Let $\mathcal{F}$ be a real-valued Hölder cocycle. If $\mathcal{F}(x,t) = u(\phi^t x) - u(x)$ a.e. *for some measurable $u : M \to \mathbb{R}$ then there exists a Hölder u' such that $u = u'$* a.e. *and $\mathcal{F}(x,t) = u'(\phi^t x) - u'(x)$ everywhere.*

Let us first discuss the periodic data criterion. Note that the converse is trivial: if $\mathcal{F}(x,t) = u(\phi^t x)u(x)^{-1}$ then clearly $\mathcal{F}(x,t) = e$ whenever $\phi^t x = x$. Observe that for two cohomologous cocycles $\mathcal{F}, \mathcal{G}$ it is not clear that $\mathcal{F}(x,t) = \mathcal{G}(x,t)$ whenever $\phi^t x = x$, at least in the case when G is non-abelian. (However, Parry [Pa2] has recently proved the following analogue of the periodic data criterion: if $\mathcal{F}(x,t) = \mathcal{G}(x,t)$ whenever $\phi^t x = x$ then $\mathcal{F}$ and $\mathcal{G}$ are cohomologous.)

More generally, it is not clear how to obtain an analogue of the periodic data criterion for α-cocycles, even if we restrict our attention to α-coboundaries.

We turn now to the regularity theorem. In Chapter 1 we prove the following.

Theorem
Let ϕ^t be a C^1 hyperbolic flow. Let G be a compact Lie group or an abelian Lie group. Let $\mathcal{F}$, $\mathcal{G}$ be Hölder cocycles such that $\mathcal{F}(x,t) = u(\phi^t x)\mathcal{G}(x,t)u(x)^{-1}$ a.e. *(with respect to a Hölder equilibrium state) for some measurable $u : \Lambda \to G$. Then there exists a Hölder continuous u' such that $u = u'$* a.e. *and $\mathcal{F}(x,t) = u'(\phi^t x)\mathcal{G}(x,t)u'(x)^{-1}$ everywhere.*

Livšic's proof of the regularity theorem in [Li2] only deals with the real-valued case. However, it has recently been observed by Nicol and Pollicott [NP] that Livšic's argument will generalise to cover the case of coboundaries taking values in an arbitrary matrix Lie group, provided the cobounding function u and its inverse u^{-1} are essentially bounded (note that this is trivially the case for compact Lie groups). The Livšic-Nicol-Pollicott argument does not extend to proving a regularity theorem for the general cocycle equation (2). (In Chapter 3 we give an example to show that the regularity theorem can fail for the cocycle equation (2) for non-compact non-connected Lie groups.) Moreover, Livšic's argument requires ϕ to be $C^{1+\varepsilon}$ to ensure that the stable and unstable foliations are absolutely continuous. If we only require ϕ to be C^1 then this is not necessarily the case [RY]. Instead, when proving the above regularity theorem, we use an approach due to Parry and Pollicott. In [PP1], a regularity theorem was proved in the context of hyperbolic diffeomorphisms for cocycle equations of the form (5) where G is a compact Lie group. The idea of the proof is to first lift the equation to a shift of finite type where the regularity theorem is already known to hold [Pa1]. The Hölder cobounding function is then 'pushed-down' from the shift to the basic set using Markov partitions so as to construct a Hölder cobounding function for ϕ. In Chapter 1, we use a similar strategy to prove the regularity theorem stated above. More specifically, we first lift the cocycle equation to a suspension of a shift of finite type. We then solve the cocycle equation in this symbolic cover, obtaining a continuous solution. This solution is then pushed down to the basic set using Markov sections. It is then checked that the function so obtained is the required Hölder cobounding function.

In Chapter 3 we discuss regularity theorems for the twisted cocycle equation (7). The regularity theorem does not always hold and we give examples to illustrate this. However, when G is a compact connected Lie group and the eigenvalues of $d\alpha$ (the derivative of α) have modulus 1, we prove that the regularity theorem does indeed hold. We call an automorphism α satisfying this condition on $d\alpha$ a *unipotent automorphism*; this somewhat generalises the usual definition of unipotent (see the remarks in Chapter 3). The unipotency assumption on α is optimal for the regularity theorem to hold, as our examples show.

Another question that can be asked is the following. If $\mathcal{F}$ and $\mathcal{G}$ are C^r cocycles ($r = 1, \infty$ or ω, say) that are cohomologous via a continuous cobounding function u, must u necessarily be C^r? There is an extensive literature on this problem for cocycle equations of the form (2), (3), (5) and (6), culminating in [NT] (indeed, Livšic [Li2] considered several special cases). For flows we have nothing new to add, but we do give a very brief expository account in §6 of Chapter 1. In Chapter 3, we adapt the method of [NT] and prove a C^r regularity theorem for the twisted cocycle equation.

§4 Applications to stable ergodicity

In Chapter 2 we study how cocycles can be used to deduce that certain dynamical systems derived from hyperbolic flows are *stably ergodic* or *stably mixing*; by this we mean that, within a certain class of perturbations, the ergodic or mixing flows form an open dense set. Throughout, ϕ^t is a hyperbolic flow. We need the technical assumption that the basic set either has finite rank first cohomology or is one-dimensional (these conditions are disjoint). It is not clear to us how restrictive these conditions are. The finite rank assumption is satisfied by all (transitive) Anosov flows and, more generally, by all attractors. However, it is easy to construct a hyperbolic flow that does not satisfy either of these conditions: just take the direct product of an attractor and a one-dimensional basic set.

Toral extensions. Let K denote the unit circle in $\mathbb{C}$ and let K^n denote the n-dimensional torus. We have already seen how a K^n-valued cocycle $\mathcal{F}$ generates a K^n-extension $\phi^t_{\mathcal{F}}$ of ϕ^t. Note that we can topologise such cocycles by identifying a cocycle with an n-tuple of real-valued functions by (4). A natural question is how do the dynamics of $\phi^t_{\mathcal{F}}$ depend on $\mathcal{F}$. From a topological viewpoint, there is little to add: the $\phi^t_{\mathcal{F}}$ are not structurally stable (although they do satisfy the weaker notion of K^n-structural stability [Fi]). Instead we consider the ergodic properties of $\phi^t_{\mathcal{F}}$ (with respect to the invariant measure $\mu \times$ Haar). The following well-known result gives a condition for $\phi^t_{\mathcal{F}}$ to be ergodic in terms of $\mathcal{F}$.

Proposition
Using (4), identify $\mathcal{F}$ with $(f_1, \ldots, f_n)$, $f_i : \Lambda \to \mathbb{R}$. Then $\phi^t_{\mathcal{F}}$ is ergodic if and only if the only solution to

$$\frac{w(\phi^t x)}{w(x)} = e^{2\pi i \int_0^t \sum_{i=1}^n l_i f_i(\phi^u x)\, du} \text{ a.e. } [\mu] \tag{8}$$

where $w : \Lambda \to K$ is measurable and $(l_1, \ldots, l_n) \in \mathbb{Z}^n$ is trivial: $l_1 = \cdots = l_n = 0$ and w is constant a.e.

Note that (8) is a K-valued cocycle equation. By the Livšic regularity theorem, we may assume that w is actually Hölder, C^1 or C^∞ (depending on how smooth the f_i are—see Chapter 1). We will show in Chapter 2 how to deduce from this fact that (8) only has a non-trivial solution for cocycles $\mathcal{F}$ in a very small (infinite codimension) closed subspace. By some 'general position' arguments, this is seen to imply the existence of an open dense collection of ergodic extensions. A similar analysis yields an open dense set of mixing extensions.[3]

How are our results related to those in the literature? It is conjectured in [PS] that partially hyperbolic dynamical systems are stably ergodic. By *partially hyperbolic* we mean that the tangent bundle splits into four $d\phi^t$-invariant sub-bundles: a stable bundle, an unstable bundle, the flow direction and a centre bundle. We require $d\phi^t$ to contract the stable bundle more rapidly than the centre bundle and to expand the unstable bundle more rapidly than the centre bundle. G-extensions are examples of partially hyperbolic dynamical systems: the centre bundle is tangent to G. Therefore our results give an example of the behaviour conjectured in [PS] (with the restriction that the perturbation fixes the base transformation ϕ^t and leaves the perturbed flow G-equivariant). The question of stable ergodicity for G-extensions for some classes of hyperbolic diffeomorphisms has been considered in [AKS, PP2] when G is a circle or K^n, respectively. The results in [PP2] have recently been generalised to the case when G is an arbitrary compact connected Lie group [FP].

Velocity changes. If ϕ^t is a flow then we can obtain a new flow ψ^t by keeping the same orbits but traversing the orbits at a different speed. (In terms of vector fields, this corresponds to multiplying the vector field of ϕ^t by a positive real-valued function.) Then we can write $\phi^t(x) = \psi^{k(x,t)}(x)$ for some $k(x,t) : \Lambda \times \mathbb{R} \to \mathbb{R}^+$ and it is easy to check that k is an $\mathbb{R}$-valued cocycle

[3] In [FP] it is shown that, for toral extensions of hyperbolic diffeomorphisms, stable ergodicity implies mixing. This is false for mixing hyperbolic flows.

for ϕ^t. A velocity change of a hyperbolic flow is always ergodic with respect to $k\,d\mu$, but it is not always mixing. Indeed, a hyperbolic flow is not mixing precisely when it is the constant suspension of a homeomorphism. We show in Chapter 2 that ψ^t is not mixing precisely when, for some $c \in \mathbb{R} \setminus \{0\}$, the circle-valued cocycle $e^{2\pi i c k(x,t)}$ is a coboundary. We can then use a similar analysis to that used for toral extensions to conclude the existence of an open dense set of mixing velocity changes.

Our result is related to one of Bowen's. In [Bo], Bowen proved that there is an open dense set of mixing suspensions of an Anosov diffeomorphism of an infranilmanifold. (In fact, Bowen's proof is valid for suspensions of an Anosov diffeomorphism ϕ on M with the condition that 1 is not an eigenvalue of ϕ^* on $H^1(M, \mathbb{Z})$. Whether this is always the case is an important open problem.) Our result is more general in that it applies to suspensions of any (transitive) Anosov diffeomorphism. We also provide a characterisation of the stably mixing velocity changes in terms of the cohomology of the basic set.

Finally, a word of caution. The three chapters that make up this thesis were originally written as preprints. Therefore there is some redundancy amongst the definitions and statements of some of the results. More importantly, the notation used is only approximately consistent from one chapter to another.

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Livšic theorems for hyperbolic flows

Abstract

We consider Hölder cocycle equations with values in certain Lie groups over a hyperbolic flow. We extend Livšic's results that measurable solutions to such equations must, in fact, be Hölder continuous.

§1 Introduction

Let ϕ^t be a C^1 hyperbolic flow on a compact smooth Riemannian manifold restricted to a basic set Λ. Let G be a Lie group. A continuous map $\mathcal{F} : \Lambda \times \mathbb{R} \to G$ is called a *cocycle* if

$$\mathcal{F}(x, t+s) = \mathcal{F}(\phi^t x, s)\mathcal{F}(x, t)$$

for all $x \in \Lambda$, $t, s \in \mathbb{R}$. Two cocycles $\mathcal{F}, \mathcal{G}$ are said to be *cohomologous* if there exists a map $u : \Lambda \to G$ such that

$$\mathcal{F}(x, t) = u(\phi^t x)\mathcal{G}(x, t)u(x)^{-1}. \tag{1}$$

(If u is only assumed to be measurable then (1) is assumed to hold a.e. with respect to an equilibrium state corresponding to a Hölder continuous function.) We call u satisfying (1) a *cobounding function*. If $\mathcal{F}$ satisfies

$$\mathcal{F}(x, t) = u(\phi^t x)u(x)^{-1} \tag{2}$$

then we call $\mathcal{F}$ a *coboundary*. Observe that if G is abelian then (1) reduces to (2).

One can obtain real and circle-valued cocycles by taking $f : \Lambda \to \mathbb{R}$ and defining

$$\begin{aligned} \mathcal{F}(x,t) &= \int_0^t f(\phi^u x)\, du, \\ \mathcal{F}(x,t) &= \exp\left(2\pi i \int_0^t f(\phi^u x)\, du\right), \end{aligned}$$

respectively.

Three questions can be asked.

1. What conditions on $\mathcal{F}$ ensure that $\mathcal{F}$ is a coboundary?

2. If $\mathcal{F}$ is a Hölder continuous cocycle (see §4 for the definition) and there is a measurable solution u to (1) (holding a.e. with respect to a Hölder equilibrium state), is the cobounding function Hölder continuous with the same exponent as $\mathcal{F}$?

3. If $\mathcal{F}$ possesses a higher degree of regularity (C^r for some $1 \leq r \leq \infty$ or $r = \omega$), is the same true of a continuous solution to (1)?

One can also formulate cocycle equations for a hyperbolic diffeomorphism ϕ. Let G be a Lie group and let $f, g : \Lambda \to G$ be Hölder continuous. We say that f, g are *cohomologous* if, for some $u : \Lambda \to G$,

$$f(x) = u(\phi x) g(x) u(x)^{-1}. \tag{3}$$

Again, f is called a *coboundary* if it is cohomologous to the trivial cocycle $g(x) \equiv e$, the group identity. We call results answering the above questions *Livšic theorems* after the celebrated work in [Li1, Li2]. In this note, we shall mostly be interested in answering question 2 in the affirmative, but we shall also find it useful to give a brief discussion of results related to questions 1 and 3. Observe that questions 2 and 3 can be combined. However, as they require different methods and as we have nothing new to add to the third question, we prefer to keep them separate.

Let us first address question 1. Let $\mathcal{F}$ be a Hölder cocycle. Clearly, if u is a Hölder continuous solution to the coboundary equation (2) then $\mathcal{F}(x,T) = e$ whenever $\phi^T x = x$. This condition is also sufficient for the existence of a Hölder continuous cobounding function. We call a result of this form a *Livšic periodic data criterion*. Such criteria continue to hold for any compact Lie group (see [KH] for a highly readable account), for finite-dimensional Lie groups [Li2] and, for certain classes of cocycles $\mathcal{F}$, for some infinite-dimensional Lie groups [Li2]. For the general cocycle equation (1), there is a partial analogue of the periodic data criterion for compact Lie groups [Pa2].

Livšic also considered question 2 for real-valued cocycles. In [Li2] he proved that if ϕ^t is a C^2 transitive Anosov flow or diffeomorphism preserving a smooth (*i.e.* equivalent to the Riemannian volume) invariant measure then any measurable solution u to the real-valued cocycle equation

$$\int_0^t f(\phi^u x)\, du = u(\phi^t x) - u(x) \text{ a.e.}$$

where f is Hölder continuous must be equal almost everywhere to a Hölder continuous function u' for which the equation holds everywhere. His proof used both the existence of a smooth invariant measure and the C^2 smoothness in an essential way (in fact, the proof given in [Li2] works for $C^{1+\varepsilon}$). We call results of this kind—where measurable solutions imply continuous or Hölder solutions—*Livšic regularity theorems*.

Recently, Nicol and Pollicott [NP] observed that Livšic's method for Anosov diffeomorphisms can be extended to coboundary equations taking values in an arbitrary matrix Lie group. The method does not extend to the general cocycle equation (1). Indeed, the regularity theorem can fail for non-compact non-connected Lie groups [Wa1].[1]

In §4 we shall prove the following. Let G be a compact Lie group. Suppose ϕ^t is a C^1 hyperbolic flow equipped with a Hölder equilibrium state m. Let $\mathcal{F}$, $\mathcal{G}$ be Hölder continuous cocycles and suppose they satisfy (1) a.e. $[m]$ for a measurable function $u : \Lambda \to G$. Then u is equal a.e. to a Hölder continuous function for which (1) holds everywhere. A precise statement of this is given in §4. In §5 we show how one may deduce the real-valued regularity theorem from the circle-valued case.

[1] Proposition 7.1 in §7 of Chapter 3.

To prove these results we first consider cocycle equations over shifts of finite type and their suspensions. By the well-known technique of Markov sections (which we sketch below), a hyperbolic flow is seen to be a factor of a suspension of a shift of finite type in a particularly nice way. This allows us to first solve the measurable cocycle equation in the symbolic cover and then push solutions down to the hyperbolic flow. We then obtain a continuous solution to the cocycle equation. For the case of a coboundary (equation (2)), this is sufficient: a Livšic periodic data criterion then implies the required degree of Hölder regularity on the solution. In the general case (equation (1)), a small further argument is necessary. The method of pushing down solutions that we use here had its origins in [PP2]. Here, hyperbolic diffeomorphisms were considered and the regularity theorem for cocycle equations of the form (3) was proved. Markov partitions were used to reduce the problem to the case of a shift of finite type, and this had already been considered in [Pa1]. A summary of the results in [Pa1] is given in §3 before we prove regularity results for suspensions of a shift of finite type.

Building on earlier work of Livšic, Guillemin and Kazhdan, Hurder and Katok (amongst others), Niţică and Török considered in [NT1, NT2] question 3. In §6 we shall make some remarks on their results and the methods used.

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§2 Preliminaries

There is a standard (albeit non-canonical) way of relating a hyperbolic flow to a shift of finite type via Markov sections. This approach goes back to Bowen [Bo] in the hyperbolic case and Ratner [Ra] in the 3-dimensional Anosov case. This technique has been well-used in the study of hyperbolic dynamics, see [Bo, BR, PP1, etc.], and we summarise the definitions and main ideas below.

Let $S = \{1, \ldots, k\}$ be a finite set. Let A be an aperiodic $k \times k$ matrix with entries 0 or 1. Define the *subshift of finite type* by

$$\Sigma = \Sigma_A = \{(x_j)_{j=-\infty}^{\infty} \in S^{\mathbb{Z}} \mid A_{x_j, x_{j+1}} = 1 \text{ for all } j \in \mathbb{Z}\}.$$

Then Σ is invariant under the shift map $\sigma : \Sigma \to \Sigma$ defined by $(\sigma x)_j = x_{j+1}$.

Let $n(x, y) = \sup\{n \mid x_j = y_j, |j| \leq n\}$. For each $\theta \in (0, 1)$ define a metric on Σ by $d^\theta(x, y) = \theta^{n(x,y)}$. For a metric space Y, let $F_\theta(\Sigma, Y)$ denote the space of functions $\Sigma \to Y$ that are Lipschitz continuous with respect to d^θ. We shall often abuse this definition and speak of functions in $F_\theta(\Sigma, Y)$ as being Hölder of exponent θ.

For a continuous $f : \Sigma \to \mathbb{R}$ define the *pressure* of f to be

$$P(f) = \sup\{h_\mu(\sigma) + \mu(f) \mid \mu \text{ is a } \sigma\text{-invariant probability}\}. \qquad (4)$$

If $f \in F_\theta(\Sigma, \mathbb{R})$ then this supremum is attained by a unique σ-invariant probability m. We call m the *equilibrium state* of f. One can check that F_θ-equilibrium states are ergodic and non-zero on non-empty open sets.

Let $r \in F_\theta(\Sigma, \mathbb{R})$ be strictly positive. Define

$$\widetilde{\Sigma}_r = \{(x, t) \in \Sigma \times \mathbb{R} \mid x \in \Sigma, 0 \leq t \leq r(x)\}/\sim$$

where $\sim$ identifies $(x, r(x))$ with $(\sigma x, 0)$. This is a compact metrisable space. We shall often denote a point in $\widetilde{\Sigma}_r$ by $\tilde{x}$ and write $\tilde{x} = (x, s)$ for $s \in [0, r(x))$. Define the *symbolic flow* σ_r^t on $\widetilde{\Sigma}_r$ by $\sigma_r^t(x, s) = (x, s + t)$ for small t and then extend to all t by using the identifications.

The pressure of a continuous function $f : \widetilde{\Sigma}_r \to \mathbb{R}$ can be defined in analogy with (4) (the entropy of a flow is defined to be the entropy of its time-one map). When f satisfies $\mathcal{I}f(x) := \int_0^{r(x)} f\sigma_r^u(x, 0)\, du \in F_\theta(\Sigma, \mathbb{R})$ for some $\theta \in (0, 1)$ the supremum is again attained by a unique σ_r^t-invariant probability m which we call the *equilibrium state* of f. One can show [PP1, for example] that $m = \mu \times l/\mu(r)$ where μ is the equilibrium state of $\mathcal{I}f - P(f)r$ and l is Lebesgue measure on the fibres $[0, r(x)]$. Clearly, m is ergodic (as μ is).

Let M be a compact C^∞ Riemannian manifold. Let $C^\alpha(\Lambda, Y)$ denote the space of Hölder continuous functions $\Lambda \to Y$ with exponent $\alpha \in (0, 1)$. Let $|f|_\theta$ denote the least possible Hölder constant and let $\|f\|_\alpha = |f|_\alpha + |f|_\infty$.

Let ϕ^t be a C^1 flow on M. A ϕ^t-invariant closed set $\Lambda \subset M$ is called a *basic set* and ϕ^t restricted to Λ is called a *hyperbolic flow* if the following conditions hold:

1. the tangent bundle $T_\Lambda M$ of M restricted to Λ has a continuous splitting into a Whitney sum $E^s \oplus E^u \oplus E^0$ of $d\phi^t$-invariant sub-bundles for which there exist constants $C > 0$, $0 < \lambda < 1$ such that

$$\|d\phi^t(v)\| \leq C\lambda^t\|v\| \text{ for } t \geq 0, \text{ if } v \in E^s_x,$$
$$\|d\phi^{-t}(v)\| \leq C\lambda^t\|v\| \text{ for } t \geq 0, \text{ if } v \in E^u_x,$$

   and E^0 is one-dimensional and tangential to the flow direction;

2. the periodic points of ϕ^t in Λ are dense in Λ;

3. Λ contains a dense orbit;

4. there exists an open neighbourhood U of Λ in M such that

$$\Lambda = \bigcap_{t=-\infty}^{\infty} \phi^t U;$$

5. Λ contains no fixed points and is larger than a single orbit.

When $\Lambda = M$, we call ϕ^t a *(transitive) Anosov flow*. The definition of a hyperbolic diffeomorphism is exactly the same, except that there is no centre sub-bundle E^0 in 1.

Again, we can use an analogue of (4) to define the pressure of a continuous function on Λ. If f is Hölder continuous then this supremum is achieved by a unique ϕ^t-invariant probability m, the *equilibrium state* of f [BR]. Moreover, m is positive on non-empty open sets and is ergodic [BR].

We now gather together some well-known facts about the existence and properties of stable and unstable manifolds for hyperbolic flows. For details, see for example [Sh] and the references cited therein.

For $\varepsilon > 0$ sufficiently small and $x \in \Lambda$, define the (local, strong) *stable* and *unstable manifolds*[2] by

$$W^s_\varepsilon(x) = \left\{ y \in M \;\middle|\; \begin{array}{l} d(\phi^t x, \phi^t y) \le \varepsilon \text{ for all } t \ge 0, \\ d(\phi^t x, \phi^t y) \to 0 \text{ as } t \to \infty \end{array} \right\},$$

$$W^u_\varepsilon(x) = \left\{ y \in M \;\middle|\; \begin{array}{l} d(\phi^{-t} x, \phi^{-t} y) \le \varepsilon \text{ for all } t \ge 0, \\ d(\phi^{-t} x, \phi^{-t} y) \to 0 \text{ as } t \to \infty \end{array} \right\}.$$

Then, provided ε is small enough, these are C^1 embedded discs in M and $T_x W^{s,u}_\varepsilon(x) = E^{s,u}_x$.

For $x \in \Lambda$ we have the following estimates:

$$\begin{array}{l} \text{if } y \in W^s_\varepsilon(x) \text{ then } d(\phi^t x, \phi^t y) \le C\lambda^t d(x,y) \text{ for all } t \ge 0; \\ \text{if } y \in W^u_\varepsilon(x) \text{ then } d(\phi^{-t} x, \phi^{-t} y) \le C\lambda^t d(x,y) \text{ for all } t \ge 0. \end{array} \tag{5}$$

We will need the fact that Λ admits a *local product structure* (or *canonical co-ordinates*). Let $\varepsilon > 0$ be small. Then there exists $\delta > 0$ such that the following holds. If $x, y \in \Lambda$ and $d(x,y) \le \delta$ then there exists a unique $\nu(x,y) \in \mathbb{R}$ such that

1. $W^s_\varepsilon(\phi^{\nu(x,y)} x) \cap W^u_\varepsilon(y) \ne \emptyset$,
2. $|\nu(x,y)| < \varepsilon$.

Moreover, the set $W^s_\varepsilon(\phi^{\nu(x,y)} x) \cap W^u_\varepsilon(y)$ consists of a single point and this point lies in Λ. Denote this point by $\langle x, y \rangle$. Notice that for any $x \in \Lambda$, $\nu(x,x) = 0$ and $\langle x, x \rangle = x$. The maps $\nu : \Delta_\delta \to \mathbb{R}$, $\langle \cdot, \cdot \rangle : \Delta_\delta \to \Lambda$ on $\Delta_\delta := \{(x,y) \in \Lambda \times \Lambda \mid d(x,y) \le \delta\}$ are continuous (and so uniformly continuous as Δ_δ is compact).

One relates a hyperbolic flow to a symbolic flow by constructing the return map on a family of suitably nice local cross-sections, called a *Markov section*. The details of this construction are well-known [Bo, BR] although somewhat complicated; we summarise them below.

A C^1-embedded codimension 1 closed disc $D \subset M$ is called a *local section* if for some $\xi > 0$ the map $D \times [-\xi, \xi] \to M : (x,t) \mapsto \phi^t x$ defines a homeomorphism onto its image $U_\xi(D)$, a neighbourhood of D in M

[2]Some authors use the notation $W^{ss}_\varepsilon(x)$ for our $W^s_\varepsilon(x)$ and use $W^s_\varepsilon(x)$ for the *weak* local stable manifold. As we shall only refer to weak stable manifolds in passing, we use the simpler notation. Similar remarks hold for the unstable manifolds.

(this is just the requirement that the flow is transverse to D). The map $\pi_D : U_\xi(D) \to D$ that projects a point in $U_\xi(D)$ along its orbit to D is differentiable. If $x, y \in \Lambda \cap D$ then $\langle x, y\rangle$ is defined and, provided ε is small enough, lies in $\Lambda \cap U_\xi(D)$. By the remarks above, $\pi_D\langle\cdot,\cdot\rangle$, when defined, is continuous.

Let D be a local section with small diameter. A subset $R \subset \operatorname{Int}(\Lambda \cap D)$ (the interior is taken as a subset of $\Lambda \cap D$) is called a *rectangle* if the following hold:

1. R is equal to the closure of its interior when viewed as a subset of $\Lambda \cap D$; in particular, ∂R is nowhere dense in $\Lambda \cap D$;

2. the diameter of R is small enough so that $\pi_D\langle\cdot,\cdot\rangle : R \times R \to \Lambda \cap D$ is defined; moreover, if $x, y \in R$ then $\pi_D\langle x, y\rangle \in R$.

Suppose $\mathcal{R} = \{R_1, \ldots, R_k\}$ is a finite collection of rectangles such that the corresponding D_i are pairwise disjoint. Define:

$$\begin{aligned}\Gamma(\mathcal{R}) &= R_1 \cup \cdots \cup R_k,\\ \Gamma^\circ(\mathcal{R}) &= \operatorname{Int} R_1 \cup \cdots \cup \operatorname{Int} R_k,\\ \partial\mathcal{R} &= \partial R_1 \cup \cdots \cup \partial R_k,\end{aligned}$$

(here, interiors and boundaries are as subsets of $\Lambda \cap D_i$). Suppose

$$\Lambda = \bigcup_{t=0}^{T} \phi^{-t}\Gamma(\mathcal{R})$$

for some small $T > 0$. Then each orbit of ϕ^t crosses $\Gamma(\mathcal{R})$ infinitely often in both forward and backward time. For $x \in \Gamma(\mathcal{R})$, let $0 < \tau(x) \le T$ be the first return time of x. This induces a return map $H : \Gamma(\mathcal{R}) \to \Gamma(\mathcal{R})$ defined by $H(x) = \phi^{\tau(x)}(x)$. In general, the maps τ and H are not continuous.

In [Bo, BR], it is shown how to construct a collection $\mathcal{R} = \{R_1, \ldots, R_k\}$ of rectangles satisfying all of the above conditions together with an additional Markov condition. This is then used to construct an aperiodic shift of finite type $\Sigma \subset \{1, \ldots, k\}^{\mathbb{Z}}$ (with transition matrix A given by $A_{i,j} = 1$ precisely when $H(\operatorname{Int} R_i) \cap \operatorname{Int} R_j \neq \emptyset$) such that the map $\pi : \Sigma \to \Gamma(\mathcal{R})$ defined by

$$\pi\left((x_j)_{j\in\mathbb{Z}}\right) = \bigcap_{j\in\mathbb{Z}} H^{-j} R_{x_j}$$

satisfies the following properties:

1. $\pi \in F_\theta(\Sigma, \Lambda)$ for some $0 < \theta < 1$,
2. $\pi\sigma = H\pi$,
3. π is bounded-to-one,
4. if $H^n x \notin \partial\mathcal{R}$ for all $n \in \mathbb{Z}$, then $\pi^{-1}\{x\}$ is a singleton.

Define, for $\pi(x) \in \Gamma(\mathcal{R})$ with orbit disjoint from $\partial\mathcal{R}$, $r(x)$ to be the first return time for x. Then it can be shown that r extends to a strictly positive $F_\theta(\Sigma, \mathbb{R})$-function for some θ. This allows us to construct a symbolic flow σ_r^t on $\widetilde{\Sigma}_r$ and a semi-conjugacy $\tilde{\pi}$ from σ_r^t to ϕ^t that is invertible on orbits that do not meet the boundary. In summary:

Theorem 2.1 ([Bo, BR])
There exists a strictly positive $r \in F_\theta(\Sigma, \mathbb{R})$ such that the map $\tilde{\pi} : \widetilde{\Sigma}_r \to \Lambda$ defined by $\tilde{\pi}(x,t) = \phi^t x$ satisfies:

1. $\tilde{\pi}\sigma_r^t = \phi^t\tilde{\pi}$;
2. *$\tilde{\pi}$ is continuous;*
3. *$\tilde{\pi}$ is bounded-to-one;*
4. *if the orbit of x is disjoint from $\bigcup_{n=-\infty}^{\infty} H^n(\partial\mathcal{R})$ then $\tilde{\pi}^{-1}\{x\}$ is a singleton.*

If $f : \Lambda \to \mathbb{R}$ is Hölder continuous (of exponent α, say) then the equilibrium state of f is given by $m\tilde{\pi}^{-1}$ where m is the equilibrium state for $f\tilde{\pi}$ (which is well-defined as $\mathcal{I}(f\tilde{\pi}) \in F_{\theta^\alpha}(\Sigma, \mathbb{R})$).

Remarks.

1. In fact, $\tilde{\pi}$ is Hölder continuous (in the sense of [Wa2]) with respect to the metric d_r^θ considered in [Wa2]. The proof is similar in spirit to those in §5 of [Wa2].[3]

[3] §5 of Chapter 2.

2. Observe that $\tilde{\pi}^{-1}\{x\}$ is a singleton if and only if $\tilde{\pi}^{-1}\{\phi^t x\}$ is a singleton for all $t \in \mathbb{R}$.

It will be important to understand how ϕ^t acts on the boundary of $\mathcal{R}$.

Lemma 2.2 ([Bo])

The boundary $\partial\mathcal{R}$ of $\mathcal{R}$ can be decomposed $\partial\mathcal{R} = \partial^s\mathcal{R} \cup \partial^u\mathcal{R}$ such that $H(\partial^s\mathcal{R}) \subset \partial^s\mathcal{R}$ and $H^{-1}(\partial^u\mathcal{R}) \subset \partial^u\mathcal{R}$. Moreover, if we define

$$\begin{aligned} \Delta^s\mathcal{R} &= \bigcup_{0\le t\le T} \phi^t(\partial^s\mathcal{R}), \\ \Delta^u\mathcal{R} &= \bigcup_{0\le t\le T} \phi^{-t}(\partial^u\mathcal{R}) \end{aligned}$$

then $\phi^t(\Delta^s\mathcal{R}) \subset \Delta^s\mathcal{R}$, $\phi^{-t}(\Delta^u\mathcal{R}) \subset \Delta^u\mathcal{R}$ for all $t \ge 0$.

Remarks.

1. Observe that $\Delta^u\mathcal{R} \cap \Gamma^\circ(\mathcal{R}) = \emptyset$. For if $z \in \Delta^u\mathcal{R} \cap \Gamma^\circ(\mathcal{R})$ then $\phi^t z \in \partial^u\mathcal{R}$ for some $t \ge 0$. Hence, $z = H^{-n}(\phi^t z) \in \partial^u\mathcal{R} \cap \Gamma^\circ(\mathcal{R})$ for some $n > 0$, a contradiction. Similarly, $\Delta^s\mathcal{R} \cap \Gamma^\circ(\mathcal{R}) = \emptyset$. However, for $t \ge 0$, $\phi^t(\partial^u\mathcal{R})$ can (and does) intersect $\Gamma^\circ(\mathcal{R})$, and similarly for the backwards iterates of $\partial^s\mathcal{R}$.

2. As $\phi^t(\partial\mathcal{R})$ intersects $\Gamma^\circ(\mathcal{R})$ only countably many times and $\partial\mathcal{R}$ is nowhere dense, it follows from Theorem 2.1.4 and the remark following it that $\tilde{\pi}^{-1}\{x\}$ is a singleton for a dense G_δ in Λ.

Let us call an orbit in Λ *good* if it never intersects $\partial\mathcal{R}$. The following is used in [PP2] in the case of a hyperbolic diffeomorphism.

Proposition 2.3

There is a dense set of good periodic orbits in Λ that only intersect $\Gamma(\mathcal{R})$ in $\Gamma^\circ(\mathcal{R})$. In particular, if γ is such a periodic orbit and $x \in \gamma$ then there is a unique $\tilde{x} \in \tilde{\Sigma}_r$ such that $\tilde{\pi}(\tilde{x}) = x$.

Proof. Let γ be a periodic orbit in Λ and let $x \in \gamma \cap \Gamma(\mathcal{R})$. Then $H^n x = x$ for some $n > 0$. Suppose $x \in \partial\mathcal{R}$. If $x \in \partial^s\mathcal{R}$ then $Hx, \ldots, H^{n-1}x \in \partial^s\mathcal{R}$ by Lemma 2.2. If $x \in \partial^u\mathcal{R}$ then $H^{-1}x = H^{n-1}x, \ldots, H^{-(n-1)}x = Hx \in \partial^u\mathcal{R}$.

In either case, $\{x, Hx, \ldots, H^{n-1}x\} \subset \partial\mathcal{R}$. Hence, if $\gamma \cap \partial\mathcal{R} \neq \emptyset$ then γ only intersects $\Gamma(\mathcal{R})$ in $\partial\mathcal{R}$. As $\partial\mathcal{R}$ is nowhere dense in $\Gamma(\mathcal{R})$ and the periodic orbits of H are dense in $\Gamma(\mathcal{R})$, the claim follows. ❑

§3 Shifts of finite type and their suspensions

In [PP2] the following is proved.

Proposition 3.1
Let G be a compact Lie group and let $f, g \in F_\theta(\Sigma, G)$. Suppose $u : \Sigma \to G$ is measurable and satisfies

$$f = u\sigma \cdot g \cdot u^{-1} \text{ a.e. } [m]$$

for some Hölder equilibrium state m. Then there exists $u' \in F_\theta(\Sigma, G)$ such that $u' = u$ a.e. and

$$f = u'\sigma \cdot g \cdot u'^{-1}$$

everywhere.

Remark. In fact, in [PP2] it is only proved that $u' \in F_{\beta^{\frac{1}{2}}}(G)$ where β denotes the maximum of θ and the Hölder exponent of the equilibrium state. To see that $u' \in F_\theta(G)$, we argue as follows. Let $x \in \Sigma$ and let $W^s_\varepsilon(x)$ denote the local stable 'manifold' through x. For $z \in W^s_\varepsilon(x)$ define $F^s(z,x) = \lim_{n\to\infty} f^n(z)^{-1} f^n(x)$ ($f^n = f\sigma^{n-1} \cdots f\sigma \cdot f$) and similarly define $G^s(z,x)$. Then one can easily check that $\rho(F^s(z,x), e) \leq \text{Const.}\, d^\theta(z,x)$ and $F^s(z,x) = u'(z)G^s(z,x)u'(x)^{-1}$ (ρ is an invariant metric on G). It follows that u' is Hölder continuous of exponent θ along stable 'manifolds'. The same is true for unstable 'manifolds'. By Proposition 19.1.1 in [KH], this shows that $u' \in F_\theta(G)$. We shall use the kind of argument many times in the sequel.

We can now prove a regularity theorem for symbolic flows. Let $\mathcal{F} : \widetilde{\Sigma}_r \times \mathbb{R} \to G$ be a continuous, compact Lie group valued cocycle. Define

$$\mathcal{I}\mathcal{F} : \Sigma \to G : x \mapsto \mathcal{F}((x,0), r(x)).$$

(In the real-valued case we write $\mathcal{I}f(x) = \int_0^{r(x)} f\sigma_r^u(x,0)\,du$ and similarly for the circle-valued case.)

Theorem 3.2

Let $\mathcal{F}, \mathcal{G}$ be continuous cocycles and suppose $\mathcal{IF}, \mathcal{IG} \in F_\theta(\Sigma, G)$ for some $\theta \in (0,1)$. Let m be a Hölder equilibrium state for σ_r^t. Suppose $\tilde{u} : \widetilde{\Sigma}_r \to G$ is a measurable solution to the cocycle equation

$$\mathcal{F}(\tilde{x}, t) = \tilde{u}\sigma_r^t(\tilde{x})\mathcal{G}(\tilde{x}, t)\tilde{u}(\tilde{x})^{-1} \text{ a.e. } [m]. \tag{6}$$

Then there exists a unique continuous $\tilde{u}' : \widetilde{\Sigma}_r \to G$ such that $\tilde{u}' = \tilde{u}$ a.e. and

$$\mathcal{F}(\tilde{x}, t) = \tilde{u}'\sigma_r^t(\tilde{x})\mathcal{G}(\tilde{x}, t)\tilde{u}'(\tilde{x})^{-1}$$

everywhere.

Proof. First note (c.f. [CFS, p. 13]) that we may take the set of measure zero on which (6) fails to hold to be independent of t.

Let $\tilde{x} = (x, s)$ and set $t = r(x)$ in (6) to obtain

$$\mathcal{F}((x,s), r(x)) = \tilde{u}\sigma_r^{r(x)}(x,s)\mathcal{G}((x,s), r(x))\tilde{u}(x,s)^{-1} \text{ a.e. } [m].$$

From the discussion in §2, $m = \mu \times l/\mu(r)$ for some Hölder equilibrium state μ for σ. By Fubini's Theorem, we can choose a 'good' small $s_0 \geq 0$ such that

$$\mathcal{F}((x,s_0), r(x)) = \tilde{u}\sigma_r^{r(x)}(x,s_0)\mathcal{G}((x,s_0), r(x))\tilde{u}(x,s_0)^{-1} \text{ a.e. } x. \tag{7}$$

As $\mathcal{F}$ is a cocycle, one can easily check that

$$\mathcal{F}((x,s_0), r(x)) = \mathcal{F}((\sigma x, 0), s_0)\mathcal{F}((x,0), r(x))\mathcal{F}((x,0), s_0)^{-1},$$

and similarly for $\mathcal{G}$. Substituting this expression into (7) we obtain

$$\begin{aligned} \mathcal{IF}(x) &= \mathcal{F}((\sigma x, 0), s_0)^{-1}\tilde{u}(\sigma x, s_0)\mathcal{G}((\sigma x, 0), s_0) \\ &\quad \cdot \mathcal{IG}(x)\mathcal{G}((x,0), s_0)^{-1}\tilde{u}(x, s_0)^{-1}\mathcal{F}((x,0), s_0) \text{ a.e. } x. \end{aligned}$$

Let $v(x) = \mathcal{F}((x,0), s_0)^{-1}\tilde{u}(x, s_0)\mathcal{G}((x,0), s_0) : \Sigma \to G$. This is measurable and

$$\mathcal{IF}(x) = v\sigma(x)\mathcal{IG}(x)v(x)^{-1}$$

a.e. with respect to a Hölder equilibrium state for σ. By Proposition 3.1, there exists $v' : \Sigma \to G \in F_\theta(G)$ such that

$$\begin{aligned} v' &= v \text{ a.e.} \\ \mathcal{IF}(x) &= v'\sigma(x)\mathcal{IG}(x)v'(x)^{-1} \text{ everywhere.} \end{aligned} \tag{8}$$

Define $\tilde{u}'$ on $\Sigma \times \{s_0\}$ by

$$\tilde{u}'(x, s_0) = \mathcal{F}((x,0), s_0)v'(x)\mathcal{G}((x,0), s_0)^{-1}.$$

Then $\tilde{u}' = \tilde{u}$ a.e. on $\Sigma \times \{s_0\}$. Define $\tilde{u}'$ on $\widetilde{\Sigma}_r$ by

$$\tilde{u}'(x,s) = \mathcal{F}((x,0),s)\mathcal{F}((x,0),s_0)^{-1}\tilde{u}'(x,s_0)\mathcal{G}((x,0),s_0)\mathcal{G}((x,0),s)^{-1}.$$

We will show $\tilde{u}'$ is the required solution.

From (8) it follows that

$$\begin{aligned} \mathcal{IF}(x) &= \mathcal{F}((\sigma x,0),s_0)^{-1}\tilde{u}'(\sigma x,s_0)\mathcal{G}((\sigma x,0),s_0)\mathcal{IG}(x) \\ &\quad \cdot \mathcal{G}((x,0),s_0)^{-1}\tilde{u}'(x,s_0)^{-1}\mathcal{F}((x,0),s_0) \end{aligned}$$

for all $x \in \Sigma$. Then

$$\begin{aligned} \tilde{u}'(x, r(x)) &= \mathcal{IF}(x)\mathcal{F}((x,0),s_0)^{-1}\tilde{u}'(x,s_0)\mathcal{G}((x,0),s_0)\mathcal{IG}(x)^{-1} \\ &= \mathcal{F}((\sigma x,0),s_0)^{-1}\tilde{u}'(\sigma x,s_0)\mathcal{G}((\sigma x,0),s_0) \\ &= \tilde{u}'(\sigma x, 0). \end{aligned}$$

Hence $\tilde{u}'$ is a well-defined continuous function on $\widetilde{\Sigma}_r$.

To see that $\tilde{u}'$ is a solution to the cocycle equation, observe that

$$\begin{aligned} &\tilde{u}\sigma_r^t(x,s) \\ &= \mathcal{F}((x,0),s+t)\mathcal{F}((x,0),s_0)^{-1}\tilde{u}'(x,s_0)\mathcal{G}((x,0),s_0)\mathcal{G}((x,0),s+t)^{-1} \\ &= \mathcal{F}((x,s),t)\mathcal{F}((x,0),s)\mathcal{F}((x,0),s_0)^{-1}\tilde{u}'(x,s_0) \\ &\quad \cdot \mathcal{G}((x,0),s_0)\mathcal{G}((x,0),s_0)^{-1}\mathcal{G}((x,s),t)^{-1} \\ &= \mathcal{F}((x,s),t)\tilde{u}'(x,s)\mathcal{G}((x,s),t)^{-1} \end{aligned}$$

everywhere.

Finally, we show that $\tilde{u}' = \tilde{u}$ a.e. Recall that $\tilde{u}'(x, s_0) = \tilde{u}(x, s_0)$ a.e. on $\Sigma \times \{s_0\}$. For any s we have

$$\begin{aligned}
\tilde{u}'(x, s+s_0) &= \tilde{u}'\sigma_r^s(x, s_0) \\
&= \mathcal{F}((x, s_0), s)\tilde{u}'(x, s_0)\mathcal{G}((x, s_0), s)^{-1} \\
&= \mathcal{F}((x, s_0), s)\tilde{u}(x, s_0)\mathcal{G}((x, s_0), s)^{-1} \text{ a.e.} \\
&= \tilde{u}(x, s+s_0) \text{ a.e.}
\end{aligned}$$

Hence $\tilde{u}' = \tilde{u}$ a.e. ❑

§4 Livšic regularity theorems for hyperbolic flows

In this section we prove the regularity theorem for cocycles taking values in a compact connected Lie group over a hyperbolic flow ϕ^t.

Let G be such a group and let ρ be a left-right invariant Riemannian metric. Let $\mathcal{F}$ be a cocycle taking values in G. We say $\mathcal{F}$ is Hölder continuous of exponent $\alpha \in (0, 1)$ if the map

$$x \mapsto \lim_{t \to 0} \frac{1}{t} \exp^{-1} \mathcal{F}(x, t)$$

with values in the Lie algebra of G is Hölder continuous of exponent α. Equivalently, if we view G as a closed subgroup of a unitary group (as we may do), then we require the map

$$x \mapsto f(x) = \lim_{t \to 0} \frac{\mathcal{F}(x, t) - I}{t}$$

to be Hölder (I denotes the identity matrix).

Lemma 4.1
Let $\mathcal{F}$ be a Hölder cocycle. Then

1. $\rho(\mathcal{F}(x, t), \mathcal{F}(y, t)) \leq \int_0^t \|f(\phi^u x) - f(\phi^u y)\| \, du$;
2. $\rho(\mathcal{F}(x, t), e) \leq \int_0^t \|f(\phi^u x)\| \, du$.

Proof. Note that $\gamma : [0, t] \to G : u \mapsto \mathcal{F}(y, u)^{-1}\mathcal{F}(x, u)$ is a smooth curve from e to $\mathcal{F}(y, t)^{-1}\mathcal{F}(x, t)$. Then an easy calculation shows that

$$\gamma'(u) = \mathcal{F}(y, u)^{-1}(f(\phi^u x) - f(\phi^u y))\mathcal{F}(x, u)$$

whence

$$\|\gamma'(u)\| = \|f(\phi^u x) - f(\phi^u y)\|$$

as $\mathcal{F}$ is unitary-valued. The claim follows from the definition of the Riemannian metric and the left-right invariance of ρ. The second part is similar.

❏

Corollary 4.2

For $z \in W^s_\varepsilon(x)$,

$$F^s(z,x) = \lim_{t\to\infty} \mathcal{F}(z,t)^{-1}\mathcal{F}(x,t)$$

exists and satisfies

$$\rho(F^s(z,x),e) \leq \text{Const.}\, d(z,x)^\alpha$$

where α is the Hölder exponent of $\mathcal{F}$ and the constant is independent of x, z.

Proof. Fix $t \geq 0$ and let $s > 0$. Then

$$\begin{aligned}
&\rho(\mathcal{F}(z,t+s)^{-1}\mathcal{F}(x,t+s), \mathcal{F}(z,t)^{-1}\mathcal{F}(x,t)) \\
&= \rho(\mathcal{F}(\phi^t z,s), \mathcal{F}(\phi^t x,s)) \\
&\leq \|f\|_\alpha \int_0^s d(\phi^u(\phi^t z), \phi^u(\phi^t x))^\alpha\, du \\
&\leq \text{Const.}\, d(\phi^t z, \phi^t x)^\alpha \\
&\leq \text{Const.}\, d(z,x)^\alpha \lambda^{\alpha t}
\end{aligned}$$

where the constant is independent of x, z and t. It follows that $\lim_{t\to\infty} \mathcal{F}(z,t)^{-1}\mathcal{F}(x,t)$ exists. Setting $t = 0$ in the above gives $\rho(F^s(z,x),e) \leq \text{Const.}\, d(z,x)^\alpha$. ❏

We can also define for $z \in W^u_\varepsilon(y)$, $F^u(z,y) = \lim_{t\to-\infty} \mathcal{F}(z,t)^{-1}\mathcal{F}(y,t)$. Again $\rho(F^u(z,y),e) \leq \text{Const.}\, d(z,y)^\alpha$.

We can now prove our main result.

Theorem 4.3

Let ϕ^t be a C^1 hyperbolic flow on a basic set Λ equipped with an equilibrium state m corresponding to a Hölder continuous function. Let $\mathcal{F}$ and $\mathcal{G}$ be

Hölder cocycles of exponent α. Suppose $u : \Lambda \to G$ is a measurable solution to

$$\mathcal{F}(x,t) = u(\phi^t x)\mathcal{G}(x,t)u(x)^{-1} \text{ a.e. } [m]. \tag{9}$$

Then there exists a Hölder $u' \in C^\alpha(G)$ such that $u = u'$ a.e. and

$$\mathcal{F}(x,t) = u'(\phi^t x)\mathcal{G}(x,t)u'(x)^{-1}$$

everywhere.

Proof. Let $\mathcal{F}$, $\mathcal{G}$ and u be as in the statement of the theorem. Let $\tilde{\pi} : \widetilde{\Sigma}_r \to \Lambda$ be as in §2. Let $\tilde{u} = u \circ \tilde{\pi}$, $\tilde{\mathcal{F}}(\tilde{x},t) = \mathcal{F}(\tilde{\pi}\tilde{x},t)$, $\tilde{\mathcal{G}}(\tilde{x},t) = \mathcal{G}(\tilde{\pi}\tilde{x},t)$ and $\tilde{m}$ be the Hölder equilibrium state for σ_r^t such that $\tilde{m}\tilde{\pi}^{-1} = m$. Then $\tilde{u}$ is measurable and

$$\tilde{\mathcal{F}}(\tilde{x},t) = \tilde{u}\sigma_r^t(\tilde{x})\tilde{\mathcal{G}}(\tilde{x},t)\tilde{u}(\tilde{x})^{-1} \text{ a.e. } [\tilde{m}].$$

Moreover, for $x, y \in \Sigma$, we have

$$\begin{aligned}
&\rho(\mathcal{I}\tilde{\mathcal{F}}(x), \mathcal{I}\tilde{\mathcal{F}}(y)) \\
&\quad = \rho(\mathcal{F}(\tilde{\pi}(x,0), r(x)), \mathcal{F}(\tilde{\pi}(y,0), r(y))) \\
&\quad \leq \rho(\mathcal{F}(\tilde{\pi}(x,0), r(x)), \mathcal{F}(\tilde{\pi}(x,0), r(y))) \\
&\qquad + \rho(\mathcal{F}(\tilde{\pi}(x,0), r(y)), \mathcal{F}(\tilde{\pi}(y,0), r(y))) \\
&\quad = \rho(\mathcal{F}(\phi^{r(y)}\tilde{\pi}(x,0), r(x) - r(y)), e) + \rho(\mathcal{F}(\tilde{\pi}(x,0), r(y)), \mathcal{F}(\tilde{\pi}(y,0), r(y))) \\
&\quad \leq \|f\|_\alpha \|r\|_\theta d^\theta(x,y) + \|f\|_\alpha \int_0^{r(y)} d(\phi^u \tilde{\pi}(x,0), \phi^u \tilde{\pi}(y,0))^\alpha \, du \\
&\quad \leq \text{Const.}\, d^\theta(x,y) + \text{Const.}\, d^{\theta^\alpha}(x,y)
\end{aligned}$$

by Lemma 4.1, where the constants are independent of x, y. Hence $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ satisfy the hypotheses of Theorem 3.2. By Theorem 3.2, $\tilde{u} = \tilde{u}'$ a.e. for some continuous $\tilde{u}' : \widetilde{\Sigma}_r \to G$ and

$$\tilde{\mathcal{F}}(\tilde{x},t) = \tilde{u}'\sigma_r^t(\tilde{x})\tilde{\mathcal{G}}(\tilde{x},t)\tilde{u}'(\tilde{x})^{-1}$$

everywhere. We shall show that $\tilde{u}' = u' \circ \pi$ for some continuous $u' : \Lambda \to G$. It then follows that (9) holds everywhere when u is replaced by u'.

Let γ be a 'good' periodic orbit. Then for each $x \in \gamma$, there exists a unique $\tilde{x} \in \widetilde{\Sigma}_r$ such that $\tilde{\pi}\tilde{x} = x$. If we define u' on γ by $u'(x) = \tilde{u}'(\tilde{x})$ then

(9) holds everywhere on the dense set of 'good' periodic orbits. It remains to show that u' can be extended to Λ.

Let γ_1 and γ_2 be two 'good' periodic orbits with least periods τ_1, τ_2, respectively. Let $x \in \gamma_1$, $y \in \gamma_2$ be sufficiently close so that $z = \langle x, y \rangle = W^s_\varepsilon(\phi^{\nu(x,y)}x) \cap W^u_\varepsilon(y)$ is defined. We claim that the orbit of z is 'good'; it will then follow from Theorem 2.1 that $z = \tilde{\pi}\tilde{z}$ for a unique $\tilde{z} \in \widetilde{\Sigma}_r$.

As $z \in W^u_\varepsilon(y)$, we have $d(\phi^{-t}z, \phi^{-t}y) \to 0$ as $t \to \infty$. Choose $t_y > 0$ such that $y' = \phi^{-t_y}y \in \operatorname{Int} R \subset \Gamma^\circ(\mathcal{R})$ for some $R \in \mathcal{R}$. Then $d(\phi^{-n\tau_2 - t_y}z, y') \to 0$ as $n \to \infty$. Thus, for all sufficiently large n, $\phi^{-n\tau_2 - t_y}z \in U_\xi(D)$. As π_D is continuous, $\pi_D\phi^{-n\tau_2 - t_y}z \to \pi_D y' = y'$. It follows that $\pi_D\phi^{-n\tau_2 - t_y}z \notin \partial\mathcal{R}$ if n is sufficiently large. Hence there exists a sequence of times $t_n^- \downarrow -\infty$ such that $\phi^{t_n^-}z \notin \partial\mathcal{R}$. Similarly, by replacing y by $\phi^{\nu(x,y)}x$ and reversing time, we can construct a sequence of times $t_n^+ \uparrow \infty$ such that $\phi^{t_n^+}z \notin \partial\mathcal{R}$. Suppose for a contradiction that, for some t, $\phi^t z \notin \partial\mathcal{R}$. If $\phi^t z \in \partial^s\mathcal{R}$ then, by Lemma 2.2, the forward orbit (under H) of z would remain in $\partial^s\mathcal{R}$, contradicting $\phi^{t_n^+}z \notin \partial\mathcal{R}$. Similarly, $\phi^t z \notin \partial^u\mathcal{R}$. By Theorem 2.1.4, the orbit of z is 'good'.

Define $u'(z) = \tilde{u}'(\tilde{z})$ and extend u' along the orbit of z. Then (9) holds for u' along the orbit of z.

We have

$$\begin{aligned} &\mathcal{F}(z,t)^{-1}\mathcal{F}(\phi^{\nu(x,y)}x, t) \\ &\quad = u'(z)\mathcal{G}(z,t)^{-1}u'(\phi^t z)^{-1}u'(\phi^{t+\nu(x,y)}x)\mathcal{G}(\phi^{\nu(x,y)}x, t)u'(\phi^{\nu(x,y)}x)^{-1}. \end{aligned} \tag{10}$$

Now

$$\begin{aligned} u'(\phi^t z)^{-1}u'(\phi^{t+\nu(x,y)}x) &= \tilde{u}'\tilde{\pi}^{-1}(\phi^t z)^{-1}\tilde{u}'\tilde{\pi}^{-1}(\phi^{t+\nu(x,y)}x) \\ &= \tilde{u}'\sigma_r^t(\tilde{\pi}^{-1}z)^{-1}\tilde{u}'\sigma_r^{t+\nu(x,y)}(\tilde{\pi}^{-1}x) \\ &\to 0 \end{aligned}$$

as $\tilde{\pi}^{-1}z \in W^s_\varepsilon(\sigma_r^{\nu(x,y)}\tilde{\pi}^{-1}x)$ and $\tilde{u}'$ is continuous. Hence letting $t \to \infty$ in (10) we see that

$$F^s(z, \phi^{\nu(x,y)}x) = u'(z)G^s(z, \phi^{\nu(x,y)}x)u'(\phi^{\nu(x,y)}x)^{-1}.$$

Similarly,

$$F^u(z,y) = u'(z)G^u(z,y)u'(y)^{-1},$$

Hence,

$$u'(\phi^{\nu(x,y)}x) = F^s(z,\phi^{\nu(x,y)}x)^{-1}F^u(z,y)u'(y)G^u(z,y)^{-1}G^s(z,\phi^{\nu(x,y)}x).$$

By the left-right invariance of ρ and Corollary 4.2 we have

$$\begin{aligned}\rho(u'(\phi^{\nu(x,y)}x),u'(y)) &\leq \rho(F^s(z,\phi^{\nu(x,y)}x),e)+\rho(F^u(z,y),e)\\ &\qquad +\rho(G^s(z,\phi^{\nu(x,y)}x),e)+\rho(G^u(z,y),e)\\ &\leq \text{Const.}\,(d(z,\phi^{\nu(x,y)}x)^\alpha + d(z,y)^\alpha)\end{aligned}$$

for some constant independent of x, y and z.

Hence,

$$\begin{aligned}\rho(u'(x),u'(y)) &\leq \rho(\tilde{u}'\tilde{\pi}^{-1}(x),\tilde{u}'\sigma_r^{\nu(x,y)}\tilde{\pi}^{-1}(x))\\ &\qquad + \text{Const.}\,(d(z,\phi^{\nu(x,y)}x)^\alpha + d(z,y)^\alpha). \qquad (11)\end{aligned}$$

As both $\tilde{u}'$ and ν are uniformly continuous, it follows that u' is uniformly continuous on the dense set of good periodic points. Therefore, u' extends uniquely to a continuous function $u' : \Lambda \to G$.

As $\tilde{\pi}$ is one-to-one a.e., $u' = u$ a.e. Moreover, u' solves (9) everywhere.

It remains to check that u' has the same degree of Hölder regularity as the cocycles $\mathcal{F}$, $\mathcal{G}$. In the case of a coboundary ($\mathcal{F}(x,t) = u(\phi^t x)u(x)^{-1}$) this is clear: a periodic data criterion gives the required degree of regularity on u. For the general case, we argue as follows. From the estimates above, it follows that if $z \in W^s_\varepsilon(x)$ then

$$F^s(z,x) = u'(z)G^s(z,x)u'(x)^{-1}.$$

Hence

$$\begin{aligned}\rho(u'(x),u'(z)) &\leq \rho(F^s(z,x),e)+\rho(G^s(z,x),e)\\ &\leq \text{Const.}\,d(z,x)^\alpha\end{aligned}$$

so that u' is Hölder continuous along stable manifolds. Similarly, u' is Hölder continuous along unstable manifolds. Along orbits it is easy to see, using flow-box co-ordinates, that for sufficiently small t,

$$\begin{aligned}\rho(u'(\phi^t x), u'(x)) &\leq \rho(\mathcal{F}(x,t), e) + \rho(\mathcal{G}(x,t), e)\\ &\leq \text{Const.}\,|t|^\alpha\\ &\leq \text{Const.}\,d(x,\phi^t x)^\alpha,\end{aligned}$$

for some constant independent of x and t. Hence u' is uniformly Hölder continuous along the leaves of three foliations whose leaves are uniformly transverse, continuously varying Lipschitz submanifolds. Proposition 19.1.1 of [KH] then shows that u' is Hölder on Λ. ❑

Remark. The above proof resembles the proof of the corresponding result for hyperbolic diffeomorphisms given in [PP2], taking into account the extra complications that arise from flows. For a hyperbolic diffeomorphism, the local product structure is given by $\langle x, y\rangle = W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ so there is no term $\nu(x,y)$ in (11). In particular, we can deduce that u' is Hölder continuous from (11) without any extra argument.

§5 More general groups

We consider some classes of non-compact Lie groups.

Abelian Lie groups. We show how to deduce the regularity theorem for real-valued cocycles from the circle-valued case. For notational convenience, we only consider diffeomorphisms.

Theorem 5.1
Let ϕ be a hyperbolic diffeomorphism on Λ. Let m be an equilibrium state corresponding to a Hölder continuous function. Suppose $f : \Lambda \to \mathbb{R} \in C^\alpha(\Lambda, \mathbb{R})$ and u is a measurable solution to $f = u\phi - u$ a.e. $[m]$. Then there exists $u' \in C^\alpha(\Lambda, \mathbb{R})$ such that $u = u'$ a.e. and $f = u'\phi - u'$ everywhere.

Proof. Let $\mu \in \mathbb{R}$. Then

$$e^{2\pi i \mu f} = \frac{e^{2\pi i \mu u\phi}}{e^{2\pi i \mu u}}.$$

By the regularity theorem for circle-valued cocycles, there exists a continuous map $u_\mu : \Lambda \to K$ such that

$$e^{2\pi i\mu f} = u_\mu \phi / u_\mu$$

everywhere. Suppose $\phi^n x = x$. Then $e^{2\pi i\mu f^n(x)} = 1$ ($f^n = \sum_{i=0}^{n-1} f\phi^i$). Hence, $\mu f^n(x)$ is an integer for each $\mu \in \mathbb{R}$. It follows that $f^n(x) = 0$ whenever $\phi^n x = x$. By the periodic data criterion for real-valued cocycles, $f = u'\phi - u'$ for some $u' \in C^\alpha(\Lambda, \mathbb{R})$. By ergodicity, we may choose u' so that $u = u'$ a.e. ❑

Remarks.

1. Hence we have the regularity theorem for cocycles taking values in an arbitrary connected abelian Lie group.
2. Livšic proved Theorem 5.1 in [Li2] for C^2 Anosov diffeomorphisms and flows preserving a smooth measure. The C^2 assumption on ϕ is needed in Livšic's approach because he uses the absolute continuity of the stable and unstable foliations. This fact is proved in, for example, [AS, An]. (In [An], Anosov remarks that one may replace C^2 by the condition that the partial derivatives of ϕ satisfy the Dini condition;[4] if ϕ is $C^{1+\alpha}$ (and, in particular, is Dini), this is proved in, for example, in [Ma].) In the C^1 case, these foliations need not be absolutely continuous [RY].
3. If u is assumed to be essentially bounded then it is easy to deduce from the periodic data criterion that u must be Hölder continuous [Li1].

Semi-direct products. For notational simplicity, we only consider the case of a hyperbolic diffeomorphism.

Theorem 5.2
Let G_1, G_2 be Lie groups and suppose that G_1 acts smoothly on G_2 by automorphisms. Let ϕ be a hyperbolic diffeomorphism and suppose the

[4] g is *Dini* if $\int_0^1 \delta^{-1} \sup\{|g(x) - g(y)| \mid d(x,y) \le \delta\}\, d\delta < \infty$.

regularity theorem holds for coboundaries taking values in G_1 and G_2. Then the regularity theorem continues to hold for coboundaries taking values in the semi-direct product $G_1 \ltimes G_2$.

Proof. Let $\alpha : G_1 \to \mathrm{Aut}(G_2)$ be the G_1-action on G_2. Then the group structure on $G_1 \ltimes G_2$ is given by $(g_1, g_2)(g_1', g_2') = (g_1 g_1', g_2 \cdot \alpha_{g_1} g_2')$.

Let $f = (f_1, f_2) : \Lambda \to G_1 \ltimes G_2$ be Hölder continuous and suppose $u = (u_1, u_2)$ is a measurable function such that

$$(f_1, f_2) = (u_1\phi, u_2\phi) \cdot (u_1, u_2)^{-1} \text{ a.e.} \tag{12}$$

It is easy to check that (12) is equivalent to

$$f_1 = u_1\phi \cdot u_1^{-1} \text{ a.e.} \tag{13}$$

$$f_2 = u_2\phi \cdot \alpha_{u_1\phi \cdot u_1^{-1}} u_2^{-1} \text{ a.e.} \tag{14}$$

As the regularity theorem holds for G_1, there exists a Hölder $u_1' : \Lambda \to G_1$ such that $u_1 = u_1'$ a.e. and $f_1 = u_1'\phi \cdot u_1'$ everywhere. Then $\alpha_{(u_1'\phi)^{-1}} f_2$ is Hölder continuous and

$$\alpha_{(u_1'\phi)^{-1}} f_2 = \alpha_{(u_1'\phi)^{-1}} u_2\phi \cdot \alpha_{u_1'^{-1}} u_2^{-1} \text{ a.e.} \tag{15}$$

Then $x \mapsto \alpha_{u_1'(x)^{-1}} u_2(x)$ is measurable and, as the regularity theorem holds for G_2, is equal a.e. to a Hölder continuous function for which (15) holds everywhere. Hence, $u_2 = u_2'$ a.e. for some Hölder continuous u_2' for which (14) holds everywhere. ❑

Remarks.

1. So the regularity theorem holds for coboundaries taking values in subgroups of $O(d) \ltimes \mathbb{R}^d$, the 'orthogonal affine' transformations of $\mathbb{R}^d$. For more general Lie subgroups of $Gl(n)$ and C^2 Anosov diffeomorphisms, the regularity theorem for coboundaries continues to hold, provided the cobounding function and its inverse are assumed to be essentially bounded [NP].

2. For the general cocycle equation (1) the regularity theorem may fail for (non-compact) semi-direct products [Wa1].

§6 Higher regularity results

Let us now briefly address question 3 in the Introduction. We have already seen that if $\mathcal{F}, \mathcal{G}$ are Hölder cocycles such that $\mathcal{F}(x,t) = u(\phi^t x)\mathcal{G}(x,t)u(x)^{-1}$ a.e. then u has a continuous version for which this equation holds everywhere.

Now let us suppose that $\mathcal{F}$ and $\mathcal{G}$ possess a higher degree of regularity. Let $1 \leq r \leq \infty$ or $r = \omega$ and let ϕ^t be a C^r Anosov flow on a manifold M. We say that $\mathcal{F}$ is a C^r cocycle if the map $\mathcal{F} : M \times \mathbb{R} \to G$ is C^r.

In [Li1, Li2], Livšic considered the C^1 case for abelian Lie groups (see also [KH, §19.2] for the details of this). He also considered $C^{r+\alpha}$, C^∞ and C^ω versions in some special cases. The C^∞ case was completely solved in [LMM] (it had previously been considered for geodesic flows on negative curved surfaces in [GK]) and since then two other proofs have been given [Jo1, HK]. In [Ll2], de la Llave used the method of [HK] to deduce the C^ω case. In all of the above work, the basic idea is to show that:

1. if u is a continuous solution to the cocycle equation then u is C^r on stable manifolds, unstable manifolds and along orbits;
2. if u is C^r on the leaves of these three foliations then u possesses some degree of regularity.

(Observe that this is how we deduced that the continuous solution u' in the proof of Theorem 4.3 was in fact Hölder continuous.) With regard to the second step, we have the following:

Proposition 6.1

Let M be a manifold with two continuous transverse foliations with uniformly smooth leaves. Suppose $f : M \to U \subset \mathbb{R}^n$ is uniformly C^r along the leaves of both foliations. Then f is $C^{r-1+\alpha}$ for any $\alpha \in [0,1)$. The same is true for $r = \omega$ if the Jacobian of each foliation is uniformly analytic on each leaf. (For $r = 1, \infty, \omega$ we define $r - 1 + \alpha = 1, \infty, \omega$, respectively.)

For $r = 1$, this is clear: if a function has continuous partial derivatives then the function is C^1. The analytic case is proved in [Ll2] and is not restricted to two foliations. Moreover, in [Ll2] it is proved that, for analytic Anosov flows, the Jacobians of the stable and unstable manifolds have the required properties. The $2 \leq r \leq \infty$ case is considered in [Jo2] and the proof given there is restricted to just two foliations. However, in our case this is no great restriction: we use Journé's theorem on, say, stable manifolds and orbits to obtain regularity on *weak* stable manifolds, and then again on weak stable manifolds and strong unstable manifolds to obtain regularity globally.

The first step is considered for cocycles taking values in a compact Lie group[5] in [NT2], building on earlier work in [NT1] for coboundaries. Indeed, in [NT2], they prove step 1 for cocycles taking values in any matrix group and in the diffeomorphism group of any compact manifold. Combining the results of [NT2] with Theorems 4.3 and 5.1, we have

Theorem 6.2
Let ϕ^t be a C^r ($1 \leq r \leq \infty$ or $r = \omega$) transitive Anosov flow on a compact manifold M. Let G be a compact Lie group or an abelian Lie group and let $\mathcal{F}, \mathcal{G} : M \times \mathbb{R} \to G$ be two C^r cocycles. Let m be the equilibrium state corresponding to a Hölder continuous function. Suppose $u : M \to G$ is a measurable solution to

$$\mathcal{F}(x,t) = u(\phi^t x)\mathcal{G}(x,t)u(x)^{-1} \text{ a.e. } [m].$$

Then there exists a continuous $u' : M \to G$ such that $u = u'$ a.e. and $\mathcal{F}(x,t) = u'(\phi^t x)\mathcal{G}(x,t)u'(x)^{-1}$ everywhere. Moreover, u' is $C^{r-1+\alpha}$ for any $\alpha \in [0,1)$.

Counterexamples [Ll1, NT2] show that, in general, we cannot assume that u' is C^r.

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Stable ergodic properties of cocycles over hyperbolic flows

Abstract

We consider flows obtained by either a toral extension or a velocity change of a hyperbolic flow on a basic set with finite rank first cohomology. A dynamic cocycle is associated to each flow so obtained. By studying the relationships between these cocycles and the cohomology of the basic set, we prove the existence (in appropriate topologies) of open dense sets of ergodic and mixing toral extensions and open dense sets of mixing velocity changes. Analogous results are proved for suspensions of shifts of finite type.

§1 Introduction and statement of results

In [PS] it is conjectured that for diffeomorphisms 'a little hyperbolicity goes a long way in guaranteeing stable ergodic behaviour'. By *stably ergodic* we mean the existence of an open dense set (in an appropriate topology) of perturbations for which the perturbed system is ergodic. An example of this phenomena can be found in [AKS]. Here, Hölder continuous circle extensions of Anosov toral automorphisms are considered and it is proved that the ergodic extensions contain a uniformly open dense set. This was generalised in [PP3] to Hölder continuous toral extensions of hyperbolic diffeomorphisms satisfying a certain condition on the first Čech cohomology group and also to toral extensions of shifts of finite type. Here, open dense sets of mixing extensions are constructed, this time using the Hölder topology. In this paper, we consider hyperbolic flows and find results analogous to those in [PP3]. We will need to assume that the first Čech cohomology group of the underlying basic set has finite rank. With this assumption, it is again true that the ergodic and mixing toral extensions contain an open dense set. When the basic set is one-dimensional, the first cohomology group does not have finite rank. However, in this case we can still prove the existence of open dense sets of ergodic and mixing extensions by using a result of Bowen's [Bo2] that such a flow is conjugate to a suspension of a shift of finite type and then using the results of [PP3]. Velocity changes of flows are also considered. Again, with the assumption that the first cohomology group of the basic set has finite rank or that the basic set is one-dimensional, we prove the existence of open dense sets of reparametrisations of time for which the velocity changed flow is mixing.

Let M be a compact C^∞ Riemannian manifold. A flow $\phi^t : M \to M$ is said to be C^1 if the vector field $x \mapsto X_\phi(x) : M \to T_xM$ is C^1. A closed ϕ^t-invariant set Λ is called a *basic set* if

1. the tangent bundle of M restricted to Λ splits continuously into a Whitney sum $T_\Lambda M = E^0 \oplus E^s \oplus E^u$ of $d\phi^t$-invariant sub-bundles such that

   - there exist constants $C > 0$, $0 < \lambda < 1$ such that

     $$\|d\phi^t(v)\| \le C\lambda^t \|v\| \text{ for all } t \ge 0, \text{ if } v \in E^s_x,$$
     $$\|d\phi^{-t}(v)\| \le C\lambda^t \|v\| \text{ for all } t \ge 0, \text{ if } v \in E^u_x,$$

   - E^0 is one-dimensional and tangential to the flow direction;

2. Λ contains a dense orbit;

3. the periodic orbits of ϕ^t in Λ are dense in Λ;

4. there exists an open neighbourhood $U \supset \Lambda$ such that

$$\bigcap_{t=-\infty}^{\infty} \phi^t U = \Lambda;$$

5. Λ has no fixed points and is not a single orbit.

When ϕ^t is restricted to Λ, we call ϕ^t a *hyperbolic flow*. If in 4. the open neighbourhood U satisfies

$$\bigcap_{t=0}^{\infty} \phi^t U = \Lambda,$$

then we call ϕ^t or Λ an *attractor*. If $\Lambda = M$ then we call ϕ^t a *(transitive) Anosov flow*. Clearly, Anosov flows are attractors.

When proving our stability results, we will need to assume that the first Čech cohomology group $H^1(\Lambda, \mathbb{Z})$ of Λ has finite rank. We call such hyperbolic flows *cohomologically finite*. As to how often this occurs appears to be an open question. For Anosov flows, $H^1(M, \mathbb{Z})$ is finitely generated and so has finite rank. More generally, it is easy to see that any attractor is cohomologically finite [FP, Theorem 2.7]. However, if Λ is one-dimensional then $H^1(\Lambda, \mathbb{Z})$ is the free abelian group with countably many generators and so does not have finite rank [PT, Theorem 4.19].

Before describing our main results, we digress slightly and discuss some of the function spaces and invariant measures considered in this paper.

Function spaces. Let d be the metric on M induced from the Riemannian metric. A function $f : M \to \mathbb{R}$ is *Hölder continuous with exponent θ*, $0 < \theta < 1$, if

$$|\, f(x) - f(y) \,| \leq C\, d(x, y)^{\theta}$$

for all $x, y \in M$ and some $C > 0$. Throughout, we shall assume θ is fixed in advance and we shall speak of Hölder continuous functions without reference to the exponent. We shall only be concerned with Hölder continuous functions defined on Λ and we denote this space by $C^{\theta}(\Lambda, \mathbb{R})$. Equip $C^{\theta}(\Lambda, \mathbb{R})$ with the norm

$$\|f\|_{\theta} := |f|_{\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^{\theta}},$$

where $|\cdot|_{\infty}$ denotes the uniform norm. Then $C^{\theta}(\Lambda, \mathbb{R})$ is a Banach space. One can easily check that two different Riemannian metrics have the same Hölder continuous functions and that the two norms on $C^{\theta}(\Lambda, \mathbb{R})$ are equivalent.

There are two properties of $C^{\theta}(\Lambda, \mathbb{R})$ functions that will prove invaluable in the sequel. Firstly, the composition (in either order) of a C^1 function on a compact set with a Hölder continuous function is Hölder continuous. Secondly, if f is 'locally Hölder continuous' (*i.e.* every point of Λ has a neighbourhood on which f is Hölder) on a compact set then f is Hölder continuous.

When $\Lambda = M$ (*i.e.* the Anosov case), we can also speak of C^r functions for $r \geq 1$. Denote the space of such functions by $C^r(\Lambda, \mathbb{R})$ (or $C^r(\Lambda, K)$ for circle-valued functions). We shall mostly be concerned with the cases $r = 1$ and $r = \infty$. In the C^1 case, we use the norm

$$\|f\|_{C^1} := |f|_\infty + \sup_{x \in \Lambda} \|df(x)\|.$$

Then $C^1(\Lambda, \mathbb{R})$ is a Banach space and $C^1(\Lambda, K)$ is complete. The topology on $C^\infty(\Lambda, \mathbb{R})$ and $C^\infty(\Lambda, K)$ is given by convergence in the C^r-norm for each $r \in \mathbb{N}$. These spaces are not Banach spaces, although they are complete.

Throughout this paper, we use C^θ to denote either Hölder continuous functions of exponent $\theta \in (0,1)$, C^1 functions or C^∞ functions. (If we refer to C^r functions for $r = 2, 3, \ldots$ then we shall mention this explicitly.) When $\theta = 1$ or ∞, we assume without mention that the flow is Anosov. When $\theta = \infty$, we always assume that the flow is itself C^∞.

Equilibrium states. For a continuous function $f : \Lambda \to \mathbb{R}$ define the *pressure* of f to be

$$P(f) = \sup\{h_\mu(\phi) + \mu(f) \mid \mu \text{ is a } \phi^t\text{-invariant probability}\} \tag{1}$$

(here $h_\mu(\phi)$ is the entropy with respect to μ of the time-one map ϕ^1). If f is Hölder continuous then this supremum is achieved by a unique ϕ^t-invariant probability m_f (denoted by m if f is fixed in advance) [BR]. We call m_f the *equilibrium state* of f. Moreover, m_f is ergodic and positive on non-empty open sets [BR]. It is not true, in general, that m_f is weak-mixing. However, if m_f is weak-mixing then (ϕ^t, m_f) is isomorphic to a Bernoulli flow (see [BR, Remark 3.5] and the references cited there); for brevity, we say such a flow is *mixing*.

In the statements of our main theorems below, we always assume that ϕ^t is equipped with a Hölder equilibrium state m.

Toral Extensions. We shall be interested in toral extensions of ϕ^t of the following form. Let K denote the unit circle in $\mathbb{C}$ and K^n the n-torus. For $1 \leq i \leq n$, let $f_i \in C^\theta(\Lambda, \mathbb{R})$ and denote $F = (f_1, \ldots, f_n)$. Define the flow $\phi_F^t : \Lambda \times K^n \to \Lambda \times K^n$ by

$$\phi_F^t(x, y_1, \ldots, y_n) = \left(\phi^t x, e^{2\pi i \int_0^t f_1(\phi^u x)\, du} y_1, \ldots, e^{2\pi i \int_0^t f_n(\phi^u x)\, du} y_n\right).$$

Then ϕ_F^t has vector field $X_\phi \oplus X_F : \Lambda \times K^n \to T_x(\Lambda \times K^n) \cong T_x\Lambda \oplus \mathbb{R}^n :$ $(x, y_1, \ldots, y_n) \mapsto (X_\phi(x), f_1(x), \ldots, f_n(x))$. Note that ϕ_F^t is not hyperbolic. It is not even C^1, merely Hölder continuous; however, it will be C^1 when all the f_i are C^1.

Let m be a Hölder equilibrium state and let ν denote the normalised Haar measure on K^n. Then $m \times \nu$ is ϕ_F^t-invariant. Moreover, if $m \times \nu$ is ergodic, then it is the only invariant measure that projects to m [KN] (this is true for *any* abelian group extension of *any* flow). When we discuss the ergodic properties of ϕ_F^t, we mean with respect to $m \times \nu$. We prove the following:

Theorem A
Let ϕ^t be a cohomologically finite hyperbolic flow equipped with a Hölder equilibrium state. Let $\theta \in (0,1)$, $\theta = 1$ or $\theta = \infty$. Then the set of n-tuples $F = (f_1, \ldots, f_n) \in C^\theta(\Lambda, \mathbb{R})^n$ for which ϕ_F^t is ergodic contains an open dense set.

If a compact group extension of a Bernoulli flow is weak-mixing then it is itself Bernoulli [Rud]. Again, for brevity, we call such flows *mixing*. We prove:

Theorem B
Let ϕ^t be a cohomologically finite hyperbolic flow equipped with a Hölder equilibrium state. Suppose ϕ^t is mixing. Let $\theta \in (0,1)$, $\theta = 1$ or $\theta = \infty$. Then the set of n-tuples $F = (f_1, \ldots, f_n) \in C^\theta(\Lambda, \mathbb{R})^n$ for which ϕ_F^t is mixing contains an open dense set.

Notice that we have not said that the ergodic toral extensions form an open set. There may be 'unstably ergodic' extensions (*i.e.* F such that an arbitrarily small perturbation (within $C^\theta(\Lambda,\mathbb{R})^n$) of F gives a non-ergodic extension). Indeed, we give criteria for the n-tuple $F=(f_1,\dots,f_n)$ to define 'stably ergodic,' 'unstably ergodic' and 'non-ergodic' extensions in terms of the linear independence, linear (but not rational) dependence and rational dependence of the f_i modulo certain spaces of functions that arise naturally from the cohomology of Λ. Similar criteria hold for the mixing case.

Velocity changes. We shall also be interested in C^1 velocity changes of ϕ^t. The details of this construction are well-known and we summarise it in the C^1 hyperbolic case here. We can carry out the same construction in the C^∞ category. For C^θ with $\theta\in(0,1)$ and any hyperbolic flow (not necessarily Anosov), we can still consider velocity changes. However, the flows we obtain will be Hölder continuous and defined only on Λ.

Let $C^1(\Lambda,\mathbb{R}^+)$ denote the space of strictly positive C^1 functions. Let $k:\Lambda\to\mathbb{R}$ be such a function, let $l(x)k(x)=1$ and define

$$k(x,t)=\int_0^t k(\phi^u x)\,du$$

(the joint use of notation should not cause any confusion); then k is a C^1 ϕ-cocycle (see §2). Moreover, for each x, the map $t\mapsto k(x,t)$ is strictly increasing and fixes 0. Hence, there is a well-defined map $l(x,t)$ defined by $t=k(x,l(x,t))$. Define $\psi_k^t(x):=\phi^{l(x,t)}(x)$. Then ψ_k^t is a flow; this amounts to checking that l is a ψ_k^t-cocycle and this is easy from the definitions. We call ψ_k^t a *flow obtained by a velocity change of* ϕ^t. Geometrically, ψ_k^t has the same oriented orbits as ϕ^t, but the orbits are traversed at different speeds.

One can invert this process, *i.e.* velocity change ψ_k^t by multiplying speeds by l to recover ϕ^t. This amounts to checking

$$l(x,t)=\int_0^t l(\psi_k^u x)\,du$$

and this follows from

$$\begin{aligned}\int_0^t l(\psi_k^u x)\,du &= \int_0^t l(\phi^{l(x,u)}x)\,du\\ &= \int_0^{l(x,t)} l(\phi^s x)k(\phi^s x)\,ds,\ \text{letting } s=l(x,u), u=k(x,s)\\ &= l(x,t).\end{aligned}$$

In particular, if X_ϕ, X_{ψ_k} denote the vector fields generating ϕ^t, ψ_k^t respectively, then $X_{\psi_k}(x) = l(x)X_\phi(x)$. Hence ψ_k^t is C^1 when k and ϕ^t are. Moreover, one can show [AS] that if the tangent bundle of Λ has a hyperbolic splitting with respect to ϕ^t, then it has a hyperbolic splitting with respect to ψ_k^t. Clearly ϕ^t and ψ_k^t have the same orbits. Hence, if ϕ^t is hyperbolic, then so is ψ_k^t.

Let m be an ergodic probability measure for ϕ^t. Then ψ_k^t will not, in general, preserve m, but it will preserve the equivalent probability measure with density $k\,dm/m(k)$ and is ergodic with respect to this measure. When we discuss the ergodic properties of ψ_k^t, we mean with respect to $k\,dm/m(k)$. It is not necessarily true that ψ_k^t is mixing, even when ϕ^t is. In [Ch] it is proved that any measurable ergodic flow can have its velocity changed by a measurable cocycle so that the new flow is weak-mixing. Similar results were proved for continuous flows in [Pa1] and for topological weak-mixing [Hm]. Here, we prove the corresponding result for C^1 and C^∞ velocity changes of Anosov flows and Hölder velocity changes of hyperbolic flows. Moreover, we show that such a velocity change can be chosen to be stable: *i.e.* there is an open dense set of mixing velocity changes.

Let m be an equilibrium state for ϕ^t corresponding to $f \in C^\theta(\Lambda, \mathbb{R})$. Observe that in assuming $m(k) = 1$, we are just rescaling $k\,dm$ so that it is a probability measure. We prove the following:

Theorem C
Let ϕ^t be a cohomologically finite hyperbolic flow and let m be a Hölder equilibrium state for ϕ^t. Let $\theta \in (0,1)$, $\theta = 1$ or $\theta = \infty$. Then the set of $k \in C^\theta(\Lambda, \mathbb{R}^+)$ for which ψ_k^t is mixing with respect to $k\,dm/m(k)$ contains an open dense set.

As is the case with toral extensions, there may be 'unstably mixing' velocity changes. Again, criteria are given for the velocity change to be 'stably mixing,' 'unstably mixing' and 'non-mixing' in terms of the congruence class of k modulo certain cohomologically defined function spaces.

Our work on hyperbolic flows is organised as follows. In §2, we examine how the first Čech cohomology group of Λ, realised as the Bruschlinsky group, embeds in the dynamic cocycles. We discuss how the 'bad' ergodic behaviour (*i.e.* not ergodic or mixing, or unstably so) corresponds to dynamic cocycles in $H^1(\Lambda, \mathbb{Z})$. 'Bad' cocycles are then seen to form a very small subgroup of the dynamic cocycles and this forms the basis of our analysis. In §3, we discuss toral extensions and prove Theorems A and B. In §4, we do the same for velocity changes.

Throughout the above, we have assumed that $H^1(\Lambda, \mathbb{Z})$ has finite rank. When this fails, it is easy to prove the existence of a dense G_δ of ergodic toral extensions (although we can say nothing about velocity changes) but so far we have been unable to show that this set contains an open set.

The simplest case when $H^1(\Lambda, \mathbb{Z})$ does not have finite rank is when Λ is one-dimensional. In this case, ϕ^t is conjugate to a suspension of a shift of finite type [Bo2] and we can still prove analogues of our main theorems, albeit using different methods. These are described below and proved in §5.

Suspensions of shifts of finite type. Let A be an aperiodic $k \times k$ matrix with entries from $\{0, 1\}$. Define

$$\Sigma = \{(x_j)_{j=-\infty}^{\infty} \in \{1, \ldots, k\}^{\mathbb{Z}} \mid A_{x_j, x_{j+1}} = 1 \text{ for } j \in \mathbb{Z}\}.$$

Then Σ is invariant under the (left) shift σ defined by $(\sigma x)_j = x_{j+1}$. For each $0 < \theta < 1$ define a metric d^θ on Σ by $d^\theta(x, y) = \theta^{n(x,y)}$ where $n(x, y) = \sup\{n \mid x_j = y_j,\ |i| \leq n\}$. We shall assume that θ is fixed in advance.

Let $F_\theta(\Sigma, \mathbb{R})$ denote the space of all functions $\Sigma \to \mathbb{R}$ that are Lipschitz continuous with respect to d^θ. Then $F_\theta(\Sigma, \mathbb{R})$ is a Banach space with respect to the norm

$$\| \cdot \|_\theta = | \cdot |_\infty + | \cdot |_\theta$$

where $|f|_\theta$ is the least Lipschitz constant of $f \in F_\theta(\Sigma, \mathbb{R})$. We shall often refer to F_θ-functions as Hölder functions of exponent θ.

For continuous $f : \Sigma \to \mathbb{R}$ we can again define the *pressure* of f to be

$$P(f) = \sup\{h_\mu(\sigma) + \mu(f) \mid \mu \text{ is a } \sigma\text{-invariant probability}\}.$$

If $f \in F_\theta(\Sigma, \mathbb{R})$ then this supremum is attained by a unique σ-invariant probability and again we call this the *equilibrium state* of f.

Let $r \in F_\theta(\Sigma, \mathbb{R})$ be strictly positive. Define

$$\widetilde{\Sigma}_r = \{(x,t) \mid x \in \Sigma, 0 \leq t \leq r(x)\}/\sim$$

where $\sim$ identifies $(x, r(x))$ with $(\sigma x, 0)$. Then $\widetilde{\Sigma}_r$ is a compact, one-dimensional space. We define the suspended flow $\sigma_r^t : \widetilde{\Sigma}_r \to \widetilde{\Sigma}_r$ by $\sigma_r^t(x,u) = (x, u+t)$ for small t and then extend to all t by using the identifications. Thus σ_r^t can be thought of as the flow that travels along the fibre $[0, r(x)]$ over $x \in \Sigma$ at unit speed and then 'jumps' to the fibre over σx. Alternatively, we can think of σ_r^t as a flow on the space $\tilde{\Sigma}_1$ where the fibre over x is traversed at speed $1/r(x)$.

We can define the notions of pressure and equilibrium states for σ_r^t in analogy with hyperbolic flows. We can also relate them to the underlying shift of finite type as follows. Let $f : \widetilde{\Sigma}_r \to \mathbb{R}$ be continuous and suppose that the map

$$\mathcal{I}f : x \to \int_0^{r(x)} f\sigma_r^u(x,0)\,du \in F_\theta(\Sigma, \mathbb{R}).$$

One can prove [PP1, for example] that the pressure of f is the unique real number $P(f)$ such that $P(\mathcal{I}f - P(f)r) = 0$. Moreover, the (unique) equilibrium state m of f is given by $m = \mu \times \nu/\mu(r)$ where μ is the equilibrium state of $\mathcal{I}f - P(f)r$ and ν is Lebesgue measure on the fibres $[0, r(x)]$. When $\mathcal{I}f \in F_\theta(\Sigma, \mathbb{R})$, we refer to m as a Hölder equilibrium state.

Let m be a Hölder equilibrium state. Then σ_r^t is ergodic with respect to m. It is not true in general that σ_r^t is mixing. However, by using results in [PP3] we prove the following analogue of Theorem C.

Theorem C′

Let μ be a Hölder equilibrium state for σ. Then the set of strictly positive functions $r : \Sigma \to \mathbb{R} \in F_\theta(\Sigma, \mathbb{R})$ such that σ_r^t is mixing with respect to $\mu \times \nu/\mu(r)$ contains an open dense set (in the $F_\theta(\Sigma, \mathbb{R})$ topology).

We can also consider toral extensions of σ_r^t defined by

$$\sigma_{r,F}^t(\tilde{x}, y_1, \ldots, y_n) = \left(\sigma_r^t \tilde{x}, e^{2\pi i \int_0^t f_1 \sigma_r^u(\tilde{x})\,du} y_1, \ldots, e^{2\pi i \int_0^t f_n \sigma_r^u(\tilde{x})\,du} y_n\right),$$

(using the identifications so that the integrals make sense) where $f_1, \ldots, f_n$ are 'Hölder continuous'. Spaces of Hölder continuous functions for suspensions do not appear to have been considered in detail in the literature. Here we propose two candidates. The first of these consists of the continuous functions $f : \widetilde{\Sigma}_r \to \mathbb{R}$ such that

$$\mathcal{I}f(x) = \int_0^{r(x)} f\sigma_r^u(x,0)\,du \in F_\theta(\Sigma, \mathbb{R})$$

(here θ is fixed in advance). Denote the space of all such functions by $\widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ and equip it with the norm

$$\|f\|_\theta^\sim = |f|_\infty + \|\mathcal{I}f\|_\theta.$$

Then one can easily check that $\widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ is a Banach space. Functions in $\widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ are easy to work with, but $\widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ may be too large a space when using symbolic flows to model other dynamical systems. For this reason, we shall also consider the space $F_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ of Lipschitz continuous functions $\widetilde{\Sigma}_r \to \mathbb{R}$ with respect to a metric d_r^θ (adapted from one given in [BW]) on $\widetilde{\Sigma}_r$. As usual, this space is a Banach space when equipped with the norm $\|\cdot\|_\theta = |\cdot|_\infty + |\cdot|_\theta$ where $|f|_\theta$ is the least Lipschitz constant of f with respect to d_r^θ. In §5, we discuss various properties of these spaces and, in particular, prove that there is a strict continuous inclusion from the latter into the former. (A third possible function space is defined in [Rue] and lies strictly and continuously between the two spaces considered here.) Nevertheless, the analogues of Theorems A and B hold for both function spaces.

Theorem A′

Let m be a Hölder equilibrium state for σ_r^t. Then the set of n-tuples $(f_1, \ldots, f_n)$ for which σ_r^t is ergodic contains an open dense set.

Theorem B′

Let m be a Hölder equilibrium state for σ_r^t and suppose σ_r^t is mixing. Then the set of n-tuples $(f_1, \ldots, f_n)$ for which σ_r^t is mixing contains an open dense set.

It is important to note that we only consider perturbations within $\widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ or $F_\theta(\widetilde{\Sigma}_r, \mathbb{R})$.

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§2 Dynamic versus Bruschlinsky cohomology

We shall be interested in the first Čech cohomology group $H^1(\Lambda, \mathbb{Z})$ of Λ and the group of $\mathbb{R}$-valued dynamic cocycles. In particular, when $H^1(\Lambda, \mathbb{Z})$ has finite rank, we show how the former embeds in the latter as a very small (*i.e.* infinite index) subgroup. We begin with some definitions.

Bruschlinsky cohomology. Let $C(\Lambda, K)$ denote the group of continuous circle-valued functions on Λ with pointwise multiplication. Let $B(\Lambda, K) = \{e^{2\pi iU} \mid U : \Lambda \to \mathbb{R} \text{ is continuous}\}$ denote the subgroup of all null-homotopic (with respect to Λ) functions. The *Bruschlinsky group* is defined to be the quotient $C(\Lambda, K)/B(\Lambda, K)$. It is known [Br] (see [Hu] for an account in English) that this is isomorphic to the first (Čech) cohomology group $H^1(\Lambda, \mathbb{Z})$ of Λ.

A function $w : \Lambda \to \mathbb{C}$ is *differentiable with respect to* ϕ if

$$\lim_{t \to 0} \frac{w(\phi^t x) - w(x)}{t} = w'_\phi(x)$$

exists for each x. We shall say that w is C^θ differentiable with respect to ϕ if w'_ϕ exists and $x \mapsto w'_\phi(x)$ is C^θ.

In [Sc], Schwartzman and Kakutani proved that any cohomology class in $H^1(\Lambda, \mathbb{Z})$ has a representative that is *continuously* differentiable with respect to ϕ. We will need the following strengthening of this result.

Proposition 2.1
Let $[w] \in H^1(\Lambda, \mathbb{Z})$. Let $\theta \in (0, 1)$, $\theta = 1$ or $\theta = \infty$. Then there exists a C^θ $w \in [w]$ that is C^θ differentiable with respect to ϕ.

Proof (sketch). Any real-valued C^∞ function $f : \Lambda \to \mathbb{R}$ is Hölder and C^∞ differentiable with respect to ϕ. It is easy to see that such functions are uniformly dense in the continuous functions $\Lambda \to \mathbb{R}$. If $w : \Lambda \to K$ is continuous then one can easily construct a C^θ function $w' : \Lambda \to K$ that is C^θ differentiable with respect to ϕ and satisfies $|w - w'| \le 1/2$. Then w' is homotopic to w and so defines the same element of $H^1(\Lambda, \mathbb{Z})$. ❑

Suppose w, w' are Hölder continuous functions and, moreover, that w is homotopic to w'. Then there exists a continuous $U : \Lambda \to \mathbb{R}$ such that $w/w' = e^{2\pi i U}$ and one can easily check (see the remarks in §1 on Hölder function spaces), that U is, in fact, Hölder continuous. Similarly, when w, w' are C^1 or C^∞, we can conclude that U is too. Hence

Corollary 2.2
Let $\theta \in (0,1)$, $\theta = 1$ or $\theta = \infty$. Let $C^\theta_\phi(\Lambda, K)$ denote the space of circle-valued C^θ functions that are C^θ differentiable with respect to ϕ. Let $B^\theta_\phi(\Lambda, K)$ denote the subspace of $C^\theta_\phi(\Lambda, K)$ of null-homotopic functions. Then $C^\theta_\phi(\Lambda, K)/B^\theta_\phi(\Lambda, K)$ is isomorphic to $H^1(\Lambda, \mathbb{Z})$.

Dynamic cohomology. Let G be an abelian group. A *G-valued ϕ-cocycle* is a continuous map $\mathcal{F} : \Lambda \times \mathbb{R} \to G$ such that

$$\mathcal{F}(x, t+s) = \mathcal{F}(\phi^t x, s) + \mathcal{F}(x, t).$$

We shall be interested in $\mathbb{R}$- and K-valued ϕ-cocycles of the form

$$\mathcal{F}(x,t) = \int_0^t f(\phi^u x)\, du, \quad \mathcal{F}(x,t) = e^{2\pi i \int_0^t f(\phi^u x)\, du} \qquad (2)$$

for some $f : \Lambda \to \mathbb{R}$, respectively. Fix $\theta \in (0,1)$, $\theta = 1$ or $\theta = \infty$. If $f \in C^\theta(\Lambda, \mathbb{R})$ then we call $\mathcal{F}$ a C^θ ϕ-cocycle. Often, we shall abuse this terminology and simply call f a C^θ ϕ-cocycle.

A function of the form $U\phi^t - U$ for measurable, Hölder, C^1 or C^∞ $U : \Lambda \to \mathbb{R}$ is called a *measurable, Hölder,* C^1 or C^∞ *ϕ-coboundary.* Two real-valued ϕ-cocycles are said to be *ϕ-cohomologous* if their difference is a ϕ-coboundary. (We use the term ϕ-cohomologous so as to avoid confusion when talking about the cohomology of Λ.) The following fundamental result allows us to always assume that the smoothness of the ϕ-coboundary is the same as that of f. The first such results occurred in the work of Livšic [Li1] in the context of Anosov flows preserving smooth measures. These have been generalised by many authors [LMM, PP2, NT, Ll2, for example] and the version we state here is adapted from the corresponding result for hyperbolic diffeomorphisms in [PP2]; a proof can be found in [Wa].[1]

Theorem 2.3 (Livšic's regularity theorem for $\mathbb{R}$ ϕ-cocycles)
Let ϕ be a hyperbolic flow on a basic set Λ equipped with a Hölder equilibrium state m. Let $\theta \in (0,1)$, $\theta = 1$ or $\theta = \infty$. Let $f \in C^\theta(\Lambda, \mathbb{R})$. Suppose

$$\int_0^t f(\phi^u x)\, du = U(\phi^t x) - U(x) \text{ a.e. } [m]$$

for a measurable $U : \Lambda \to \mathbb{R}$. Then there exists $U' \in C^\theta(\Lambda, \mathbb{R})$ such that $U = U'$ a.e. and

$$\int_0^t f(\phi^u x)\, du = U'(\phi^t x) - U'(x) \text{ everywhere.}$$

Moreover, U' is C^θ differentiable with respect to ϕ and $(U')'_\phi = f$.

Remarks.

1. The Hölder case is proved in [Wa]. The C^1 case can be deduced from the Hölder case as in [Li1, Theorem 2]. The C^∞ case is proved in [LMM].

2. Suppose the flow is C^r. If f is C^r then U is $C^{r-1+\varepsilon}$ for any $\varepsilon \in (0,1)$ [Ll1].[2]

For K-valued ϕ-cocycles, the situation is entirely analogous. Two functions $f, f' : \Lambda \to \mathbb{R}$ are said to generate *ϕ-cohomologous ϕ-cocycles* if there exists $w : \Lambda \to K$ such that

$$\frac{w(\phi^t x)}{w(x)} = e^{2\pi i \int_0^t f(\phi^u x) - f'(\phi^u x)\, du}.$$

Again we have a Livšic regularity theorem:

[1] Theorem 5.1 in Chapter 1.

[2] See the remarks in §6 of Chapter 1.

Theorem 2.4 (Livšic's regularity theorem for K ϕ-cocycles) *Let ϕ be a hyperbolic flow on a basic set Λ equipped with a Hölder equilibrium state m. Let $\theta \in (0,1)$, $\theta = 1$ or $\theta = \infty$. Let $f \in C^\theta(\Lambda, \mathbb{R})$. Suppose*

$$e^{2\pi i \int_0^t f(\phi^u x)\, du} = \frac{w(\phi^t x)}{w(x)} \text{ a.e. } [m]$$

for a measurable $w : \Lambda \to K$. Then there exists $w' \in C^\theta(\Lambda, K)$ such that $w = w'$ a.e. and

$$e^{2\pi i \int_0^t f(\phi^u x)\, du} = \frac{w'(\phi^t x)}{w'(x)} \text{ everywhere.}$$

Moreover, w' is C^θ differentiable with respect to ϕ and

$$\frac{1}{2\pi i} \frac{(w')'_\phi}{w'} = f.$$

Remark. The Hölder case is proved in [Wa].[3] The C^1 case follows from an easy modification to the proof of Theorem 2 in [Li1]. See the remarks in [Wa, §6] for higher regularity results.

Let

$$\mathcal{B}_\theta = \left\{ U'_\phi \;\middle|\; \begin{array}{l} U : \Lambda \to \mathbb{R} \in C^\theta(\Lambda, \mathbb{R}) \text{ and is } C^\theta \\ \text{differentiable with respect to } \phi \end{array} \right\}.$$

Then, for example, $f, f' \in C^\theta(\Lambda, \mathbb{R})$ generate ϕ-cohomologous ϕ-cocycles if and only if $f - f' \in \mathcal{B}_\theta$.

Both the real and circle valued ϕ-cocycles modulo ϕ-coboundaries form an abelian group. We denote these by $\mathcal{H}^1_\theta(\phi, \mathbb{R})$ and $\mathcal{H}^1_\theta(\phi, K)$, respectively. See Corollary 2.7 below for one relation between the real and circle valued groups.

We shall also be interested in C^θ ϕ-cocycles modulo essential ϕ-coboundaries. An *essential ϕ-coboundary* is a ϕ-cocycle of the form

$$\int_0^t f(\phi^u x)\, du = U(\phi^t x) - U(x) + ct,$$

for some constant $c \in \mathbb{R}$. An easy modification to Theorem 2.3 allows us to conclude that U is C^θ if f is C^θ. Moreover, if this is the case, then U'_ϕ exists

[3] Theorem 4.3 in Chapter 1.

and $f = U'_\phi + c$. Let

$$\mathcal{B}_\theta^{\mathrm{ess}} = \left\{ U'_\phi + c \;\middle|\; \begin{array}{l} U \in C^\theta(\Lambda, \mathbb{R}) \text{ and is } C^\theta \text{ differen-} \\ \text{tiable with respect to } \phi \text{ and } c \in \mathbb{R} \end{array} \right\}.$$

We denote the group of real-valued C^θ ϕ-cocycles modulo $\mathcal{B}_\theta^{\mathrm{ess}}$ by $\mathcal{H}_\theta^{1,\mathrm{ess}}(\phi, \mathbb{R})$.

Finally, we shall need the following criteria for ϕ-cohomological triviality.

Theorem 2.5 (Livšic's periodic data criteria, [Li1, Li2]) *Let ϕ be a hyperbolic flow on a basic set Λ. Let $\theta \in (0,1)$, $\theta = 1$ or $\theta = \infty$. Let $f \in C^\theta(\Lambda, \mathbb{R})$.*

1. *Suppose that whenever $\phi^T x = x$, we have $\int_0^T f(\phi^u x)\, du = 0$. Then $f \in \mathcal{B}_\theta$.*

2. *Suppose that whenever $\phi^T x = x$, we have $\int_0^T f(\phi^u x)\, du \in \mathbb{Z}$. Then there exists $w \in C^\theta(\Lambda, \mathbb{R})$ such that*

$$e^{2\pi i \int_0^t f(\phi^u x)\, du} = \frac{w(\phi^t x)}{w(x)}.$$

   Moreover, w'_ϕ exists and $w'_\phi = 2\pi i w f$.

Cohomological inclusions. For the remainder of this section, we assume that $H^1(\Lambda, \mathbb{Z})$ has finite rank. Recall that the rank of a free abelian group is the maximum number of integrally independent elements. If $H^1(\Lambda, \mathbb{Z})$ is trivial then all of the results below go through with only minor changes, so we shall assume that $H^1(\Lambda, \mathbb{Z}) \neq 0$.

Let $w : \Lambda \to K$ be continuous. Then for fixed t, $w\phi^t$ is homotopic to w. Hence, there exists a (unique) continuous function $\mathcal{G} : \Lambda \times \mathbb{R} \to \mathbb{R}$ such that

$$\frac{w(\phi^t x)}{w(x)} = e^{2\pi i \mathcal{G}(x,t)}. \tag{3}$$

Then $\mathcal{G}$ is easily seen to be a real-valued cocycle. Suppose further that w is differentiable with respect to ϕ. Then locally we can write $w(\phi^t x) = e^{2\pi i W(x,t)}$ for small t and x in a small neighbourhood. Moreover, W is continuous and differentiable in the t coordinate. Substituting this into (3) we see that

$$\frac{W(x,t)-W(x,0)}{t}=\frac{\mathcal{G}(x,t)}{t}.$$

Hence, letting $t\to 0$, we see that

$$G(x):=\lim_{t\to 0}\frac{\mathcal{G}(x,t)}{t} \tag{4}$$

exists and is continuous. Moreover, one can easily check that

$$\frac{w(\phi^t x)}{w(x)}=e^{2\pi i\int_0^t G(\phi^u x)\,du}. \tag{5}$$

Then $w'_\phi=2\pi i w G$. Hence if w is C^θ and C^θ differentiable with respect to ϕ, then G is C^θ. Conversely, if G is C^θ, then w is C^θ and C^θ differentiable with respect to ϕ; this follows from Theorem 2.5.

Let $[w]\in H^1(\Lambda,\mathbb{Z})$ and choose a C^θ representative $w\in[w]$ that is C^θ differentiable with respect to ϕ. Then by the discussion above, there exists a unique $G\in C^\theta(\Lambda,\mathbb{R})$ such that

$$\frac{w(\phi^t x)}{w(x)}=e^{2\pi i\int_0^t G(\phi^u x)\,du}.$$

Thus to every C^θ circle-valued function (that is also C^θ differentiable with respect to ϕ), we can associate a real-valued C^θ cocycle.

Suppose instead we had chosen another representative $w'\in[w]$ with $w'\in C^\theta(\Lambda,K)$ and C^θ differentiable with respect to ϕ. Then there exists a continuous map $U:\Lambda\to\mathbb{R}$ such that $w'=we^{2\pi iU}$. Let G' be the C^θ function appearing in (5) for w'. Then

$$e^{2\pi i\int_0^t (G'-G)(\phi^u x)\,du}=\frac{e^{2\pi i U(\phi^t x)}}{e^{2\pi i U(x)}}.$$

Hence, by Theorem 2.5, U'_ϕ exists and $U'_\phi\in\mathcal{B}_\theta$. Then it is easy to see that $G'-G=U'_\phi$. Thus to each element of $H^1(\Lambda,\mathbb{Z})$ we can associate a real-valued dynamic cocycle modulo a ϕ-coboundary. Hence we have a well-defined map

$$\eta_\theta: H^1(\Lambda,\mathbb{Z})\to\mathcal{H}^1_\theta(\phi,\mathbb{R}): w\mapsto G.$$

Proposition 2.6

The map η_θ is a group homomorphism. Indeed, η_θ is an isomorphism onto its image.

Proof. Let $w, w' \in C^\theta(\Lambda, K)$ be C^θ differentiable with respect to ϕ. Let $G, G' \in C^\theta(\Lambda, \mathbb{R})$ be the functions associated to w, w' by (5). Then $ww' \in C^\theta(\Lambda, K)$ is C^θ differentiable with respect to ϕ and, moreover,

$$\begin{aligned}
\frac{(ww')(\phi^t x)}{(ww')(x)} &= \frac{w(\phi^t x)}{w(x)} \frac{w'(\phi^t x)}{w'(x)} \\
&= e^{2\pi i \int_0^t G(\phi^u x)\, du} e^{2\pi i \int_0^t G'(\phi^u x)\, du} \\
&= e^{2\pi i \int_0^t (G+G')(\phi^u x)\, du}.
\end{aligned}$$

Hence $\eta_\theta(ww') = \eta_\theta(w) + \eta_\theta(w')$.

To show that η_θ is injective, suppose $\eta_\theta(w) = 0 \bmod \mathcal{B}_\theta$. Then

$$\frac{w(\phi^t x)}{w(x)} = e^{2\pi i \int_0^t U'_\phi(\phi^u x)\, du} = \frac{e^{2\pi i U(\phi^t x)}}{e^{2\pi i U(x)}}.$$

Hence $we^{-2\pi i U}$ is ϕ^t-invariant, thus constant by the ergodicity of ϕ. So $w = \mu e^{2\pi i U}$, for some $\mu \in \mathbb{C}$ of unit modulus. Thus w is null-homotopic. ❑

Corollary 2.7

The following sequence is exact:

$$0 \to H^1(\Lambda, \mathbb{Z}) \xrightarrow{\eta_\theta} \mathcal{H}^1_\theta(\phi, \mathbb{R}) \xrightarrow{\pi} \mathcal{H}^1_\theta(\phi, K) \to 0.$$

Proof. We use the realisation of $H^1(\Lambda, \mathbb{Z})$ given by Corollary 2.2.

We first define the map π. Any $f \in C^\theta(\Lambda, \mathbb{R})$ determines both a real ϕ-cocycle and a circle-valued ϕ-cocycle by (2). Let π be the map that takes the former to the latter. It is easy to see that π is a homomorphism. If f is a real-valued ϕ-coboundary, $f = U'_\phi$ say, then

$$\frac{\left(e^{2\pi i U}\right)(\phi^t x)}{e^{2\pi i U(x)}} = e^{2\pi i \int_0^t f(\phi^u x)\, du},$$

whence f is a K-valued ϕ-coboundary. It follows that the induced map on the dynamic cohomology group is well-defined.

By Proposition 2.6, η_θ is injective and it is clear from the definitions that π is onto. It remains to check that $\operatorname{im}\eta_\theta = \ker\pi$. Let $w \in [w] \in H^1(\Lambda, \mathbb{Z})$. Then

$$\frac{w(\phi^t x)}{w(x)} = e^{2\pi i \int_0^t \eta_\theta(w)(\phi^u x)\, du}$$

and it follows that $\eta_\theta(w) \in \ker\pi$. Similarly, $\ker\pi \subset \operatorname{im}\eta_\theta$. ❏

There is also a well-defined map

$$\eta_\theta^{\mathrm{ess}} : H^1(\Lambda, \mathbb{Z}) \to \mathcal{H}_\theta^{1,\mathrm{ess}}(\phi, \mathbb{R}).$$

Proposition 2.8

If ϕ^t is mixing then $\eta_\theta^{\mathrm{ess}}$ is an isomorphism onto its image.

Proof. The same argument as in the proof of Proposition 2.6 shows that $\eta_\theta^{\mathrm{ess}}$ is an onto homomorphism. It remains to check that $\eta_\theta^{\mathrm{ess}}$ is injective. If $\eta_\theta^{\mathrm{ess}}(w) = 0 \bmod \mathcal{B}_\theta^{\mathrm{ess}}$ then

$$\begin{aligned} \frac{w(\phi^t x)}{w(x)} &= e^{2\pi i c t} e^{2\pi i \int_0^t U'_\phi(\phi^u x)\, du} \\ &= e^{2\pi i c t} e^{2\pi i (U(\phi^t x) - U(x))}, \end{aligned}$$

for some $c \in \mathbb{R}$ and $U : \Lambda \to \mathbb{R}$. Hence $we^{-2\pi i U}$ has $2\pi i c$ as an eigenfrequency. As ϕ^t is mixing, $c = 0$ and $we^{2\pi i U}$ is a constant. It follows that w is null-homotopic, whence $[w] = 0 \in H^1(\Lambda, \mathbb{Z})$. ❏

Remark. We can also define K-valued essential ϕ-coboundaries to be functions of the form

$$\frac{c\, w(\phi^t x)}{w(x)}$$

for some $c \in K$. These form a subgroup and we denote the group of ϕ-cocycles modulo essential ϕ-coboundaries by $\mathcal{H}_\theta^{1,\mathrm{ess}}(\phi, K)$. When ϕ is mixing there is an exact sequence

$$0 \to H^1(\Lambda, \mathbb{Z}) \xrightarrow{\eta_\theta^{\mathrm{ess}}} \mathcal{H}_\theta^{1,\mathrm{ess}}(\phi, \mathbb{R}) \xrightarrow{\pi} \mathcal{H}_\theta^{1,\mathrm{ess}}(\phi, K) \to 0.$$

For each $[w] \in H^1(\Lambda, \mathbb{Z})$, fix a C^θ representative $w \in [w]$ that is C^θ differentiable with respect to ϕ. Let G_w be the real-valued function associated to w by (5). For $R = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$, let

$$\mathcal{Z}_R = \left\{ \sum \lambda_i G_{w_i} \mid \lambda_i \in R,\, [w_i] \in H^1(\Lambda, \mathbb{Z}) \right\};$$

i.e. $\mathcal{Z}_R$ is the R-linear span of the G_w. Thus, identifying functions in $(\mathcal{Z}_\mathbb{Z} + \mathcal{B}_\theta)/\mathcal{B}_\theta$ with the real-valued ϕ-cocycles they generate (modulo ϕ-coboundaries), we see that $\eta_\theta(H^1(\Lambda, \mathbb{Z})) = (\mathcal{Z}_\mathbb{Z} + \mathcal{B}_\theta)/\mathcal{B}_\theta$. Note also that $\mathcal{Z}_R$ depends on θ, but we suppress this for notational clarity. We urge the reader to take careful note of the suffixes on the $\mathcal{Z}$'s.

We shall see that 'bad' dynamical behaviour corresponds to ϕ-cocycles in $(\mathcal{Z}_\mathbb{R} + \mathcal{B}_\theta)/\mathcal{B}_\theta$ and $(\mathcal{Z}_\mathbb{R} + \mathcal{B}_\theta/\mathcal{B}_\theta^{\text{ess}})$. We have just seen that $H^1(\Lambda, \mathbb{Z})$ corresponds to a subgroup of these 'bad' ϕ-cocycles. The following results show that the 'bad' ϕ-cocycles correspond to a very small subgroup of $\mathcal{H}^1_\theta(\phi, \mathbb{R})$ (and the other groups), both algebraically and topologically.

Proposition 2.9

1. *$\mathcal{B}_\theta$ is closed in $C^\theta(\Lambda, \mathbb{R})$.*
2. *$\mathcal{Z}_\mathbb{R} + \mathcal{B}_\theta$ is closed in $C^\theta(\Lambda, \mathbb{R})$.*
3. *$\mathcal{Z}_\mathbb{R} + \mathcal{B}_\theta$ has infinite codimension in $C^\theta(\Lambda, \mathbb{R})$.*

Proof. 1. Let $H_n = (U_n)'_\phi \in \mathcal{B}_\theta$ be such that $H_n \to H$ in the C^θ topology. Suppose $\phi^T x = x$; then clearly $\int_0^T H_n(\phi^u x)\, du = 0$. By Theorem 2.5, to show that $H \in \mathcal{B}_\theta$, it suffices to prove that $\int_0^T H(\phi^u x)\, du = 0$. This follows from:

$$\begin{aligned} \left| \int_0^T H(\phi^u x)\, du - \int_0^T H_n(\phi^u x)\, du \right| &\leq \int_0^T |H(\phi^u x) - H_n(\phi^u x)|\, du \\ &\leq T|H - H_n|_\infty. \end{aligned}$$

As the uniform norm is dominated by the Hölder norm, as well as the C^r norm for each $r \in \mathbb{N}$, we see that $|H - H_n|_\infty \to 0$ as $n \to \infty$. Hence $\int_0^T H(\phi^u x)\, du = 0$.

2. It is sufficient to show that $(\mathcal{Z}_\mathbb{R} + \mathcal{B}_\theta)/\mathcal{B}_\theta$ is finite dimensional, hence closed. Let β denote the rank of $H^1(\Lambda, \mathbb{Z})$ and let $G_1 + \mathcal{B}_\theta, \ldots, G_\beta + \mathcal{B}_\theta$ be a maximal set of integrally independent elements in $(\mathcal{Z}_\mathbb{Z} + \mathcal{B}_\theta)/\mathcal{B}_\theta = \eta_\theta(H^1(\Lambda, \mathbb{Z}))$. It is easy to see that any $G + \mathcal{B}_\theta \in (\mathcal{Z}_\mathbb{Z} + \mathcal{B}_\theta)/\mathcal{B}_\theta$ is a rational linear combination of the $G_i + \mathcal{B}_\theta$ and it follows that any $G + \mathcal{B}_\theta \in (\mathcal{Z}_\mathbb{R} + \mathcal{B}_\theta)/\mathcal{B}_\theta$ is a real linear combination of the $G_i + \mathcal{B}_\theta$. Hence $\dim(\mathcal{Z}_\mathbb{R} + \mathcal{B}_\theta)/\mathcal{B}_\theta \leq$

β. (The inequality arises from the fact that the $G_i + \mathcal{B}_\theta$, although integrally independent, may be linearly dependent.)

3. Observe from the proof of 2. that $(\mathcal{Z}_{\mathbb{R}} + \mathcal{B}_\theta)/\mathcal{B}_\theta$ is finite dimensional. Therefore, to show $\mathcal{Z}_{\mathbb{R}} + \mathcal{B}_\theta$ has infinite codimension in $C^\theta(\Lambda, \mathbb{R})$, it is sufficient to prove that $\mathcal{B}_\theta$ has infinite codimension in $C^\theta(\Lambda, \mathbb{R})$.

Any function $U'_\phi \in \mathcal{B}_\theta$ has zero integral with respect to m. This follows from the fact that

$$\int U'_\phi \, dm = \lim_{t\to\infty} \frac{1}{t} \int U\phi^t - U \, dm = 0,$$

by dominated convergence and the ϕ^t-invariance of m. Hence any invariant measure on Λ determines a linear functional on $C^\theta(\Lambda, \mathbb{R})/\mathcal{B}_\theta$. If the dimension of $C^\theta(\Lambda, \mathbb{R})/\mathcal{B}_\theta$ was finite, then its dual space would also be finite dimensional. However, as there are infinitely many periodic orbits, the invariant measures supported on periodic orbits generate an infinite dimensional space of linear functionals. ❏

Let us consider $\mathcal{Z}_{\mathbb{Z}} + \mathcal{B}_\theta^{\mathrm{ess}}$. First observe that it is easy to modify Theorem 2.5 to give a criterion for essential ϕ-cohomological triviality in terms of periodic orbits. This allows us to conclude, in exactly the same way as above, that $\mathcal{B}_\theta^{\mathrm{ess}}$ is closed in $C^\theta(\Lambda, \mathbb{R})$. Also, observe that $\mathcal{B}_\theta$ has codimension 1 in $\mathcal{B}_\theta^{\mathrm{ess}}$. Hence

Proposition 2.10

The subspace $\mathcal{B}_\theta^{\mathrm{ess}}$ is closed in $C^\theta(\Lambda, \mathbb{R})$. Furthermore, $\mathcal{Z}_{\mathbb{R}} + \mathcal{B}_\theta^{\mathrm{ess}}$ is closed in $C^\theta(\Lambda, \mathbb{R})$ and has infinite codimension.

We shall need some results on functions lying in 'general position'. The method of proof is surely well-known, but see in particular [PP3] where similar results have been used in similar circumstances.

Proposition 2.11

Let $n \geq 1$ be arbitrary. Let $(f_1, \ldots, f_n) \in C^\theta(\Lambda, \mathbb{R})^n$. Then there exists $(f'_1, \ldots, f'_n) \in C^\theta(\Lambda, \mathbb{R})^n$ arbitrarily close to $(f_1, \ldots, f_n)$ such that $(f'_1, \ldots, f'_n)$ is linearly independent mod $\mathcal{Z}_{\mathbb{R}} + \mathcal{B}_\theta$.

Proof. The proof is similar to, but easier than, the proof of the Proposition 5.14 below. ❑

Remark. The proposition above remains true with $\mathcal{B}_\theta$ replaced by $\mathcal{B}_\theta^{\text{ess}}$, using the observation that the former has codimension 1 in the latter.

Proposition 2.12
Let $n \geq 1$ be arbitrary. Then the set of n-tuples $(f_1, \ldots, f_n) \in C^\theta(\Lambda, \mathbb{R})^n$ that are linearly independent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}_\theta$ forms an open set in $C^\theta(\Lambda, \mathbb{R})^n$. The same remains true of n-tuples in $C^\theta(\Lambda, \mathbb{R})^n$ that are linearly dependent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}_\theta^{\text{ess}}$.

Proof. We show that the linearly dependent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}_\theta$ n-tuples form a closed set. Suppose $F_m = (f_{1,m}, \ldots, f_{n,m}) \in C^\theta(\Lambda, \mathbb{R})^n$ are linearly dependent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}_\theta$. Suppose $F_m \to F \in C^\theta(\Lambda, \mathbb{R})^n$. Then there exist real numbers $\lambda_{i,j}$ (not all zero) such that

$$\lambda_{1,m} f_{1,m} + \cdots + \lambda_{n,m} f_{n,m} = 0 \bmod \mathcal{Z}_\mathbb{R} + \mathcal{B}_\theta. \tag{6}$$

By re-ordering the indices, if necessary, we can assume that $\lambda_{1,m} \neq 0$ and $|\lambda_{1,m}| \geq |\lambda_{i,m}|$, $2 \leq i \leq n$, for infinitely many m. Dividing (6) by $\lambda_{1,m}$ for such m, we obtain:

$$f_{1,m} + \lambda'_{2,m} f_{2,m} + \cdots + \lambda'_{n,m} f_{n,m} = 0 \bmod \mathcal{Z}_\mathbb{R} + \mathcal{B}_\theta,$$

and $|\lambda'_{i,m}| \leq 1$ for $2 \leq i \leq n$. By taking a subsequence, we can assume that $\lambda'_{i,m} \to \lambda'_i$ as $m \to \infty$, for each j. Hence (writing $F = (f_1, \ldots, f_n)$),

$$f_1 + \lambda'_2 f_2 + \cdots + \lambda'_n f_n = 0 \bmod \mathcal{Z}_\mathbb{R} + \mathcal{B}_\theta,$$

as $\mathcal{Z}_\mathbb{R} + \mathcal{B}_\theta$ is closed. Thus $f_1, \ldots, f_n$ are linearly dependent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}_\theta$. ❑

One can say more about the relationship between $H^1(\Lambda, \mathbb{Z})$ and the various dynamical cohomology groups, but we postpone this to the following sections where their context makes them clearer.

§3 Toral extensions of hyperbolic flows

Let $\theta \in (0,1)$, $\theta = 1$ or $\theta = \infty$. Recall that for an n-tuple $F = (f_1, \ldots, f_n) \in C^\theta(\Lambda, \mathbb{R})^n$ we have a toral extension ϕ_F^t of ϕ^t given by

$$\phi_F^t(x, y_1, \ldots, y_n) = \left(\phi^t x, e^{2\pi i \int_0^t f_1(\phi^u x)\, du} y_1, \ldots, e^{2\pi i \int_0^t f_n(\phi^u x)\, du} y_n\right).$$

Observe that each f_i determines an equivalence class $[f_i] \in \mathcal{H}_\theta^1(\phi, K)$. If $f_i' \in [f_i]$, with $f_i - f_i' = w_i{}'_\phi / w_i$, then the two extensions $\phi_F^t, \phi_{F'}^t$ are metrically and topologically conjugate via the map

$$(x, y_1, \ldots, y_n) \mapsto (x, w_1(x) y_1, \ldots, w_n(x) y_n).$$

Thus, the ergodic properties of ϕ_F^t depend only on the ϕ-cohomology class of F.

We shall need the following well-known criterion (valid for arbitrary flows) for ϕ_F^t to be ergodic, given that ϕ^t is. See [KN], for example, for a proof.

Proposition 3.1

Suppose ϕ^t is ergodic with respect to m. Then the following are equivalent:

1. *ϕ_F^t is not ergodic with respect to $m \times \nu$;*
2. *there exist integers $(l_1, \ldots, l_n) \in \mathbb{Z}^n \setminus \{(0, \ldots, 0)\}$ and a measurable function $w : \Lambda \to K$ such that, for each t,*

$$\frac{w(\phi^t x)}{w(x)} = e^{2\pi i \int_0^t \sum_{i=1}^n l_i f_i(\phi^u x)\, du} \text{ a.e.}$$

Remark. *A priori* the set of measure zero in 2. above may depend on t. However, a standard measure-theoretic argument allows us to change w on a set of measure zero so that the modified equation holds a.e., independent of t.

By Theorem 2.4, the above proposition becomes:

Proposition 3.2

Let m be a Hölder equilibrium state. Then the following are equivalent:

1. *ϕ_F^t is not ergodic with respect to $m \times \nu$;*

2. *there exist integers $(l_1, \ldots, l_n) \in \mathbb{Z}^n \setminus \{(0, \ldots, 0)\}$ and $w \in C^\theta(\Lambda, K)$ such that*
$$\frac{w(\phi^t x)}{w(x)} = e^{2\pi i \int_0^t \sum_{i=1}^n l_i f_i(\phi^u x)\, du} \tag{7}$$
everywhere.

Observe that equation (7) says that an integral combination of the f_i lies in $\eta_\theta(H^1(\Lambda, \mathbb{Z}))$.

There is an analogue of Proposition 3.1 giving a criterion for ϕ_F^t to be mixing. Again, combining this with Theorem 2.4, we have

Proposition 3.3
Suppose ϕ^t is mixing with respect to the Hölder equilibrium state m. Then the following are equivalent:

1. *ϕ_F^t is not mixing with respect to $m \times \nu$;*

2. *there exist integers $(l_1, \ldots, l_n) \in \mathbb{Z}^n \setminus \{(0, \ldots, 0)\}$, a constant $0 \neq c \in \mathbb{R}$ and $w \in C^\theta(\Lambda, K)$ such that*
$$\frac{w(\phi^t x)}{w(x)} = e^{2\pi i \int_0^t \left(c + \sum_{i=1}^n l_i f_i(\phi^u x)\right) du} \tag{8}$$
everywhere.

We first show that the non-ergodic extensions correspond to the ϕ-cocycles in $\mathcal{Z}_\mathbb{Q} + \mathcal{B}_\theta$. In what follows, we urge the reader to take careful note of the suffixes on the $\mathcal{Z}$'s.

Proposition 3.4
The following are equivalent:

1. *ϕ_F^t is not ergodic;*

2. *$f_1, \ldots, f_n$ are rationally dependent mod $\mathcal{Z}_\mathbb{Q} + \mathcal{B}_\theta$.*

In particular, if $f_1, \ldots, f_n$ are linearly independent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}_\theta$ then ϕ_F^t is ergodic.

Proof. Suppose ϕ_F^t is not ergodic. Then, for suitable integers $(l_1, \ldots, l_n) \in \mathbb{Z}^n \setminus \{(0, \ldots, 0)\}$ and some $w \in C^\theta(\Lambda, K)$, (7) holds. Applying η_θ to (7), we obtain

$$l_1 f_1 + \cdots + l_n f_n = \eta_\theta(w) + U'_\phi, \tag{9}$$

for some $U'_\phi \in \mathcal{B}_\theta$. Hence, $f_1, \ldots, f_n$ are integrally dependent mod $\mathcal{Z}_{\mathbb{Z}} + \mathcal{B}_\theta$, hence rationally dependent mod $\mathcal{Z}_{\mathbb{Q}} + \mathcal{B}_\theta$, hence linearly dependent mod $\mathcal{Z}_{\mathbb{R}} + \mathcal{B}_\theta$.

Conversely, if $f_1, \ldots, f_n$ are rationally dependent mod $\mathcal{Z}_{\mathbb{Q}} + \mathcal{B}_\theta$ then, by multiplying by a common denominator, they are integrally dependent mod $\mathcal{Z}_{\mathbb{Z}} + \mathcal{B}_\theta$. Hence (9) holds for suitable integers $l_1, \ldots, l_n, w : \Lambda \to K$ and $U'_\phi \in \mathcal{B}_\theta$. Moreover, not all the l_i are zero. It follows that

$$\frac{(e^{2\pi i U} w)\,(\phi^t x)}{(e^{2\pi i U} w)\,(x)} = e^{2\pi i \int_0^t \sum_{i=1}^n l_i f_i(\phi^u x)\, du}.$$

By Proposition 3.2, ϕ_F^t is not ergodic. ❑

Remark. Observe that if the $f_1, \ldots, f_n$ are rationally dependent mod $\mathcal{Z}_{\mathbb{R}} + \mathcal{B}_\theta$ (as opposed to $\mathcal{Z}_{\mathbb{Q}} + \mathcal{B}_\theta$), then we cannot conclude that the associated extension ϕ_F^t is not ergodic.

By applying η_θ^{ess} to (8), we have the corresponding result for mixing extensions:

Proposition 3.5

The following are equivalent:

1. ϕ_F^t *is not mixing;*

2. $f_1, \ldots, f_n$ *are rationally dependent mod* $\mathcal{Z}_{\mathbb{Q}} + \mathcal{B}_\theta^{\text{ess}}$.

In particular, if $f_1, \ldots, f_n$ *are linearly independent mod* $\mathcal{Z}_{\mathbb{R}} + \mathcal{B}_\theta^{\text{ess}}$ *then* ϕ_F^t *is mixing.*

Again, rational dependence mod $\mathcal{Z}_{\mathbb{R}} + \mathcal{B}_\theta^{\text{ess}}$ need not imply that ϕ_F^t is not mixing. The next result shows that linear dependence mod $\mathcal{Z}_{\mathbb{R}} + \mathcal{B}_\theta$ corresponds to 'unstable' ergodic behaviour: if the associated extension is ergodic (or mixing), then it can be perturbed by an arbitrarily small amount to a non-ergodic (or non-mixing) system.

Proposition 3.6

Let $f_1, \ldots, f_n$ be linearly dependent mod $\mathcal{Z}_{\mathbb{R}}+\mathcal{B}_\theta$. Then there exist functions $f'_1, \ldots, f'_n$, arbitrarily close to $f_1, \ldots, f_n$ such that $f'_1, \ldots, f'_n$ are rationally dependent mod $\mathcal{Z}_{\mathbb{Q}}+\mathcal{B}_\theta$. In particular, $\phi^t_{F'}$ is not ergodic. The same is true with $\mathcal{B}_\theta^{\mathrm{ess}}$ and 'mixing' replacing $\mathcal{B}_\theta$ and 'ergodic,' respectively.

Proof. Suppose $f_1, \ldots, f_n$ are linearly dependent mod $\mathcal{Z}_{\mathbb{R}}+\mathcal{B}_\theta$. Choose a maximal set $G_1, \ldots, G_\beta \in \mathcal{Z}_{\mathbb{Z}}$ that are integrally independent mod $\mathcal{B}_\theta$. Then, by the proof of Proposition 2.9.2, there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ (not all zero), $\mu_1, \ldots, \mu_\beta \in \mathbb{R}$ and $U'_\phi \in \mathcal{B}_\theta$ such that

$$\lambda_1 f_1 + \cdots + \lambda_n f_n = \mu_1 G_1 + \cdots + \mu_\beta G_\beta + U'_\phi. \tag{10}$$

Let $\varepsilon > 0$ be arbitrary. Choose non-zero rationals p_i/q_i, r_j/s_j such that

$$\begin{aligned} \lambda_i &= \frac{p_i}{q_i} + \varepsilon_i, \quad |\varepsilon_i| < \varepsilon, \quad 1 \le i \le n, \\ \mu_j &= \frac{r_j}{s_j} + \delta_j, \quad |\delta_j| < \varepsilon, \quad 1 \le j \le \beta. \end{aligned}$$

Then (10) takes the form

$$\begin{aligned} &\tfrac{p_1}{q_1} f_1 \left(1 + \tfrac{q_1}{p_1}\varepsilon_1\right) + \cdots + \tfrac{p_n}{q_n} f_n \left(1 + \tfrac{q_n}{p_n}\varepsilon_n\right) - (\delta_1 G_1 + \cdots + \delta_\beta G_\beta) \\ &\qquad = \tfrac{r_1}{s_1} G_1 + \cdots + \tfrac{r_\beta}{s_\beta} G_\beta + U'_\phi. \end{aligned} \tag{11}$$

Let

$$\begin{aligned} f'_1 &= f_1 \left(1 + \frac{q_1}{p_1}\varepsilon_1\right) - \frac{q_1}{p_1}(\delta_1 G_1 + \cdots + \delta_\beta G_\beta), \\ f'_i &= f_i \left(1 + \frac{q_i}{p_i}\varepsilon_i\right), \ 2 \le i \le n. \end{aligned}$$

Then the f'_i can be made arbitrarily close to f_i by choosing ε (hence ε_i) arbitrarily small (f'_1 is close to f_1 as the G_j are continuous and therefore bounded and the δ_j can be made arbitrarily small). By (11), we see that the $f'_1, \ldots, f'_n$ are rationally dependent mod $\mathcal{Z}_{\mathbb{Q}}+\mathcal{B}_\theta$.

The $\mathcal{B}_\theta^{\mathrm{ess}}$ case is similar to the above, with the addition of a constant term to (10) and (11). ❑

Proof of Theorems A and B. The set of n-tuples in $C^\theta(\Lambda, \mathbb{R})^n$ that are linearly independent mod $\mathcal{Z}_{\mathbb{R}}+\mathcal{B}_\theta$ forms an open (by Proposition 2.12), dense (by Proposition 2.11) set. By Proposition 3.4, all such n-tuples correspond to ergodic extensions. This proves Theorem A.

The mixing case (Theorem B) is exactly the same, with $\mathcal{B}_\theta^{\mathrm{ess}}$ replacing $\mathcal{B}_\theta$. ❏

Remarks.

1. Consider the C^r case for $2 \leq r < \infty$. If $F \in C^r(\Lambda, \mathbb{R})^n$ is such that ϕ_F^t is not ergodic (or mixing) then for any $\varepsilon \in (0,1)$ there is an n-tuple $F' \in C^{r-1+\varepsilon}(\Lambda, \mathbb{R})^n$ arbitrarily close to F in the $C^{r-1+\varepsilon}$ topology for which $\phi_{F'}^t$ is ergodic (or mixing).

2. For a circle extension of a hyperbolic flow on a basic sets with $H^1(\Lambda, \mathbb{Z})$ trivial,[4] there are no unstably ergodic or unstably mixing extensions. This is because all of the $\mathcal{Z}$'s drop out of the above discussion and linear dependence mod $\mathcal{B}_\theta$ coincides with rational dependence mod $\mathcal{B}_\theta$.

§4 Velocity changes of hyperbolic flows

Throughout this section, we work primarily with C^1 or C^∞ functions and Anosov flows. For notational simplicity, we usually only consider the C^1 case and we write $\mathcal{B}_{C^1}$ for $\mathcal{B}_1$, etc. Our results continue to hold for Hölder velocity changes of hyperbolic flows, although the flows we obtain are Hölder continuous and defined only on Λ.

Before proving Theorem C, we first discuss the extent to which a velocity change ψ_k^t of ϕ^t depends only on the ϕ-cohomology class of k in $\mathcal{H}_{C^1}^1(\phi, \mathbb{R})$.

Suppose ψ_k^t is obtained from ϕ^t by a velocity change. If there is a C^1 diffeomorphism $h : \Lambda \to \Lambda$ such that $h\phi^t = \psi_k^t h$ and h maps each (oriented) orbit to itself, then we say ψ_k^t and ϕ^t are *flow equivalent*. (Note: this is not the usual definition of flow equivalence given in, say, [KH, p.64].)

The following proposition says that the ϕ-cohomology class of k determines a velocity changed flow up to flow equivalence (c.f. [KH, §2.2]).

[4] We do not know of any examples of basic sets Λ for which $H^1(\Lambda, \mathbb{Z}) = 0$.

Proposition 4.1
Let k, k' be strictly positive C^1 functions. Then (in the notation of §§1,2) the following are equivalent:

1. *ψ_k^t and $\psi_{k'}^t$ are flow equivalent;*

2. *$k = k' \bmod \mathcal{B}_{C^1}$.*

Proof. Suppose 1. holds. Let h be a C^1 conjugacy such that $\psi_k^t h = h\psi_{k'}^t$ and h maps each orbit to itself, preserving its orientation. Then for each $x \in \Lambda$, $h(x) = \psi_k^{U(x)}(x)$ for some $U(x) \geq 0$, thus defining a map $U : \Lambda \to \mathbb{R}$. It is easy to see that U is C^1 (as h is).

Clearly, we have

$$\psi_k^t h(x) = \psi_k^{t+U(x)}(x), \tag{12}$$
$$h\psi_{k'}^t(x) = \psi_k^{U(\psi_{k'}^t x)}\psi_{k'}^t(x). \tag{13}$$

Now $\psi_{k'}^t(x) = \phi^{l'(x,t)}(x) = \psi_k^{k(x,l'(x,t))}(x)$ (using the notation of §1), and combining this with (12) and (13) it follows that

$$t + U(x) = U(\psi_{k'}^t x) + k(x, l'(x,t)). \tag{14}$$

Let $t = k'(x,s)$ so that $s = l'(x,t)$ and $\psi_{k'}^t(x) = \psi_{k'}^{k'(x,s)}(x) = \phi^s(x)$. Then (14) is equivalent to

$$k'(x,s) + U(x) = U(\phi^s x) + k(x,s).$$

Dividing by s and letting $s \to 0$ we see that $k' = k + U'_\phi$. Hence $U'_\phi \in \mathcal{B}_{C^1}$ and $k = k' \bmod \mathcal{B}_{C^1}$.

The converse follows easily by reversing the argument. ❑

The following gives a ϕ-cohomological criterion for a velocity changed flow to be mixing. Again, we urge the reader to carefully note the suffixes on the $\mathcal{Z}$'s.

Proposition 4.2
The flow ψ_k^t is not mixing if and only if, for some $0 \neq c \in \mathbb{R}$, we have $ck = 0 \bmod \mathcal{Z}_{\mathbb{Q}} + \mathcal{B}_{C^1}$. In particular, if $k \neq 0 \bmod \mathcal{Z}_{\mathbb{R}} + \mathcal{B}_{C^1}$ then ψ_k^t is mixing.

Proof. Recall that ψ_k^t is always ergodic with respect to $k\,dm/m(k)$. Hence, ψ_k^t is not weak-mixing (equivalently, not mixing) if and only if there exists a measurable eigenfunction $w : \Lambda \to K$ corresponding to a non-zero eigenfrequency c, *i.e.* for each t,

$$\frac{w(\psi_k^t x)}{w(x)} = e^{2\pi i c t} \text{ a.e.}$$

By the definition of ψ_k^t, this becomes

$$\frac{w(\phi^{l(x,t)} x)}{w(x)} = e^{2\pi i c t} \text{ a.e.}$$

(using the notation of §1). Substituting $s = l(x,t)$ and $t = k(x,s)$, this is equivalent to

$$\frac{w(\phi^s x)}{w(x)} = e^{2\pi i \int_0^s ck(\phi^u x)\,du} \text{ a.e.} \tag{15}$$

By Theorem 2.4, we may assume that w is C^1 and C^1 differentiable with respect to ϕ and that (15) holds everywhere. Applying η_{C^1} to this equation, we see that there exists $U'_\phi \in \mathcal{B}_{C^1}$ such that

$$ck = \eta_{C^1}(w) + U'_\phi. \tag{16}$$

Hence $ck = 0 \bmod \mathcal{Z}_{\mathbb{Z}} + \mathcal{B}_{C^1}$ and, in particular, $ck = 0 \bmod \mathcal{Z}_{\mathbb{Q}} + \mathcal{B}_{C^1}$.

Conversely, if $ck = 0 \bmod \mathcal{Z}_{\mathbb{Q}} + \mathcal{B}_{C^1}$ and $c \neq 0$ then by multiplying by a suitable integer, we have (for a possibly different c) $ck = 0 \bmod \mathcal{Z}_{\mathbb{Z}} + \mathcal{B}_{C^1}$. Applying $\eta_{C^1}^{-1}$ to this equation, we recover an equation of the form (15) (see the proof of Proposition 3.4). Hence, ψ_k^t has an eigenfunction corresponding to a non-zero eigenfrequency and thus cannot be mixing.

Finally, observe that if $k \neq 0 \bmod \mathcal{Z}_{\mathbb{R}} + \mathcal{B}_{C^1}$ then any non-zero multiple of k is also not congruent to $0 \bmod \mathcal{Z}_{\mathbb{R}} + \mathcal{B}_{C^1}$ and, in particular, is not congruent to $0 \bmod \mathcal{Z}_{\mathbb{Q}} + \mathcal{B}_{C^1}$. ❑

Remarks.

1. It could happen that $k = 0 \bmod \mathcal{Z}_{\mathbb{R}} + \mathcal{B}_{C^1}$ but no non-zero multiple of k is congruent to $0 \bmod \mathcal{Z}_{\mathbb{Q}} + \mathcal{B}_{C^1}$, so that ψ_k^t is mixing. Using the same method as Proposition 3.6, one can easily show that such velocity changes are unstably mixing: there are arbitrarily small perturbations

(in the C^1 sense) of k such that the perturbed velocity changed flow is not mixing.

2. Suppose $H^1(\Lambda, \mathbb{Z})$ is trivial. Then any velocity change of ϕ^t is mixing. This is because if $k = 0 \bmod \mathcal{B}_{C^1}$ then k cannot be strictly positive as it must vanish on periodic orbits.

Proof of Theorem C. First observe that if $k \in C^1(\Lambda, \mathbb{R}^+)$ then any sufficiently small C^1 perturbation of k also lies in $C^1(\Lambda, \mathbb{R}^+)$.

Consider the subset $\{k \in C^1(\Lambda, \mathbb{R}^+) \mid k \neq 0 \mod \mathcal{Z}_{\mathbb{R}} + \mathcal{B}_{C^1}\}$. By Proposition 4.2, if k lies in this set then ψ_k^t is mixing. By Proposition 2.11, this set is dense in $C^1(\Lambda, \mathbb{R}^+)$. By Proposition 2.12, it is also open. This proves Theorem C. ❏

Remark. We could also have considered C^r velocity changes of C^r flows for $2 \leq r < \infty$. In this case, given $k \in C^r(\Lambda, \mathbb{R}^+)$, we can obtain an open neighbourhood of k in $C^{r-1+\varepsilon}$ for any $\varepsilon \in (0,1)$ of mixing velocity changes.

There may be no non-mixing velocity changes. This happens precisely when no function in $\mathcal{Z}_{\mathbb{R}} + \mathcal{B}_{C^1}$ is strictly positive. As an example, it is remarked in [An] that any velocity change of the geodesic flow on the unit tangent bundle of a compact, negatively curved surface is mixing. If there is a non-mixing velocity change then there must be an associated strictly positive k in $\mathcal{Z}_{\mathbb{Z}} + \mathcal{B}_{C^1}$. Applying $\eta_{C^1}^{-1}$ to k, we see that there exists a C^1 $w : \Lambda \to K$ such that

$$\frac{1}{2\pi i} \frac{w'_\phi}{w} > 0.$$

By a result in [Sc], the sub-manifold $\Lambda_w = \{x \in \Lambda \mid w(x) = 1\}$ is a global cross-section to the flow. Thus ϕ^t is the suspension of an Anosov diffeomorphism $\phi : \Lambda_w \to \Lambda_w$, recovering a familiar result of Plante [Pl]. In the hyperbolic case, when there is a non-mixing velocity change, we again obtain a Hölder $w : \Lambda \to K$ for which $w'_\phi / 2\pi i w$ is strictly positive and again Λ_w is a cross-section. Thus ϕ^t is a suspension of a homeomorphism $\phi : \Lambda_w \to \Lambda_w$ satisfying Axiom A^* (see [Bo1]). Thus we recover a result of Bowen's [Bo3]:

a hyperbolic flow is not mixing if and only if it is the constant suspension of an Axiom A^* homeomorphism.

We can use the above to give a criterion for the existence of a cross-section in terms of the pressure functional. Observe from (1) that the pressure $P(G)$ of a continuous function $G : \Lambda \to \mathbb{R}$ depends only on the ϕ-cohomology class of G. We define the *cohomological pressure* to be the map

$$P_H : H^1(\Lambda, \mathbb{Z}) \to \mathbb{R} : [w] \mapsto P(\eta_\theta[w]).$$

The following is proved in [Sh] in the Anosov case. Sharp uses the first de Rham cohomology group whereas we are using the Bruschlinsky group. Nevertheless, the proof given here is identical to that in [Sh].

Proposition 4.3

Let ϕ^t be a hyperbolic flow. Then ϕ^t has a cross-section if and only if P_H is unbounded below.

Proof. Suppose there is a cross-section. Then by the remarks above, we can find a Hölder $w : \Lambda \to K$ that is Hölder differentiable with respect to the flow and such that

$$G = \frac{1}{2\pi i}\frac{w'_\phi}{w}$$

is strictly positive. Suppose $G \geq \varepsilon > 0$. Let $n \in \mathbb{N}$ and let μ_n denote the equilibrium state corresponding to nG. Then for $n \leq 0$ we have

$$P_H([w^n]) = h_{\mu_n}(\phi) + \mu_n(nG) \leq h(\phi) + n\varepsilon \tag{17}$$

($h(\phi)$ is the topological entropy of ϕ). The right-hand side of (17) tends to $-\infty$ as $n \to -\infty$, whence P_H is unbounded below.

Conversely, suppose no cross-section exists. A result in [Sc] allows us to choose a ϕ-invariant probability measure μ_0 such that $\mu_0(\eta_\theta([w])) = 0$ for all $[w] \in H^1(\Lambda, \mathbb{Z})$. Then

$$P_H([w]) = \sup_\mu \{h_\mu(\phi) + \mu(\eta_\theta([w]))\} \geq h_{\mu_0}(\phi) \geq 0.$$

So P_H is bounded below. ❑

As in [Sh], we see that if P_H is bounded below then it is bounded below by 0.

It is interesting to consider how the dynamic cohomology groups behave as invariants of the flow. It is easy to see that if $f, f' : \Lambda \to \mathbb{R}$ generate ϕ-cohomologous ϕ-cocycles then they also generate ψ_k-cohomologous ψ_k-cocycles. Hence $\mathcal{H}^1_\theta(\phi, \mathbb{R}) \cong \mathcal{H}^1_\theta(\psi_k, \mathbb{R})$ (and similarly for the circle-valued groups). However, if f, f' are *essentially* ϕ-cohomologous then it is not in general true that they are essentially ψ_k-cohomologous. Hence $\mathcal{H}^{1,\mathrm{ess}}_\theta(\phi, \mathbb{R})$ is not necessarily isomorphic to $\mathcal{H}^{1,\mathrm{ess}}_\theta(\psi_k, \mathbb{R})$.

An example of this phenomena can be found in [Pa2]. It is known that the equilibrium state of $f \in C^\theta(\Lambda, \mathbb{R})$ depends only on the essential ϕ-cohomology class of f [PP1, for example]. In [Pa2] it is shown how the velocity of a hyperbolic attractor ϕ^t (with C^1 unstable distribution) can be changed to ψ^t_k so that two *a priori* distinct equilibrium states for ϕ^t (namely the measure of maximal entropy and the S.R.B. measure) coincide for ψ^t_k.

Finally, one may observe that a velocity change of a toral extension is a toral extension of a velocity change. Specifically, suppose we start with ϕ^t, torally extend ϕ^t to ϕ^t_F and then velocity change ϕ^t_F by $k' : \Lambda \times K^n \to \mathbb{R} : (x, y) \mapsto k(x)$, $k \in C^\theta(\Lambda, \mathbb{R})$, to $(\phi^t_F)_k$. Then this is the same flow that we would have obtained had we first velocity changed ϕ^t by k to ψ^t_k and then torally extended ψ^t_k by $Fl = (f_1 l, \ldots, f_n l)$ to $(\psi^t_k)_{Fl}$.

§5 Suspensions of shifts of finite type

Here we prove our main theorems in the context of suspensions of shifts of finite type. We first discuss function spaces for $\widetilde{\Sigma}_r$ and their relationships with functions spaces for Σ. This allows us to prove Theorems A′ and B′. We then turn to Theorem C′. The proof of this is independent of the rest of this section and only depends on the properties of $F_\theta(\Sigma, \mathbb{R})$ proved in [PP3].

Function spaces for $\boldsymbol{\Sigma}$. In proving our main theorems, it will often be useful to replace functions defined on $\widetilde{\Sigma}_r$ by functions defined on Σ. Here we gather together some useful properties of $F_\theta = F_\theta(\Sigma, \mathbb{R})$ which will allow us to deduce properties of function spaces on $\widetilde{\Sigma}_r$.

Define the following subspaces of $F_\theta(\Sigma, \mathbb{R})$:

$$\begin{aligned}
B_\theta &= \{u\sigma - u \mid u \in F_\theta\}, \\
B_\theta^{\mathrm{ess}} &= \{u\sigma - u + c \mid u \in F_\theta, c \in \mathbb{R}\}, \\
V_\theta &= \mathrm{cl}\,\{v : \Sigma \to \mathbb{R} \mid v \text{ is locally constant}\}, \\
R &= \{cr \mid c \in \mathbb{R}\},
\end{aligned}$$

(the closure of the locally constant functions is taken with respect to the F_θ topology).

The following is proved in [PP3].

Theorem 5.1

1. *$B_\theta^{\mathrm{ess}} + V_\theta$ is a closed subspace of F_θ;*

2. *$F_\theta / B_\theta^{\mathrm{ess}} + V_\theta$ is non-separable.*

Remark. An argument similar to the one in the proof of Proposition 2.9.1 shows that B_θ and B_θ^{ess} are closed subspaces; here we need to use the Livšic periodic data criterion for shifts of finite type [Li2].

Corollary 5.2

1. *$B_\theta + V_\theta$, $B_\theta + V_\theta + R$ are closed subspaces of F_θ;*

2. *$F_\theta / B_\theta + V_\theta$, $F_\theta / B_\theta + V_\theta + R$ are non-separable.*

Proof. 1. Let $b_n + v_n \in B_\theta + V_\theta$ and suppose $b_n + v_n \to f$ as $n \to \infty$. As $B_\theta + V_\theta \subset B_\theta^{\mathrm{ess}} + V_\theta$ and this is closed, $f = b + c + v$ with $b \in B_\theta, c \in \mathbb{R}$ and $v \in V_\theta$. Then $c + v \in V_\theta$ whence $f \in B_\theta + V_\theta$. Hence $B_\theta + V_\theta$ is closed. That $B_\theta + V_\theta + R$ is closed follows from the fact that R is one-dimensional.

2. Clearly $B_\theta + V_\theta \subset B_\theta^{\mathrm{ess}} + V_\theta$ so $F_\theta / B_\theta + V_\theta \supset F_\theta / B_\theta^{\mathrm{ess}} + V_\theta$. Hence $F_\theta / B_\theta + V_\theta$ is non-separable. To see that $F_\theta / B_\theta + V_\theta + R$ is non-separable, observe that $B_\theta + V_\theta$ has codimension at most 1 in $B_\theta + V_\theta + R$. ❑

Function spaces for $\tilde{\Sigma}_r$. Recall that we have the space

$$\tilde{F}_\theta(\tilde{\Sigma}_r, \mathbb{R}) := \{f : \tilde{\Sigma}_r \to \mathbb{R} \mid f \text{ is continuous and } \mathcal{I}f \in F_\theta(\Sigma, \mathbb{R})\}$$

(see below for the definition of $\mathcal{I}$) with norm $\|f\|_\theta^\sim = |f|_\infty + \|\mathcal{I}f\|_\theta$.

We shall also be interested in Lipschitz continuous functions defined on $\widetilde{\Sigma}_r$. In [BW] a metric is constructed on the constant suspension space $\widetilde{\Sigma}_1$. We modify their construction to obtain a metric on $\widetilde{\Sigma}_r$.

For each $0 \leq t \leq 1$ define the t^{th} r-section of Σ to be

$$\Sigma_{r,t} = \{(x, tr(x)) \mid x \in \Sigma\}.$$

Then $\widetilde{\Sigma}_r$ is the disjoint union on the $\Sigma_{t,r}$ for $0 \leq t < 1$ and $\Sigma_{1,r}$ is identified with $\Sigma_{0,r}$. Let $r_0 = \inf\{r(x) \mid x \in \Sigma\}$. We define a metric d_t^θ on each $\Sigma_{t.r}$ by

$$d_t^\theta((x, tr(x)), (y, tr(y))) = r_0((1-t)\, d^\theta(x, y) + t\, d^\theta(\sigma x, \sigma y)),$$

(d^θ is defined in §1). Observe that d_0^θ and d_1^θ are compatible with the identifications between $\Sigma_{0,r}$ and $\Sigma_{1,r}$. For $0 \leq t < 1$, each $\Sigma_{t,r}$ can be canonically identified with Σ and it will often simplify our notation to regard d_t^θ as a metric on Σ. It is easy to see that the d_t^θ are all pairwise equivalent. Indeed, more is true:

Lemma 5.3
There exists a constant $C > 0$ such that for all $0 \leq t', t \leq 1$ we have $d_{t'}^\theta(x, y) \leq C\, d_t^\theta(x, y)$ for all $x, y \in \Sigma$.

The proof of this is routine and we postpone it until the end of this section.

Let $\tilde{x}, \tilde{y} \in \widetilde{\Sigma}_r$. A *path* from $\tilde{x}$ to $\tilde{y}$ is a finite sequence of points

$$\tilde{x} = w_0, w_1, \ldots, w_{n-1}, w_n = \tilde{y}$$

such that for each i either:

1. w_i, w_{i+1} lie in the same r-section—in this case, set $|w_i - w_{i+1}| = d_t^\theta(w_i, w_{i+1})$ if $w_i, w_{i+1} \in \Sigma_{t,r}$; or

2. w_i, w_{i+1} lie in the same σ_r^t-orbit—in this case, set $|w_i - w_{i+1}| = \inf\{|\tau| \mid \sigma_r^\tau w_i = w_{i+1}\}$.

It could happen that both possibilities occur: w_i, w_{i+1} lie in the same r-section and the same σ_r^t-orbit. In this case, we always take the first definition of $|w_i - w_{i+1}|$ (indeed, it is readily checked that $d_t^\theta(w_i, w_{i+1}) \leq \inf\{|\tau| \mid$

$\sigma_r^\tau w_i = w_{i+1}\}$ and this is the reason for introducing the factor r_0 in the definition of d_t^θ). If w is a path from $\tilde{x}$ to $\tilde{y}$ then we set

$$|w| = \sum_{i=0}^{n-1} |w_i - w_{i+1}|.$$

Define

$$d_r^\theta(\tilde{x}, \tilde{y}) = \inf\{|w| \mid w \text{ is a path from } \tilde{x} \text{ to } \tilde{y}\}.$$

It is easy to see that d_r^θ is a metric and gives $\widetilde{\Sigma}_r$ the correct topology.

The following lemma says that the time-t map σ_r^t is Lipschitz continuous on $\Sigma \times \{0\}$.

Lemma 5.4

Let $t \geq 0$. There exists a constant $C(t)$ such that for all $x, y \in \Sigma$,

$$d_r^\theta(\sigma_r^t(x, 0), \sigma_r^t(y, 0)) \leq C(t)\, d^\theta(x, y).$$

The proof will be given at the end of this section.

Let $F_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ denote the space of functions $\widetilde{\Sigma}_r \to \mathbb{R}$ that are Lipschitz continuous with respect to d_r^θ. Equip $F_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ with the norm $\|\cdot\|_\theta := |\cdot|_\infty + |\cdot|_\theta$ where $|f|_\theta$ is the least Lipschitz constant of $f \in F_\theta(\widetilde{\Sigma}_r, \mathbb{R})$:

$$|f|_\theta := \sup_{\tilde{x} \neq \tilde{y}} \frac{|f(\tilde{x}) - f(\tilde{y})|}{d_r^\theta(\tilde{x}, \tilde{y})}.$$

Then $F_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ is a Banach space.

The following relates $\widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ and $F_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ and will be proved later.

Lemma 5.5

There is a continuous inclusion $F_\theta(\widetilde{\Sigma}_r, \mathbb{R}) \hookrightarrow \widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})$.

It is easy to construct examples to show that, in general, this inclusion is strict.

One should note that $F_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ depends on the metric d_r^θ. If one changes d_r^θ (even just on σ_r^t orbits) then one may change $F_\theta(\widetilde{\Sigma}_r, \mathbb{R})$. This is in contrast to the smooth hyperbolic case where the space of Hölder continuous functions of a given exponent is independent of the choice of Riemannian metric.

As we have already remarked, it will be useful to replace functions defined on $\widetilde{\Sigma}_r$ by functions defined on Σ. To do this, we use the linear operator

$$\begin{aligned} \mathcal{I} &: \widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R}) \to F_\theta(\Sigma, \mathbb{R}) \\ \mathcal{I}f(x) &= \int_0^{r(x)} f\sigma_r^u(x,0)\, du. \end{aligned}$$

Clearly, $\mathcal{I}$ is continuous. The most important property of $\mathcal{I}$ is the following:

Proposition 5.6

The operator $\mathcal{I} : \widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R}) \to F_\theta(\Sigma, \mathbb{R})$ is onto. The same is true when $\mathcal{I}$ is restricted to $F_\theta(\widetilde{\Sigma}_r, \mathbb{R})$.

Proof. Let $f \in F_\theta(\Sigma, \mathbb{R})$. Take a C^1 map $\tau : [0,1] \to \mathbb{R}$ such that $\tau \geq 0$, $\tau(0) = 0 = \tau(1)$ and $\int_0^1 \tau(u)\, du = 1$.

Define

$$\tilde{f}(x,t) = \frac{f(x)}{r(x)} \tau\left(\frac{t}{r(x)}\right).$$

Then $\tilde{f}(x,0) = 0 = \tilde{f}(x, r(x))$ so $\tilde{f}$ is a well-defined continuous function on $\widetilde{\Sigma}_r$. A simple calculation shows that $\mathcal{I}\tilde{f} = f$.

To see that $\mathcal{I}(F_\theta(\widetilde{\Sigma}_r, \mathbb{R})) = F_\theta(\Sigma, \mathbb{R})$, it is sufficient to check that $\tilde{f} \in F_\theta(\widetilde{\Sigma}_r, \mathbb{R})$. This will be proved later. ❑

Toral extensions of symbolic flows. Recall that we have a toral extension of σ_r^t of the form

$$\sigma_{r,F}^t(\tilde{x}, y_1, \ldots, y_n) = \left(\sigma_r^t(\tilde{x}), e^{2\pi i \int_0^t f_1 \sigma_r^u(\tilde{x})\, du}, \ldots, e^{2\pi i \int_0^t f_n \sigma_r^u(\tilde{x})\, du}\right).$$

Here, $f_i \in \widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ or $F_\theta(\widetilde{\Sigma}_r, \mathbb{R})$. We shall prove the existence of open dense sets (in both function spaces with their corresponding topologies) of ergodic and mixing extensions. We assume that σ_r^t is equipped with a Hölder equilibrium state. The following regularity theorem is proved in [Wa].[5]

Theorem 5.7 (Livšic regularity theorem for symbolic flows)

Let m be a Hölder equilibrium state for σ_r^t and let $f \in \widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})$. Suppose $\tilde{w} : \widetilde{\Sigma}_r \to K$ is a measurable solution to the cohomological equation

$$\frac{\tilde{w}\sigma_r^t(\tilde{x})}{\tilde{w}(\tilde{x})} = e^{2\pi i \int_0^t f\sigma_r^u(\tilde{x})\, du} \text{ a.e. } [m]. \tag{18}$$

[5] Theorem 3.2 in Chapter 1.

Then there exists a unique continuous $\tilde{w}'$ *such that* $\tilde{w}' = \tilde{w}$ a.e.,

$$\frac{\tilde{w}'\sigma_r^t(\tilde{x})}{\tilde{w}'(\tilde{x})} = e^{2\pi i \int_0^t f\sigma_r^u(\tilde{x})\,du} \text{ everywhere,} \tag{19}$$

and $\tilde{w}'(\cdot, 0) \in F_\theta(\Sigma, K)$.

It is not immediately clear from [Wa] that $\tilde{w}'(\cdot, 0) \in F_\theta(\Sigma, K)$. To see this, we need the following result from [Pa3].

Proposition 5.8

Let μ *be a Hölder equilibrium state for* σ *and let* $g \in F_\theta(\Sigma, K)$. *Suppose* $w : \Sigma \to K$ *is a measurable function such that* $w(\sigma x) = w(x)g(x)$ a.e. $[\mu]$. *Then there exists* $w' \in F_\theta(\Sigma, K)$ *such that* $w = w'$ a.e. *and* $w'(\sigma x) = w'(x)g(x)$ *everywhere.*

In (19), set $\tilde{x} = (x, 0)$ and $t = r(x)$ to obtain $\tilde{w}'(\sigma x, 0)/\tilde{w}'(x, 0) = e^{2\pi i \mathcal{I}f(x)}$. Proposition 5.8 then applies.

Recall from [KN] that Proposition 3.1 and its analogue for the mixing case remain valid for σ_r^t. Combining this with the above regularity theorem we have the following analogues of Propositions 3.2 and 3.3.

Proposition 5.9

If $(\mathcal{I}f_1, \ldots, \mathcal{I}f_n)$ *is linearly independent mod* $B_\theta + V_\theta$ *then* $\sigma_{r,F}^t$ *is ergodic.*

Proof. Suppose $\sigma_{r,F}^t$ is not ergodic. By Proposition 3.1, there exist integers $(l_1, \ldots, l_n) \in \mathbb{Z}^n \setminus \{(0, \ldots, 0)\}$ and a measurable $\tilde{w} : \widetilde{\Sigma}_r \to K$ such that

$$\frac{\tilde{w}\sigma_r^t(\tilde{x})}{\tilde{w}(\tilde{x})} = e^{2\pi i \int_0^t \sum_{i=1}^n l_i f_i \sigma_r^u(\tilde{x})\,du} \text{ a.e. } [m]. \tag{20}$$

By Theorem 5.7, we may assume that $\tilde{w}$ is continuous, (20) holds everywhere and $x \mapsto \tilde{w}(x, 0) \in F_\theta(\Sigma, K)$. As Σ is zero-dimensional, we can write $\tilde{w}(x, 0) = e^{2\pi i W(x)}$ for some $W \in F_\theta(\Sigma, \mathbb{R})$. Setting $t = r(x)$ it follows that

$$e^{2\pi i(W(\sigma x) - W(x))} = e^{2\pi i \sum_{i=1}^n l_i \mathcal{I}f_i(x)}$$

for all $x \in \Sigma$. Hence, $W\sigma - W + V = \sum_{i=1}^n l_i \mathcal{I}f_i$ for some continuous integer-valued, hence locally constant, $V \in V_\theta$. Thus $(\mathcal{I}f_1, \ldots, \mathcal{I}f_n)$ is integrally dependent, hence linearly dependent, mod $B_\theta + V_\theta$. ❑

A similar result holds for the mixing case; recall that R is the subspace $\{cr \mid c \in \mathbb{R}\}$.

Proposition 5.10

If $(\mathcal{I}f_1, \ldots, \mathcal{I}f_n)$ is linearly independent mod $B_\theta + V_\theta + R$ then $\sigma^t_{r,F}$ is mixing.

We can use these results to give an analogue of Proposition 2.12; in particular, we see that the ergodic and mixing extensions contain an open set.

Proposition 5.11

The set of n-tuples $(f_1, \ldots, f_n) \in \widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})^n$ for which $(\mathcal{I}f_1, \ldots, \mathcal{I}f_n)$ is linearly independent mod $B_\theta + V_\theta$ forms an open set in $\widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})^n$. The same remains true if $\widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ is replaced by $F_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ and also if $B_\theta + V_\theta$ is replaced by $B_\theta + V_\theta + R$.

Proof. We prove only the first case; the proofs of the other parts are identical.

We will prove that the set of n-tuples $(f_1, \ldots, f_n) \in \widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})^n$ with $(\mathcal{I}f_1, \ldots, \mathcal{I}f_n)$ linearly dependent mod $B_\theta + V_\theta$ is closed. Suppose $F_m = (f_{1,m}, \ldots, f_{n,m}) \in \widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})^n$ is such that $F_m \to F = (f_1, \ldots, f_n)$ in $\widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})^n$ and, for each m, $(\mathcal{I}f_{1,m}, \ldots, \mathcal{I}f_{n,m})$ is linearly dependent mod $B_\theta + V_\theta$. Hence, there exist real numbers $\lambda_{1,m}, \ldots, \lambda_{n,m}$, not all zero, such that

$$\lambda_{1,m}\mathcal{I}f_{1,m} + \cdots + \lambda_{n,m}\mathcal{I}f_{n,m} = 0 \bmod B_\theta + V_\theta.$$

If necessary, we can re-order the indices so that $\lambda_{1,m} \neq 0$ and $|\lambda_{1,m}| \geq |\lambda_{i,m}|$, $2 \leq i \leq n$, for infinitely many m. Dividing by $\lambda_{1,m}$ we see that

$$\mathcal{I}f_{1,m} + \mu_{2,m}\mathcal{I}f_{2,m} + \cdots + \mu_{n,m}\mathcal{I}f_{n,m} = 0 \bmod B_\theta + V_\theta$$

and $|\mu_{i,m}| \leq 1$ for $2 \leq i \leq n$. By taking a subsequence, we can assume that $\mu_{i,m} \to \mu_i$ as $m \to \infty$. As $\mathcal{I}$ is continuous and $f_{i,m} \to f_i$, it follows that

$$\mathcal{I}f_1 + \mu_2\mathcal{I}f_2 + \cdots + \mu_m\mathcal{I}f_m = 0 \bmod B_\theta + V_\theta$$

(as $B_\theta + V_\theta$ is closed). This proves the proposition. ❑

Thus, the ergodic extensions contain the set of n-tuples $(f_1, \ldots, f_n)$ for which $(\mathcal{I}f_1, \ldots, \mathcal{I}f_n)$ is linearly independent mod $B_\theta + V_\theta$. We claim that this set is dense. Fix $(f_1, \ldots, f_n) \in \widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})^n$ and let $V = B_\theta + V_\theta + \text{span}\{\mathcal{I}f_1, \ldots, \mathcal{I}f_n\}$; then V is a closed subspace of $F_\theta(\Sigma, \mathbb{R})$. As $\mathcal{I}$ is continuous, $\mathcal{I}^{-1}V$ is a closed subspace of $\widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})$.

Lemma 5.12
$\widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})/\mathcal{I}^{-1}V$ is non-separable.

Proof. As $B_\theta + V_\theta$ has finite index in V, it follows from Corollary 5.2 that $F_\theta(\Sigma, \mathbb{R})/V$ is non-separable.

As $\mathcal{I}(\mathcal{I}^{-1}V) \subset V$, we have a well-defined map $\mathcal{I}_V : \widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})/\mathcal{I}^{-1}V \to F_\theta(\Sigma, \mathbb{R})/V$. As $\mathcal{I}$ is onto, so is $\mathcal{I}_V$. The lemma follows from this, together with the fact that $\mathcal{I}_V$ is continuous. ❑

Corollary 5.13
$F_\theta(\widetilde{\Sigma}_r, \mathbb{R})/\mathcal{I}^{-1}V$ is non-separable.

Proof. As the operator $\mathcal{I} : F_\theta(\widetilde{\Sigma}_r, \mathbb{R}) \to F_\theta(\Sigma, \mathbb{R})$ is continuous and onto, the above proof goes through. ❑

Remark. Observe that we could replace V by $V' = B_\theta + V_\theta + R + \text{span}\{\mathcal{I}f_1, \ldots, \mathcal{I}f_n\}$ in the above and conclude that $\widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})/\mathcal{I}^{-1}V'$ and $F_\theta(\widetilde{\Sigma}_r, \mathbb{R})/\mathcal{I}^{-1}V'$ are non-separable.

Proposition 5.14
The set of n-tuples $(f_1, \ldots, f_n) \in \widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})^n$ or $F_\theta(\widetilde{\Sigma}_r, \mathbb{R})^n$ such that $(\mathcal{I}f_1, \ldots, \mathcal{I}f_n)$ is linearly independent mod $B_\theta + V_\theta$ is dense. A similar statement holds for $B_\theta + V_\theta + R$.

Proof. We prove only the $\widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ case. Fix $(f_1, \ldots, f_n) \in \widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})^n$ and let $V = \mathcal{B}_\theta + V_\theta + \text{span}\{f_1, \ldots, f_n\}$. Choose small functions $\varepsilon_i \in \widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ such that the ε_i are linearly independent mod $\mathcal{I}^{-1}V$ (this is possible as $\widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})/\mathcal{I}^{-1}V$ is non-separable, hence infinite dimensional). Let $f_i' = f_i + \varepsilon_i$. Then f_i' is $\|\cdot\|_\theta^\sim$-close to f_i. Moreover, $(\mathcal{I}f_1', \ldots, \mathcal{I}f_n')$ is linearly independent mod $B_\theta + V_\theta$. To see this, suppose $\sum_{i=1}^n \lambda_i \mathcal{I}f_1' = 0 \bmod B_\theta + V_\theta$.

Then $\sum \lambda_i \mathcal{I}\varepsilon_i = \mathcal{I}\left(\sum \lambda_i \varepsilon_i\right) = 0 \bmod V$. Hence $\sum \lambda_i \varepsilon_i = 0 \bmod \mathcal{I}^{-1}V$ and it follows that $\lambda_i = 0$ for $1 \leq i \leq n$. ❑

Proof of Theorems A′ and B′. By Proposition 5.9, the set of n-tuples $F = (f_1, \ldots, f_n)$ for which $\sigma^t_{r,F}$ is ergodic contains the n-tuples for which $(\mathcal{I}f_1, \ldots, \mathcal{I}f_n)$ is linearly independent mod $B_\theta + V_\theta$. By Proposition 5.11, this set is open. By Proposition 5.14, it is also dense. This proves Theorem A′.

The mixing case (Theorem B′) is exactly the same, with $B_\theta + V_\theta + R$ replacing $B_\theta + V_\theta$. ❑

Velocity changes of symbolic flows. Here we prove Theorem C′. As we consider functions defined on Σ, and not $\widetilde{\Sigma}_r$, we need only a very weak Livšic regularity theorem. Such a theorem is contained implicitly in the proof of the following proposition. Again, we assume that σ^t_r is equipped with a Hölder equilibrium state m.

Proposition 5.15

If $r \neq 0 \bmod B_\theta + V_\theta$ then σ^t_r is mixing.

Proof. Suppose σ^t_r is not mixing. Then there exist $0 \neq c \in \mathbb{R}$ and a measurable $\tilde{w} : \widetilde{\Sigma}_r \to K$ such that

$$\tilde{w}\sigma^t_r(x, s) = e^{2\pi i c t}\tilde{w}(x, s) \text{ a.e. } [m].$$

Then setting $t = r(x)$ we have

$$\tilde{w}\sigma^t_r(x, s) = e^{2\pi i c r(x)}\tilde{w}(x, s) \text{ a.e. } [m].$$

Let μ be the Hölder equilibrium state for σ associated to m. By Fubini's Theorem, choose a 'good' small $s_0 \geq 0$ such that

$$\tilde{w}(\sigma x, s_0) = e^{2\pi i c r(x)}\tilde{w}(x, s_0) \text{ a.e. } [\mu]. \tag{21}$$

and $w(x) = \tilde{w}(x, s_0)$ is a measurable function $\Sigma \to K$. By Proposition 5.8, we may assume that $w \in F_\theta(\Sigma, K)$ and $w(\sigma x) = e^{2\pi i c r(x)} w(x)$ everywhere. As Σ is totally disconnected, $w = e^{2\pi i W}$ for some $W \in F_\theta(\Sigma, \mathbb{R})$. Thus, (21) becomes $W\sigma - W + V = cr$ for some continuous integer-valued, hence locally constant, $V \in V_\theta$. It follows that $r = 0 \bmod B_\theta + V_\theta$. ❑

Remarks.

1. The implicit proof above of the regularity result for $\widetilde{\Sigma}_r$ is the same as the proof (given in [Wa][6]) of Theorem 5.7 in the case when f is constant.

2. There is also a partial converse. Suppose $r = v \bmod B_\theta$ where v takes values in a discrete subgroup of $\mathbb{R}$ (and so is locally constant). Then σ_r^t is not mixing.

Proof of Theorem C′. We use the same arguments as in the proof of Theorem C, using Corollary 5.2 to see that the set of strictly positive $r \in F_\theta(\Sigma, \mathbb{R})$ that are not congruent to 0 mod $B_\theta + V_\theta$ forms an open dense set. ❑

Technicalities. It remains to prove the various claims stated above without proof.

Proof of Lemma 5.3. We prove the following: $\exists C > 0$ s.t. $\forall 0 \le t' \le t \le 1$, $\forall x, y \in \Sigma$, $d_{t'}^\theta(x, y) \le C\, d_t^\theta(x, y)$. The corresponding statement for $0 \le t \le t' \le 1$ is proved similarly.

Observe that $\theta\, d^\theta(x, y) \le d^\theta(\sigma x, \sigma y) \le \theta^{-1}\, d^\theta(x, y)$. Then

$$\begin{aligned}
d_{t'}^\theta(x,y) &= r_0((1-t')\, d^\theta(x,y) + t'\, d^\theta(\sigma x, \sigma y)) \\
&\le r_0((2-t)\, d^\theta(x,y) + \theta^{-1} t'\, d^\theta(x,y)) \\
&\le r_0 \frac{(2-t+\theta^{-1}t)}{(\theta^{-1}(1-t)+t)}\, \theta^{-1}((1-t)\, d^\theta(x,y) + \theta t\, d^\theta(x,y)) \\
&\le \frac{(2-t+\theta^{-1}t)}{(\theta^{-1}(1-t)+t)}\, \theta^{-1}\, d_t^\theta(x,y).
\end{aligned}$$

The map $t \mapsto (2 - t + \theta^{-1})/(\theta^{-1}(1-t) + t)$ is strictly increasing on $[0,1]$ and so is bounded above on $[0,1]$ by $\theta^{-1} + 1$. Hence

$$d_{t'}^\theta(x,y) \le \theta^{-1}(\theta^{-1} + 1)\, d_t^\theta(x,y).$$

❑

[6] Theorem 3.2 in Chapter 1

Proof of Lemma 5.4. Let $x, y \in \Sigma$ and fix $t \geq 0$. Let n be the unique integer such that $r^n(x) \leq t < r^{n+1}(x)$ (here and throughout this proof $r^n(x) = \sum_{j=0}^{n-1} r(\sigma^j x)$). Then

$$\sigma_r^t(x,0) = \left(\sigma^n x, \frac{t - r^n(x)}{r(\sigma^n x)} r(\sigma^n x)\right) \in \Sigma_{\frac{t-r^n(x)}{r(\sigma^n x)}, r}.$$

Let $\tilde{y}$ denote the point

$$\tilde{y} = \left(\sigma^n y, \frac{t - r^n(x)}{r(\sigma^n x)} r(\sigma^n y)\right) \in \Sigma_{\frac{t-r^n(x)}{r(\sigma^n x)}, r}.$$

Then $\sigma_r^t(x,0)$ and $\tilde{y}$ are in the same r-section. By Lemma 5.3, there exists a constant $C_1 > 0$, independent of x, y, such that

$$d^\theta_{\frac{t-r^n(x)}{r(\sigma^n x)}}(\sigma_r^t(x,0), \tilde{y}) \leq C_1\, d_0^\theta(\sigma^n x, \sigma^n y) \leq \frac{C_1 r_0}{\theta^n}\, d^\theta(x,y).$$

Clearly, $\sigma_r^t(y,0)$ and $\tilde{y}$ lie in the same σ_r^t-orbit. Now, $\sigma_r^t(y,0)$ is identified with (y,t) and $\tilde{y}$ is identified with

$$\left(y, \frac{(t - r^n(x))}{r(\sigma^n x)} r(\sigma^n y) + r^n(y)\right).$$

Hence, the 'vertical' distance between $\tilde{y}$ and $\sigma_r^t(y,0)$ is no more than

$$\begin{aligned} &\left|\frac{(t - r^n(x))}{r(\sigma^n x)} r(\sigma^n y) + r^n(y) - t\right| \\ &\leq \frac{1}{r_0}\left(|t(r(\sigma^n x) - r(\sigma^n y))| + |r^n(y) r(\sigma^n x) - r^n(x) r(\sigma^n y)|\right). \end{aligned}$$

The first term is bounded above by $t|r|_\theta \theta^{-n}\, d^\theta(x,y)$ as $r \in F_\theta(\Sigma, \mathbb{R})$. The second term is bounded by

$$|r^n(y)|\,|r(\sigma^n x) - r(\sigma^n y)| + |r(\sigma^n y)|\,|r^n(x) - r^n(y)|. \tag{22}$$

An easy computation shows that

$$|r^n(x) - r^n(y)| \leq C_2\, d^\theta(x,y),$$

for some constant $C_2 > 0$ independent of x, y. Hence

$$\begin{aligned} (22) &\leq |r|_\infty |r|_\theta \theta^{-n}\, d^\theta(x,y) + |r|_\infty C_2\, d^\theta(x,y) \\ &\leq C_3\, d^\theta(x,y). \end{aligned}$$

for some constant $C_3 > 0$. Thus the vertical distance between $\tilde{y}$ and $\sigma_r^t(y,0)$ is bounded above by $C_4\, d^\theta(x,y)$ where $C_4 > 0$ is independent of x, y (although it will depend on t).

The sequence $\sigma_r^t(x,0), \tilde{y}, \sigma_r^t(y,0)$ is a path from $\sigma_r^t(x,0)$ to $\sigma_r^t(y,0)$. Moreover, it has length at most $C_1 r_0 \theta^{-n}\, d^\theta(x,y) + C_4\, d^\theta(x,y) = C(t)\, d^\theta(x,y)$. From the definition of d_r^θ it follows that

$$d_r^\theta(\sigma_r^t(x,0), \sigma_r^t(y,0)) \leq C(t)\, d^\theta(x,y)$$

for some constant $C(t) > 0$, independent of x, y, as claimed. ❑

Proof of Lemma 5.5. Let $f \in F_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ be Lipschitz with respect to d_r^θ. We need to show that $f \in \widetilde{F}_\theta(\widetilde{\Sigma}_r, \mathbb{R})$ and that this inclusion is continuous.

Now,

$$\begin{aligned}
|\mathcal{I}f(x) - \mathcal{I}f(y)| &= \left| \int_0^{r(x)} f\sigma_r^u(x,0)\, du - \int_0^{r(y)} f\sigma_r^u(y,0)\, du \right| \\
&\leq \int_0^{r(x)} |f\sigma_r^u(x,0) - f\sigma_r^u(y,0)|\, du + \left| \int_{r(x)}^{r(y)} |f\sigma_r^u(y,0)|\, du \right| \\
&\leq |f|_\theta C(|r|_\infty) \int_0^{r(x)} d^\theta(x,y)\, du + |f|_\infty |r(x) - r(y)| \\
&\leq (|f|_\theta C(|r|_\infty)|r|_\infty + |f|_\infty |r|_\theta)\, d^\theta(x,y),
\end{aligned}$$

(here $C(|r|_\infty)$ is as in Lemma 5.4). Hence $\mathcal{I}f \in F_\theta(\Sigma, \mathbb{R})$ so the inclusion is well-defined. It follows from the above that $|\mathcal{I}f|_\theta \leq C_1 \|f\|_\theta$ for some constant $C_1 > 0$. Moreover,

$$|\mathcal{I}f(x)| \leq \int_0^{r(x)} |f\sigma_r^u(x,0)|\, du \leq |r|_\infty |f|_\infty \leq |r|_\infty \|f\|_\theta$$

so that $\|f\|_\theta^\sim = |f|_\infty + \|\mathcal{I}f\|_\theta \leq C\, \|f\|_\theta$ for some constant $C > 0$. ❑

Proof of Proposition 5.6. It remains to check that $\tilde{f} \in F_\theta(\widetilde{\Sigma}_r, \mathbb{R})$. Let $\tilde{x}, \tilde{y} \in \widetilde{\Sigma}_r$ and let $\tilde{x} = w_0, w_1, \ldots, w_{n-1}, w_n = \tilde{y}$ be a path between them.

Suppose first that w_i, w_{i+1} belong to the same r-section, $w_i = (x, tr(x)), w_{i+1} = (y, tr(y))$, say. Then,

$$|\tilde{f}(w_i) - \tilde{f}(w_{i+1})| = \left| \frac{f(x)}{r(x)}\tau(t) - \frac{f(y)}{r(y)}\tau(t) \right|$$

$$
\begin{aligned}
&\le |\tau|_\infty |f/r|_\theta \, d^\theta(x,y) \\
&\le C_1 \, d_t^\theta(x,y) \\
&= C_1 \, |w_i - w_{i+1}|
\end{aligned}
$$

for some constant $C_1 > 0$, by Lemma 5.3.

Suppose w_i, w_{i+1} are in the same σ_r^t-orbit. Then we may assume that $w_i = (x,t), w_{i+1} = (x, t+u)$ with $|u|$ least (if $t+u > r(x)$ then one considers the sub-path $(x,t),(x,r(x)) \sim (\sigma x, 0), (\sigma x, t+u-r(x))$). Then

$$
\begin{aligned}
\left|\tilde f(w_i) - \tilde f(w_{i+1})\right| &= \left|\tau\left(\frac{t+u}{r(x)}\right) - \tau\left(\frac{t}{r(x)}\right)\right| \left|\frac{f(x)}{r(x)}\right| \\
&\le |f/r|_\infty C_2 \left|\frac{u}{r(x)}\right| \\
&\le r_0^{-1} |f/r|_\infty C_2 \, |u| \\
&\le C_3 \, |w_i - w_{i+1}|
\end{aligned}
$$

as τ is C^1, hence Lipschitz. Here, C_3 is a constant independent of the path.

Hence,

$$
\begin{aligned}
|\tilde f(\tilde x) - \tilde f(\tilde y)| &\le \sum_{i=0}^{n-1} |\tilde f(w_i) - \tilde f(w_{i+1})| \\
&\le C \sum_{i=0}^{n-1} |w_i - w_{i+1}| \\
&= C \, |w|
\end{aligned}
$$

for some constant $C > 0$, independent of the path. Taking the infimum over all paths, we see that $\tilde f \in F_\theta(\widetilde\Sigma_r, \mathbb{R})$. ❑

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Livšic regularity theorems for twisted cocycle equations over hyperbolic systems

Abstract

Let ϕ be a hyperbolic map. We consider cocycle equations of the form $f = u\phi \cdot g \cdot \alpha u^{-1}$ with f, g, u taking values in a compact connected Lie group G, α an automorphism of G and f, g Hölder continuous. When the eigenvalues of the derivative of α have modulus one, we prove that any measurable solution u has a Hölder continuous version. When f, g are C^k then u may be taken to be $C^{k-1+\varepsilon}$, for any $\varepsilon \in (0,1)$.

§1 Introduction

Let $\phi : \Lambda \to \Lambda$ be a hyperbolic map. By a *hyperbolic map* we mean that either

1. ϕ is a (mixing) shift of finite type, or

2. ϕ is a hyperbolic diffeomorphism restricted to a basic set

(definitions of these are given in §2). In either case, we equip ϕ with a fixed Hölder continuous equilibrium state.

Let G be a compact connected Lie group. Equip G with a left-right invariant Riemannian metric. We can then speak of Hölder continuous functions $\Lambda \to G$. Denote the space of Hölder continuous functions of exponent θ by $F_\theta(G)$.

Let α be a (continuous, algebraic) automorphism of G. Two functions $f, g \in F_\theta(G)$ are said to be *α-cohomologous* if there exists a function $u : \Lambda \to G$ for which the *twisted cocycle equation* $f = u\phi \cdot g \cdot \alpha u^{-1}$ holds a.e. (here and throughout, we often denote multiplication in G by a dot '$\cdot$' for clarity). We call u the *cobounding function*. It is usually important to specify the class of functions to which u belongs. Indeed, one aim of this paper is to give conditions on α for when the existence of a measurable solution u implies the existence of a Hölder continuous solution u' equaling u a.e. (and then the above cocycle equation holds everywhere for u'). We call such results *regularity theorems*.

The classical case is when α is the identity. This was first considered by Livšic [Lv] under various assumptions. The general case is due to Parry and Pollicott [PP].[1]

Theorem 1.1 ([PP])
Let ϕ be a hyperbolic map preserving a Hölder equilibrium state. Suppose $f, g \in F_\theta(G)$ satisfy $f = u\phi \cdot g \cdot u^{-1}$ a.e. for some measurable u. Then there exists a Hölder continuous $u' \in F_\theta(G)$ such that $u' = u$ a.e. and $f = u'\phi \cdot g \cdot u'^{-1}$ everywhere.

Cohomologous functions occur when studying group extensions of ϕ. Specifically, let us fix $f \in F_\theta(G)$ and define the map $\phi_f : \Lambda \times G \to \Lambda \times G : (x, y) \mapsto (\phi x, f(x)\alpha(y))$. The condition that f is α-cohomologous to g translates to the condition that ϕ_f and ϕ_g are metrically isomorphic via an isomorphism of the form $(x, y) \mapsto (x, u(x)y)$; *i.e.* an isomorphism that maps fibres to fibres. When α is the identity, we can use Theorem 1.1 to conclude that u must actually be Hölder continuous. In particular, ϕ_f and ϕ_g are topologically conjugate.

Consider now the case when α is an ergodic automorphism (with respect to Haar measure). In [Th] it is proved that G must be a torus. A theorem due to Lind [Li] then says that *any* two functions $f, g \in F_\theta(G)$ are measurably α-cohomologous. More specifically, we have:

[1] For a shift of finite type, Parry and Pollicott only proved that $u \in F_{\theta^{\frac{1}{2}}}(G)$. However, it is easy to see that $u \in F_\theta(G)$—see the remark after Proposition 3.1 in Chapter 1.

Theorem 1.2 ([Li])

Let ϕ be a hyperbolic map preserving a Hölder equilibrium state. Let G be a torus and let α be an ergodic automorphism of G. Let $f, g \in F_\theta(G)$. Then there exists a measurable $u : \Lambda \to G$ such that $f = u\phi \cdot g \cdot \alpha u^{-1}$ a.e. In particular, the extension ϕ_f is metrically isomorphic to the direct product $\phi \times \alpha$.

(In fact, Lind proved considerably more: ϕ need only be an (aperiodic) invertible measure preserving transformation (not even ergodic), G need only be compact and abelian and f, g need only be measurable.) An observation due to Lind [Li], which we make precise in §3, then shows that the regularity theorem always fails when α is a hyperbolic toral automorphism.

We shall be interested in the case when α is a unipotent automorphism of a compact connected Lie group. We say that α is *unipotent* if all the eigenvalues of the derivative $d\alpha$ on the Lie algebra of G have modulus 1. We shall see later that this is equivalent to the eigenvalues being roots of unity. (*Caution*: this is not the usual meaning of the term 'unipotent' which is usually only defined for abelian groups. However, it is a generalisation appropriate for Lie groups. See the remark preceding Theorem 3.1 in §3.)

Our main result is the following.

Theorem A

Let ϕ be a hyperbolic map preserving a Hölder equilibrium state. Let α be a unipotent automorphism of a compact connected Lie group G. Let $f, g \in F_\theta(G)$. Suppose there exists a measurable solution $u : \Lambda \to G$ to $f = u\phi \cdot g \cdot \alpha u^{-1}$ a.e. Then there exists a Hölder continuous $u' \in F_\theta(G)$ such that $u' = u$ a.e. and $f = u'\phi \cdot g \cdot \alpha u'^{-1}$ everywhere.

Consider now the case of a (transitive) Anosov diffeomorphism ϕ. In [NT] it is proved that if f, g are C^k functions and u is a continuous solution to the untwisted equation $f = u\phi \cdot g \cdot u^{-1}$ then u is $C^{k-1+\varepsilon}$ for any $\varepsilon \in (0, 1)$. We adapt their proof to obtain:

Theorem B

Let ϕ be a C^k (transitive) Anosov diffeomorphism ($1 \leq k \leq \infty$ or $k = \omega$) on a compact manifold M. Let α be a unipotent automorphism of a compact connected Lie group G. Let $f, g : M \to G$ be C^k. Suppose $f = u\phi \cdot g \cdot \alpha u^{-1}$ a.e. (with respect to a Hölder equilibrium state) for a measurable u. Then there exists a continuous u' such that $u = u'$ a.e. and $f = u'\phi \cdot g \cdot \alpha u'^{-1}$ everywhere. Moreover u' is $C^{k-1+\varepsilon}$ for any $\varepsilon \in (0, 1)$. (When $k = 1, \infty, \omega$ we define $k - 1 + \varepsilon = 1, \infty, \omega$ respectively, for any $\varepsilon \in (0, 1)$.)

This note is organised as follows. In §2, we briefly review some definitions. In §3, we consider the case of a torus and prove the regularity theorem for a unipotent toral automorphism. We also discuss whether this is the best possible result. In §4, we prove the regularity theorem for an inner automorphism of a compact Lie group. This allows us to deduce the regularity theorem for any automorphism of a compact semi-simple Lie group. In §5, we 'glue' the results of §§3,4 together to obtain Theorem A. We prove Theorem B in §6. Some applications are given in §7.

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§2 Hyperbolic systems

We quickly review some definitions and basic properties of hyperbolic dynamical systems in order to establish notation.

Let A be an aperiodic $k \times k$ matrix with entries from $\{0, 1\}$. Let

$$\Sigma = \{(x_j)_{j=-\infty}^{\infty} \in \{1, \ldots, k\}^{\mathbb{Z}} \mid A_{x_j x_{j+1}} = 1\};$$

then Σ is a totally disconnected zero-dimensional compact metrisable space. Define the shift $\sigma : \Sigma \to \Sigma$ by $(\sigma x)_j = x_{j+1}$ and note that $\sigma(\Sigma) = \Sigma$. We call σ a *shift of finite type*.

For each $\theta \in (0, 1)$ we have a metric

$$d^\theta((x_j), (y_j)) = \theta^{n(x,y)},$$

where $n(x,y) = \sup\{|n| \mid x_j = y_j, |j| \leq n\}$. For a metric space (Y,ρ), let $F_\theta(Y)$ denote the space of all Lipschitz continuous functions with respect to ρ and d^θ:

$$F_\theta(Y) = \{f : \Sigma \to Y \mid \rho(fx, fy) \leq C d^\theta(x,y) \text{ for some } C > 0\}.$$

We shall abuse this definition and call $F_\theta(Y)$ the space of Hölder continuous functions of exponent θ. For $f \in F_\theta(Y)$, we define $|f|_\theta$ to be the least constant $C > 0$ for which $\rho(fx, fy) \leq C d^\theta(x,y)$.

Let $f \in F_\theta(\mathbb{R})$ and consider the quantity

$$h_\mu(\sigma) + \int f \, d\mu \tag{1}$$

as μ ranges over all σ-invariant probability measures. It is well-known that (1) is maximised for a unique invariant probability measure m. We call m the *equilibrium state of f*. Hölder equilibrium states enjoy various nice properties. For example, σ is a Bernoulli automorphism with respect to m, m is positive on non-empty open sets and has no atoms.

Let M be a compact smooth Riemannian manifold and let ϕ be a C^1 diffeomorphism of M. We call a compact ϕ-invariant subset Λ a *basic set* if the following hold:

1. the tangent bundle restricted to Λ can be written as a continuous Whitney sum $E^s \oplus E^u$ of $d\phi$-invariant sub-bundles for which there exist constants $C > 0, \lambda \in (0,1)$ such that

$$\|d\phi^n|_{E^s}\| \leq C\lambda^n, \; n \geq 0,$$
$$\|d\phi^{-n}|_{E^u}\| \leq C\lambda^n, \; n \geq 0;$$

2. the periodic orbits in Λ are dense in Λ and Λ is not a single orbit;

3. there is a dense orbit in Λ;

4. there is an open neighbourhood $U \supset \Lambda$ such that $\Lambda = \bigcap_{n=-\infty}^{\infty} \phi^n U$.

We call ϕ, when restricted to Λ, a *hyperbolic diffeomorphism*. Furthermore, we shall always assume that $\phi|_\Lambda$ is topologically mixing.

Give Λ the metric d induced from the Riemannian metric. Let (Y, ρ) be a metric space. For $\theta \in (0,1]$ we let $F_\theta(Y)$ denote the space of Hölder continuous functions $\Lambda \to Y$ of exponent θ. Again, we define $|f|_\theta$ to be the least possible Hölder constant.

If $f \in F_\theta(\mathbb{R})$ we can again define the equilibrium state of f as above. They have the same properties as those stated above.

Throughout §§3,4 it makes no difference whether we consider shifts of finite type or hyperbolic diffeomorphisms. Therefore we refer to them together as *hyperbolic maps*. However, in §5 we shall need to consider them separately and, in particular, use the fact that a hyperbolic diffeomorphism is a quotient of a shift of finite type in a nice way via Markov partitions. We give a quick review of the parts of this construction that will be needed in §5.

For $x \in \Lambda$ and small $\varepsilon > 0$, define

$$W_\varepsilon^s(x) = \{y \in M \mid d(\phi^n x, \phi^n y) \leq \varepsilon,\ n \geq 0\},$$
$$W_\varepsilon^u(x) = \{y \in M \mid d(\phi^{-n} x, \phi^{-n} y) \leq \varepsilon,\ n \geq 0\}.$$

These are tangent to E^s and E^u, respectively. If $z \in W_\varepsilon^s(x)$ then we have the estimate $d(\phi^n x, \phi^n z) \leq C\lambda^n d(x,z)$ for $n \geq 0$; if $z \in W_\varepsilon^u(y)$ then $d(\phi^{-n} y, \phi^{-n} z) \leq C\lambda^n d(y,z)$ for $n \geq 0$. If $x, y \in \Lambda$ are sufficiently close then $\langle x, y\rangle = W_\varepsilon^s(x) \cap W_\varepsilon^u(y)$ is a single point and this point lies in Λ.

The following describes the properties of Markov partitions that we shall need in the sequel.

Proposition 2.1

Let ϕ be a hyperbolic diffeomorphism. Then there exists a shift of finite type σ and a semi-conjugacy $\pi : \Sigma \to \Lambda$ (so that $\pi\sigma = \phi\pi$) such that the following hold.

1. *π is one-to-one on a residual subset of Λ and on a set of full measure for any equilibrium state. (Call a point $x \in \Lambda$ 'good' if it has a unique pre-image under π.)*

2. *There is a dense set of good periodic points for ϕ.*

3. *If $x, y \in \Lambda$ are good periodic points such that $\langle x, y\rangle$ is defined, then $\langle x, y\rangle$ is good.*

4. *π is Lipschitz continuous with respect to d^λ. In particular, if $f : \Lambda \to G \in F_\theta(G)$ then $f\pi : \Sigma \to G \in F_{\lambda^\theta}(G)$.*

Statements 1 and 4 are well-known ([Bo], for example). Statements 2 and 3 are easy consequences of well-known properties of Markov partitions, see [PP].

§3 Toral automorphisms

Let $G = \mathbb{R}^k/\mathbb{Z}^k$ be the k-torus for which we use additive notation. Let $\alpha \in \mathrm{Aut}(G)$ be an automorphism. Then we can identify α with a matrix in $Sl(k, \mathbb{Z})$.

We first consider the case when α is unipotent. In our terminology, this means that all the eigenvalues of α have modulus 1. It is well-known that this forces all the eigenvalues to be roots of unity. Hence, for some $p \geq 1$, α^p has 1 as its only eigenvalue. For definiteness, we fix p to be the least such $p \geq 1$ with this property. It follows that for some $n \geq 1$,

$$(\alpha^p - \mathrm{id})^n G = \{e\}; \tag{2}$$

here, id is the identity map on G and e is the group identity. We call the least positive integer n for which (2) holds the *unipotency order* of α^p.

Remark. Observe that (2) is equivalent to

$$\bigcap_{n=0}^{\infty} (\alpha^p - \mathrm{id})^n G = \{e\}.$$

This is the usual definition of what it means for α^p to be a unipotent automorphism of the abelian group G. Thus our definition of unipotent can be viewed as a generalisation of the usual definition to the case of a non-abelian Lie group. Note also that we say α is unipotent whereas one would usually say that some power of α is unipotent.

We prove the following.

Theorem 3.1
Let ϕ be a hyperbolic map preserving a Hölder equilibrium state. Let α be a unipotent toral automorphism. If $f \in F_\theta(G)$ is Hölder continuous and satisfies $f = u\phi - \alpha u$ a.e. for some measurable $u : \Lambda \to G$ then there exists $u' \in F_\theta(G)$ such that $u' = u$ a.e. and $f = u'\phi - \alpha u'$ everywhere.

Define
$$f^p = \sum_{j=0}^{p-1} \alpha^j f \phi^{p-1-j}.$$
Then $f^p \in F_\theta(G)$ as α is smooth. It is easy to see that if $f = u\phi - \alpha u$ a.e. then $f^p = u\phi^p - \alpha^p u$ a.e. Observe that ϕ^p is also a hyperbolic map. Hence it is sufficient to prove Theorem 3.1 under the assumption that all the eigenvalues of α are equal to 1. From now on, we assume that this is indeed the case.

Let us indicate how such an α may be built up as a finite number of toral extensions of the identity. For $0 \leq i \leq n$ define $G_i = (\alpha - \mathrm{id})^i G$. Then

$$G = G_0 \supset G_1 \supset \cdots \supset G_{n-1} \supset G_n = \{e\}. \tag{3}$$

By the minimality of n, each inclusion is strict. As

$$\alpha G_i = \alpha(\alpha - \mathrm{id})^i G = (\alpha - \mathrm{id})^i \alpha G = (\alpha - \mathrm{id})^i G = G_i,$$

we see that each G_i is α-invariant. Hence, there is a well-defined quotient automorphism

$$a_{i+1}^i : G_i/G_{i+1} \to G_i/G_{i+1} : g + G_{i+1} \mapsto \alpha(g) + G_{i+1},\ g \in G_i.$$

Lemma 3.2
We have $\alpha_{i+1}^i = \mathrm{id}$.

Proof. Let $g = (\alpha - \mathrm{id})^i h \in G_i$, $h \in G$. Then

$$\begin{aligned}
a_{i+1}^i(g + G_{i+1}) &= \alpha(\alpha - \mathrm{id})^i h - (\alpha - \mathrm{id})^i h + (\alpha - \mathrm{id})^i h + G_{i+1} \\
&= (\alpha - \mathrm{id})^{i+1} h + (\alpha - \mathrm{id})^i h + G_{i+1} \\
&= (\alpha - \mathrm{id})^i h + G_{i+1} \\
&= g + G_{i+1}
\end{aligned}$$

as $(\alpha - \mathrm{id})^{i+1} h \in G_{i+1}$. ❑

Lemma 3.3
Let $H = (\alpha - \mathrm{id})G$. Then H is a torus, G/H is a torus and $G = G/H \oplus H$. Moreover, α, when restricted to H, is a toral automorphism with all eigenvalues equal to 1 and unipotency order $n-1$.

Proof. Observe that H is the continuous image of a compact connected set, hence is itself compact and connected. It follows that H is a torus, as these are the only compact connected abelian Lie groups.

View $\alpha - \mathrm{id}$ as a linear map on $\mathbb{R}^k$. Then $(\alpha - \mathrm{id})\mathbb{R}^k$ is a subspace of $\mathbb{R}^k$ and projects to H under the natural projection $\pi_G : \mathbb{R}^k \mapsto \mathbb{R}^k/\mathbb{Z}^k = G$. Choose a complementary subspace V to $(\alpha - \mathrm{id})\mathbb{R}^k$. Then $\pi_G V$ can be identified with G/H. Hence, $G = G/H \oplus H$.

That α has unipotency order $n-1$ when restricted to H is clear from (3). ❑

Proof of Theorem 3.1. Suppose $f : \Lambda \to G$ is Hölder and α is an automorphism of G, all of whose eigenvalues are equal to 1. Let n denote the unipotency order of α. Suppose $f = u\phi - \alpha u$ a.e. for some measurable $u : \Lambda \to G$. We use induction on the unipotency order n of α to construct a Hölder version of u.

When $n = 1$ we have $(\alpha - \mathrm{id})G = \{e\}$ whence α is the identity. The result follows from Theorem 1.1.

Suppose the result is true for automorphisms with unipotency order $n-1$.

Let $H = (\alpha - \mathrm{id})G$. Then by Lemma 3.3, the restriction α_H of α to H has unipotency order $n-1$. By Lemma 3.2, the induced automorphism $\alpha_{G/H}$ on the quotient torus G/H is the identity. Let $\pi_{G/H} : G \to G/H$ be the natural projection.

Applying $\pi_{G/H}$ to our cocycle equation we have

$$\begin{aligned} \pi_{G/H} f &= \pi_{G/H} u\phi - \pi_{G/H} \alpha u \text{ a.e.} \\ &= (\pi_{G/H} u)\phi - \pi_{G/H} u \text{ a.e.} \end{aligned}$$

By Theorem 1.1, there exists a Hölder continuous $v'' : \Lambda \to G/H$ such that $\pi_{G/H} u = v''$ a.e. and $\pi_{G/H} f = v''\phi - v''$ everywhere.

Viewing G as $G/H \oplus H$, define $v'(x) = (v''(x), e_H)$, where e_H is the group identity in H. Then $v' \in F_\theta(G)$. Consider

$$f - v'\phi + \alpha v' = (u - v')\phi - \alpha(u - v') \text{ a.e.} \tag{4}$$

Now $\pi_{G/H}(u - v') = \pi_{G/H}u - v'' = 0$ a.e. whence $u - v' \in H$ a.e. Also, $\pi_{G/H}(f - v'\phi + \alpha v') = \pi_{G/H}f - v''\phi + v'' = 0$ everywhere (as $\pi_{G/H}\alpha = \pi_{G/H}$). Thus we can view (4) as a cocycle equation taking values in H.

By the inductive hypothesis, $u - v' = v$ a.e. where $v : \Lambda \to H \in F_\theta(H)$ and $f - v'\phi + \alpha v' = v\phi - \alpha v$ everywhere. Set $u' = v + v'$. Then $u \in F_\theta(G)$ and $u' = u$ a.e. Moreover, $f = u\phi - \alpha u$ everywhere. By induction, this proves the theorem. ❑

Remark. Let α be a unipotent toral automorphism. Consider the toral skew product $\phi_f(x, y) = (\phi x, f(x) + \alpha(y))$. Then one can view ϕ_f as a skew product (with trivial automorphism part) of a skew product of ϕ. We can then use Theorem 3.1 to obtain a version of Theorem 1.1 for certain cocycle equations over certain non-hyperbolic systems. Let us give an example to illustrate this. Let $G = \mathbb{R}/\mathbb{Z}$ and let $w \in F_\theta(G)$. Define the skew product

$$\phi_w : \Lambda \times G \to \Lambda \times G : (x, y) \mapsto (\phi x, w(x) + y).$$

Note that ϕ_w is not a hyperbolic map. Suppose $g : \Lambda \times G \to G \in F_\theta(\Lambda \times G)$ is independent of the second co-ordinate and that

$$g = v\phi_w - v \text{ a.e.} \tag{5}$$

for some measurable $v : \Lambda \times G \to G$. Suppose further that $v(x, y) = u(x) + dy$ for some measurable $u : \Lambda \to G$ and $d \in \mathbb{Z}$. Then (5) is equivalent to

$$\begin{pmatrix} w\phi - w \\ g \end{pmatrix} = \begin{pmatrix} w\phi \\ u\phi \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} \text{ a.e.}$$

By Theorem 3.1, u, hence v, has a Hölder continuous version.

To what extent is Theorem 3.1 the best possible? We can use an observation of Lind's [Li] to show that the regularity theorem *always* fails for hyperbolic toral automorphisms (*i.e. no* eigenvalues of modulus 1) in the following sense. Let α be a hyperbolic toral automorphism. Then there exists a hyperbolic map ϕ and a Hölder continuous (indeed analytic) $f \in F_\theta(G)$ such that $f = u\phi - \alpha u$ a.e. for a measurable function u, but u does not have a continuous version.

Proposition 3.4 ([Li])
Let α be a hyperbolic automorphism of the k-torus G (so α is a hyperbolic map in the sense of §1). Define $f : G \to G$ by $f(x) = x$, the identity map. Then f can be written as $f = u\alpha - \alpha u$ a.e. for some measurable $u : G \to G$ but u never has a continuous version.

Proof. As α is hyperbolic and therefore ergodic with respect to Haar measure, $f = u\alpha - \alpha u$ a.e. for some measurable u. Hence the automorphisms α_f and $\alpha \times \alpha$ (using the notation of §1) of the $2k$-torus are metrically isomorphic. Moreover, α_f and $\alpha \times \alpha$ can be represented by matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ \mathrm{id} & \alpha \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},$$

respectively (here, id denotes the $k \times k$ identity matrix). If u had a continuous version then α_f and $\alpha \times \alpha$ would be topologically conjugate, hence, by a result of Adler and Palais [AP], algebraically conjugate. We show that this never happens.

For a linear map L on $\mathbb{R}^l$ and an eigenvalue μ of L, define the generalised eigenspaces $E_n^\mu(L) = E_n(L) = \{v \in \mathbb{R}^l \mid (L - \mu)^n v = 0\}$. Let dim $E_n(L) = m_n(L)$.

Let μ be an eigenvalue of α and choose n least such that $E_n(\alpha) = E_{n+1}(\alpha)$. It is clear that $E_n(\alpha \times \alpha) = E_n(\alpha) \oplus E_n(\alpha)$ whence dim $E_n(\alpha \times \alpha) = 2m_n(\alpha)$. If $(a, b)^T \in E_n(\alpha_f)$ then

$$\begin{pmatrix} (\alpha - \mu)^n & 0 \\ n(\alpha - \mu)^{n-1} & (\alpha - \mu)^n \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

hence $a \in E_n(\alpha)$ and $-n(\alpha - \mu)^{n-1} a = (\alpha - \mu)^n b$. It follows that $b \in E_{n+1}(\alpha) = E_n(\alpha)$, hence $a \in E_{n-1}(\alpha)$. So $m_n(\alpha_f) = m_{n-1}(\alpha) + m_n(\alpha) < 2m_n(\alpha) = m_n(\alpha \times \alpha)$. Thus α_f is not similar to $\alpha \times \alpha$. ❑

Now consider the case when α has an eigenvalue of modulus one but no roots of unity occur as eigenvalues, the so-called *quasi-hyperbolic* case. Such an α is still ergodic, so Theorem 1.2 still holds and any f is a measurable α-coboundary. We now show that the regularity theorem always fails for quasi-hyperbolic toral automorphisms in the same sense as above.

Our example is easier to give using multiplicative notation. Let K denote the unit circle in $\mathbb{C}$. Let ϕ be a hyperbolic automorphism of the m-torus $\Lambda = K^m$ (so ϕ is a hyperbolic map). Let α be a quasi-hyperbolic automorphism of K^n. We show how to construct a Hölder continuous (indeed, analytic) $f : \Lambda \to K^n$ that cannot be written $f = u\phi \cdot \alpha u^{-1}$ for a continuous u. The key observation is the following.

Lemma 3.5
Suppose $f = u\phi \cdot \alpha u^{-1}$ where $f, u : \Lambda \to K^n$ are continuous. If f is null-homotopic then u is null-homotopic.

Proof. Write

$$u(x) = \gamma_u(x)(\exp(2\pi i U_1(x)), \ldots, \exp(2\pi i U_n(x)))$$

where $\gamma_u : \Lambda \to K^n$ is a homomorphism. Then it is easy to check that $\gamma_u \phi = \alpha \gamma_u$, and this is impossible unless $\gamma_u = 0$, as ϕ and α have different eigenvalues. ❑

Remark. In fact, Lemma 3.5 holds provided the characteristic polynomials of ϕ and α are coprime. The example we give below relies only on the validity of Lemma 3.5 and so continues to hold in this case.

Suppose f is continuous and null-homotopic and u is a continuous solution to $f = u\phi \cdot \alpha u^{-1}$. As u is null-homotopic, we can lift this equation to $\mathbb{R}^n$ to obtain (after possibly subtracting an integer)

$$F = U\phi - \alpha U \tag{6}$$

where F is a lift of f, U is a lift of u and α is viewed as a linear map on $\mathbb{R}^n$. Conversely, if $F, U : \Lambda \to \mathbb{R}^n$ are continuous and satisfy (6) then we can obtain a continuous solution $u = e^{2\pi i U}$ to $\exp(2\pi i F) = u\phi\cdot\alpha u^{-1}$. Therefore, it is sufficient to choose a Hölder $F : \Lambda \to \mathbb{R}^n$ such that (6) does not have a continuous solution U.

Let $\mu = e^{2\pi i\theta}$ be an eigenvalue of α of unit modulus. By diagonalising (6) and considering the eigenspace of μ, we obtain a solution to

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} U_1\phi \\ U_2\phi \end{pmatrix} - R(\theta)\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \tag{7}$$

where $F_i, U_i : \Lambda \to \mathbb{R}$ and $R(\theta)$ denotes the rotation of $\mathbb{R}^2$ by θ. Alternatively, (7) can be viewed as an equation in $\mathbb{C}$:

$$\hat{F} = \hat{U}\phi - \mu\hat{U}. \tag{8}$$

Let $\eta \in \hat{\Lambda}$ be an arbitrary non-zero character of Λ. Define $\hat{F} = \eta : \Lambda \to \mathbb{C}$ and observe that we can recover a Hölder continuous (indeed analytic) function $F : \Lambda \to \mathbb{R}^2/\mathbb{Z}^2$ by taking real and imaginary parts. Write

$$\hat{U} = \sum_{\gamma\in\hat{\Lambda}} u_\gamma\gamma$$

where $u_\gamma \in \mathbb{C}$ and $\sum_{\gamma\in\hat{\Lambda}} |u_\gamma|^2 < \infty$. Then (8) becomes

$$\eta = \sum_{\gamma\in\hat{\Lambda}} u_{\gamma\phi^{-1}}\gamma - \mu\sum_{\gamma\in\hat{\Lambda}}\gamma$$

so that

$$u_{\gamma\phi^{-1}} = \begin{cases} \mu u_\gamma & \gamma \neq \eta, \\ 1 + \mu u_\gamma & \gamma = \eta \end{cases} \tag{9}$$

Hence $u_{\eta\phi^{-p}} = \mu^{p-1}(1 + \mu u_\eta)$ for $p > 0$ so that $|u_{\eta\phi^{-p}}| = |1 + \mu u_\eta|$. As ϕ is hyperbolic, and in particular ergodic, all the $\eta\phi^{-p}$ are distinct. Hence, $u_\eta = -\mu^{-1}$. However, it also follows from (9) that for $p \geq 0$, $u_\eta = \mu^p u_{\eta\phi^p}$, whence $|u_{\eta\phi^P}| = |u_\eta|$ for $p \geq 0$. Hence $u_\eta = 0$, a contradiction.

Problem. Given a toral automorphism α, can one give a condition on $f \in F_\theta(G)$ that ensures f is of the form $f = u\phi - \alpha u$ for some Hölder continuous u? In particular, is there an analogue of Livšic's periodic data criterion (see [Lv, KH], etc.)?

§4 Inner automorphisms and semi-simple Lie groups

We first prove the regularity theorem for an inner automorphism of a compact Lie group. This follows from a simple modification to the proof of Theorem 1.1 given in [PP]. Throughout, we use the notation $U(Y)$ to denote the group of unitary operators on a Euclidean space Y. The key fact is the following result.

Proposition 4.1

Let $M(d)$ denote the space of $d \times d$ matrices over $\mathbb{C}$ with the Euclidean structure induced from $\mathbb{C}^{d^2}$. Let $\Psi : \Lambda \to U(M(d)) \in F_\theta(U(M(d)))$ and let $u : \Lambda \to M(d)$ be measurable. Let ϕ be a hyperbolic map preserving a Hölder equilibrium state. Suppose

$$u(\phi x) = \Psi(x)u(x) \text{ a.e.}$$

Then there exists $u' \in F_\theta(M(d))$ such that $u' = u$ a.e. and

$$u'(\phi x) = \Psi(x)u'(x)$$

everywhere.

Remarks.

1. Parry and Pollicott only proved this for a shift of finite type. Moreover, they concluded that $u' \in F_{\theta^{\frac{1}{2}}}(M(d))$. To see that $u' \in F_\theta(M(d))$, it is sufficient to show that $u' \in F_\theta(M(d))$ along stable and unstable 'manifolds' (Proposition 19.1.1 in [KH]). Let $\Psi^n(x) = \Psi(\phi^{n-1}x)\cdots\Psi(x) \in U(M(d))$. Then

   $$\Psi^n(y)(u'(x) - u'(y)) = u'(\phi^n x) - u'(\phi^n y) - (\Psi^n(x) - \Psi^n(y))u'(x).$$

   By ergodicity, we may assume that $\|u'\| = 1$ so that

   $$\|u'(x) - u'(y)\| \leq \|u'(\phi^n x) - u'(\phi^n y)\| + \|\Psi^n(x) - \Psi^n(y)\|. \quad (10)$$

   It is easy to see that when y belongs to the same stable 'manifold' as x, we have $\|\Psi^n(x) - \Psi^n(y)\| \leq \sum_{j=0}^{n-1} \|\Psi\|_\theta \theta^n d^\theta(x, y) \leq \text{Const.}\, d^\theta(x, y)$

for some constant independent of n. Letting $n \to \infty$ in (10) we see that

$$\|u'(x) - u'(y)\| \leq \text{Const.}\, d^{\theta}(x,y).$$

2. To deduce the hyperbolic diffeomorphism case, one can use the 'push-down' method described in [PP], a modification of which appears in §5.

Let G be a compact Lie group. Then for some d, G is a closed subgroup of $U(d)$, the $d \times d$ unitary matrices. Let $N(G)$ denote the normaliser of G in $Gl(d)$, the general linear group. Then any $H \in N(G)$ determines an automorphism α of G by conjugation: $\alpha(A) = HAH^{-1}$.

Theorem 4.2
Let ϕ be a hyperbolic map preserving a Hölder equilibrium state. Let G be a compact Lie group and α an automorphism of G arising from an element of the normaliser, as described above. Let $f, g \in F_\theta(G)$ and suppose $f = u\phi \cdot g \cdot \alpha u^{-1}$ a.e. for some measurable u. Then there exists $u' \in F_\theta(G)$ such that $u' = u$ a.e. and $f = u'\phi \cdot g \cdot \alpha u'^{-1}$ everywhere.

Proof. We modify the corresponding argument given in [PP] for the case when α is the identity.

Equip $M(d)$ with the inner product $\langle A, A' \rangle = \text{trace}\, AA'^*$ ($*$ denotes the Hermitian conjugate). Define $\Psi : \Lambda \to U(M(d))$ by

$$\Psi(x)A = f(x)HAH^{-1}g(x)^{-1}.$$

It is easy to check that Ψ is linear. To see that Ψ is an unitary operator, observe that

$$\begin{aligned} \langle \Psi(x)A, \Psi(x)A' \rangle &= \text{trace}\, f(x)HAH^{-1}g(x)^{-1}g(x)(H^{-1})^*A'^*H^*f(x)^{-1} \\ &= \text{trace}\, AA'^* \\ &= \langle A, A' \rangle \end{aligned}$$

(using the facts that $(H^{-1})^* = (H^*)^{-1}$ and $\text{trace}\, BC = \text{trace}\, CB$).

If $f = u\phi \cdot g \cdot \alpha u^{-1}$ a.e. then

$$\begin{aligned} u(\phi x) &= f(x)\alpha u(x) g(x)^{-1} \text{ a.e.} \\ &= f(x) H u(x) H^{-1} g(x)^{-1} \text{ a.e.} \\ &= \Psi(x) u(x) \text{ a.e.} \end{aligned}$$

and Proposition 4.1 applies. ❏

Corollary 4.3

The regularity theorem holds for an inner automorphism of a compact Lie group.

We now consider automorphisms of compact connected semi-simple Lie groups. Our first observation is that *all* automorphisms of such groups are unipotent (in the sense defined above). This fact is surely known, but we have been unable to locate a reference.

Proposition 4.4

Let α be an automorphism of a compact connected semi-simple Lie group G. Then the eigenvalues of $d\alpha$ are roots of unity.

Proof. It is is well-known ([Ti], for example) that the inner automorphisms have finite index in the automorphism group of a compact semi-simple Lie group. Thus we may choose n such that α^n is inner. It is clear that the eigenvalues of $d\alpha^n$ have modulus 1; this follows from the existence of a left-right invariant metric. We prove that, in fact, the eigenvalues are roots of unity. A standard result allows us to assume that, for some d, G is a closed subgroup of $O(d)$, the $d \times d$ orthogonal matrices (previously we had embedded G in a unitary group $U(d)$). Let $\mathfrak{g}$ be the Lie algebra of G and note that it is a sub-algebra of $\mathfrak{o}(d) = \{X \in M(d) \mid X^T + X = 0\}$.

Suppose $\alpha(A) = HAH^{-1}$ for some $H \in G$. Then for $X \in \mathfrak{g}$,

$$\alpha(\exp X) = H \sum_{r=0}^{\infty} \frac{X^r}{r!} H^{-1} = \sum_{r=0}^{\infty} \frac{(HXH^{-1})^r}{r!} = \exp HXH^{-1}.$$

Hence $d\alpha(X) = HXH^{-1}$.

Suppose μ is an eigenvalue of $d\alpha$ with eigenvector $X \neq 0$. Then

$$\mu X = d\alpha(X) = HXH^{-1}.$$

Thus, when viewed as matrices, μX and X are similar. Therefore they have the same eigenvalues. We need the following.

Sub-lemma. *If $X \in \mathfrak{o}(d)$ has zero as its only eigenvalue then $X = 0$.*

Hence multiplying the spectrum of X by μ merely permutes the eigenvalues of X. As this is a finite set containing a non-zero entry, some power of μ is equal to 1. Hence μ is a root of unity. ❑

Proof of sublemma. Let $X = (x_{ij}) \in \mathfrak{o}(d)$. Then $x_{ii} = 0$ and $x_{ij} = -x_{ji}$ for $i \neq j$. An easy calculation shows that the coefficient of ν^{d-2} in $p(\nu)$, the characteristic polynomial of X, is given by $\sum_{i>j} x_{ij}^2$. If the eigenvalues of X are all zero then $p(\nu) = \nu^d$ whence $x_{ij} = 0$ for $i \neq j$. Hence $X = 0$. ❑

We now prove the regularity theorem for automorphisms of compact connected semi-simple Lie groups.

Theorem 4.5
Let ϕ be a hyperbolic map preserving a Hölder equilibrium state. Let G be a compact connected semi-simple Lie group and α an automorphism of G. Let $f, g \in F_\theta(G)$ and suppose $f = u\phi \cdot g \cdot \alpha u^{-1}$ a.e. for some measurable $u : \Lambda \to G$. Then there exists $u' \in F_\theta(G)$ such that $u' = u$ a.e. and $f = u'\phi \cdot g \cdot \alpha u'^{-1}$ everywhere.

Proof. Choose $n > 0$ such that α^n is inner. Define

$$\begin{aligned} f^n(x) &= f(\phi^{n-1}x)\alpha f(\phi^{n-2}x)\cdots\alpha^{n-1}f(x), \\ g^n(x) &= g(\phi^{n-1}x)\alpha g(\phi^{n-2}x)\cdots\alpha^{n-1}g(x). \end{aligned}$$

Note that $f^n, g^n \in F_\theta(G)$. Then it is easily seen that $f^n = u\phi^n \cdot g^n \cdot \alpha^n u^{-1}$ a.e. As ϕ^n is a hyperbolic map, Corollary 4.3 applies. ❑

§5 Unipotent automorphisms of compact connected Lie groups

In this section, we complete the proof of Theorem A. Our strategy is as follows. We first make some preliminary remarks on the structure of compact connected Lie groups. This allows us to prove Theorem A for a shift of finite type. To obtain Theorem A for a hyperbolic diffeomorphism, we want to employ the 'push-down' technique used to prove Theorem 1.1 in [PP]. However, a crucial fact in [PP] is the existence of a left-right invariant metric. For the method in [PP] to work verbatim, we would need a left-right invariant metric that is also invariant under α. Such metrics do not always exist. (For example, the toral automorphism given by

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

does not preserve any invariant metric.) However, when α is unipotent, a left-right invariant metric can only expand distances at an arbitrarily small exponential rate, and this suffices to prove the theorem.

Let G be a compact connected Lie group. Let $\hat{A}$ denote the identity component of the centre of G; hence $\hat{A}$ is a torus. Let $\hat{S}$ denote the product of the simple normal subgroups of G (of which there are only finitely many); hence $\hat{S}$ is semi-simple. Then a standard result allows us to write

$$G = \frac{\hat{A} \times \hat{S}}{D}$$

where D is a discrete normal subgroup contained in the centre of $\hat{A} \times \hat{S}$. Let $\pi_D : \hat{A} \times \hat{S} \to G$ denote the natural projection. Let α be an automorphism of G. Then α lifts to an automorphism $\hat{\alpha}$ of $\hat{A} \times \hat{S}$ such that $\pi_D \hat{\alpha} = \alpha \pi_D$. Moreover, $\hat{A}$ and $\hat{S}$ are $\hat{\alpha}$-invariant.

Let $A = \pi_D \hat{A}$, $S = \pi_D \hat{S}$. Then A is a torus and S is semi-simple. The properties of A and S that are important for our work are listed below.

Lemma 5.1

1. *A and S are α-invariant normal subgroups of G;*
2. *$A \cap S$ is a discrete α-invariant normal subgroup of G;*
3. *$G/A \cap S \cong A/A \cap S \times S/A \cap S$;*
4. *$A/A \cap S$ is a torus and $S/A \cap S$ is semi-simple.*

By Lemma 5.1.2, there is a well-defined quotient automorphism on $G/A \cap S$. We continue to denote this automorphism by α. By Lemma 5.1.1, both $A/A \cap S$ and $S/A \cap S$ are α-invariant.

Remark. As both D and $A \cap S$ are discrete, the groups $\hat{A} \times \hat{S}$, G and $G/A \cap S$ are all locally isomorphic. Hence, the eigenvalues of $d\alpha$ are the same, irrespective of whether α is viewed as an automorphism of $\hat{A} \times \hat{S}$, G or $G/A \cap S$.

Proof of Theorem A (for a shift of finite type). Let $\sigma : \Sigma \to \Sigma$ be a shift of finite type. Our cocycle equation takes the form

$$f = u\sigma \cdot g \cdot \alpha u^{-1} \text{ a.e.} \tag{11}$$

where $f, g \in F_\theta(G)$ and $u : \Sigma \to G$ is measurable. Apply the projection $\pi_{A\cap S} : G \to G/A \cap S$ to (11) to obtain

$$\pi_{A\cap S} f = (\pi_{A\cap S} u)\sigma \cdot \pi_{A\cap S} g \cdot \alpha(\pi_{A\cap S} u)^{-1} \text{ a.e.}$$

By Lemma 5.1.3, this is an equation holding in the direct product of a torus and a semi-simple Lie group. Observe that α is the direct product of a unipotent toral automorphism and an automorphism of a semi-simple Lie group. By Theorems 3.1 and 4.5, there exists a Hölder continuous $v \in F_\theta(G/A \cap S)$ such that

$$\begin{aligned} \pi_{A\cap S} f &= v\sigma \cdot \pi_{A\cap S} g \cdot \alpha v^{-1} \text{ everywhere} \\ v &= \pi_{A\cap S} u \text{ a.e.} \end{aligned}$$

As Σ is totally disconnected, v lifts continuously to any covering space of $G/A \cap S$, and in particular to G. Hence, we can find a continuous $w : \Sigma \to G$ such that $\pi_{A\cap S} w = v$. As $\pi_{A\cap S}$ is a local isomorphism, $w \in F_\theta(G)$. By construction, $\pi_{A\cap S} w = \pi_{A\cap S} u$ a.e. Hence, there exists a measurable $p : \Sigma \to A \cap S$ such that

$$p \cdot w = u \text{ a.e.} \tag{12}$$

We claim that p has a continuous version. First observe that, as $A \cap S$ is a discrete (hence finite) α-invariant subgroup of G, some power of α restricted to $A \cap S$ is the identity. Suppose $\alpha^n|_{A\cap S} = \mathrm{id}$. Define, as usual,

$$f^n(x) = f(\sigma^{n-1}x)\alpha f(\sigma^{n-2}x) \cdots \alpha^{n-1} f(x)$$

and similarly for g^n. Then

$$f^n = u\sigma^n \cdot g^n \cdot \alpha^n u^{-1} \text{ a.e.} \tag{13}$$

Substituting (12) into (13), we see that

$$\begin{aligned} f^n &= p\sigma^n \cdot w\sigma^n \cdot g^n \cdot \alpha^n w^{-1} \cdot \alpha^n p^{-1} \text{ a.e.} \\ &= p\sigma^n \cdot w\sigma^n \cdot g^n \cdot \alpha^n w^{-1} \cdot p^{-1} \text{ a.e.} \end{aligned}$$

As f^n, $w\sigma^n \cdot g^n \cdot \alpha^n w^{-1} \in F_\theta(G)$ and σ^n is a shift of finite type, we can use Theorem 1.1 to find $p' \in F_\theta(A \cap S)$ such that $p' = p$ a.e. and the above equation holds everywhere for p'.

Define $u = p' \cdot w$. Then $u' \in F_\theta(G)$, $u' = u$ a.e. and $f = u'\sigma \cdot g \cdot \alpha u'^{-1}$ everywhere. ❑

We now want to prove Theorem A for hyperbolic diffeomorphisms. As stated above, we will need the fact that, in a sense, α^n expands distances at an arbitrarily small exponential rate.

Proposition 5.2
Let ρ be a left-right invariant Riemannian metric on G. Let $\delta > 0$. Then there exists $C_\delta > 0$ such that for all $n \in \mathbb{Z}$ and for all $g, h \in G$,

$$\rho(\alpha^n g, \alpha^n h) \leq C_\delta (1+\delta)^{|n|} \rho(g, h).$$

Proof. For $g \in G$ let $T_g G$ denote the tangent space of g and identify $T_e G$ with $\mathfrak{g}$, the Lie algebra of G. Let $\|\cdot\|_g$ denote the norm on $T_g G$ induced by the Riemannian metric giving rise to ρ. For $h \in G$, let $l_h, r_h : G \to G$ denote left and right multiplication by h, respectively. Then for $X \in T_g G$,

$$\|dl_h X\|_{hg} = \|X\|_g = \|dr_h X\|_{gh}.$$

Define a norm $|||\cdot|||_e$ on $\mathfrak{g}$ by

$$|||X|||_e = \sum_{n=0}^{\infty} \frac{\|d\alpha^n X\|_e}{(1+\delta)^n}.$$

This is well-defined by the n^{th} root test, as

$$\left(\frac{\|d\alpha^n X\|_e}{(1+\delta)^n}\right)^{1/n} \leq \frac{\|d\alpha^n\|^{1/n} \|X\|_e^{1/n}}{(1+\delta)} \to \frac{1}{1+\delta} < 1$$

by the spectral radius formula (recall that, by definition, the spectral radius of the derivative of a unipotent automorphism is 1). Let

$$C_\delta^+ = \sum_{n=0}^{\infty} \frac{\|d\alpha^n\|}{(1+\delta)^n} < \infty.$$

Then it is easy to see that for $X \in \mathfrak{g}$,

$$\|X\|_e \leq |||X|||_e \leq C_\delta^+ \|X\|_e. \tag{14}$$

Moreover,

$$|||d\alpha^n X|||_e \leq (1+\delta)^n |||X|||_e \tag{15}$$

for $n \geq 0$. Combining (14) and (15) we see that

$$\|d\alpha^n X\|_e \leq C_\delta^+ (1+\delta)^n \|X\|_e \tag{16}$$

for $n \geq 0$.

Let $g, h \in G$ and let $\gamma : [0,1] \to G$ be a path between them. Then $\alpha^n \gamma$ is a path between $\alpha^n g$ and $\alpha^n h$. We have

$$\begin{aligned}
\int_0^1 \|(\alpha^n\gamma)'(t)\|_{\alpha^n\gamma(t)}\, dt &= \int_0^1 \|d\alpha^n(\gamma'(t))\|_{\alpha^n\gamma(t)}\, dt \\
&= \int_0^1 \|dl_{\alpha^n\gamma(t)^{-1}} d\alpha^n(\gamma'(t))\|_e\, dt \\
&= \int_0^1 \|d\alpha^n dl_{\gamma(t)^{-1}}(\gamma'(t))\|_e\, dt \\
&\leq C_\delta^+(1+\delta)^n \int_0^1 \|dl_{\gamma(t)^{-1}}(\gamma'(t))\|_e\, dt \\
&= C_\delta^+(1+\delta)^n \int_0^1 \|\gamma'(t)\|_{\gamma(t)}\, dt.
\end{aligned}$$

Taking the infimum over all such paths γ, we have

$$\rho(\alpha^n g, \alpha^n h) \leq C_\delta^+(1+\delta)^n \rho(g,h) \tag{17}$$

for $n \geq 0$. Repeating the above construction using α^{-1} in place of α, we obtain another constant C_δ^- for which (17) is true for $n \leq 0$. Take $C_\delta = \max\{C_\delta^-, C_\delta^+\}$. ❑

We are nearly in a position to prove Theorem A for hyperbolic diffeomorphisms. We first prove some preliminary estimates.

Let $f, g \in F_\theta(G)$. Recall that λ is the exponential rate of contraction and expansion of ϕ on stable and unstable manifolds. Choose $\delta > 0$ such that

$$(1+\delta)\lambda^\theta < 1; \tag{18}$$

this is possible as both $\lambda, \theta \in (0,1)$. Define, as usual,

$$f^n(x) = f(\phi^{n-1}x)\alpha f(\phi^{n-2}x)\cdots\alpha^{n-1}f(x)$$

so that

$$f^{n+1}(x) = f(\phi^n x)\alpha f^n(x).$$

For $z \in W^s_\varepsilon(x)$ and $n \geq 0$, define

$$F^n(z,x) = \alpha^{-n}(f^n(z)^{-1}f^n(x)).$$

Similarly, define g^n and G^n.

Lemma 5.3

Let $z \in W^s_\varepsilon(x)$. Then

1. $F(z,x) = \lim_{n\to\infty} F^n(z,x)$ *exists;*
2. $\rho(F(z,x), e) \leq \mathrm{Const.}\, d(x,z)^\theta$, *where the constant is independent of* x, z.

Proof. We prove 1. Observe that

$$\begin{aligned}
&\rho(F^{n+1}(z,x), F^n(z,x)) \\
&\quad = \rho(\alpha^{-(n+1)}(f^{n+1}(z)^{-1}f^{n+1}(x)), \alpha^{-n}(f^n(z)^{-1}f^n(x))) \\
&\quad = \rho(\alpha^{-(n+1)}(\alpha f^n(z)^{-1}f(\phi^n z)^{-1}f(\phi^n x)\alpha f^n(x)), \alpha^{-n}(f^n(z)^{-1}f^n(x))) \\
&\quad \leq C_\delta(1+\delta)^{n+1}\rho(\alpha f^n(z)^{-1}f(\phi^n z)^{-1}f(\phi^n x)\alpha f^n(x), \alpha f^n(z)^{-1}\alpha f^n(x)) \\
&\quad = C_\delta(1+\delta)^{n+1}\rho(f(\phi^n x), f(\phi^n z)) \\
&\quad \leq C_\delta(1+\delta)^{n+1}|f|_\theta d(\phi^n x, \phi^n z)^\theta \\
&\quad \leq C_\delta(1+\delta)^{n+1}|f|_\theta C^\theta\lambda^{n\theta}d(x,z)^\theta \\
&\quad = \mathrm{Const.}((1+\delta)\lambda^\theta)^n d(x,z)^\theta
\end{aligned}$$

where the constant is independent of x, z and n. As $(1+\delta)\lambda^\theta < 1$, this estimate tends to zero exponentially fast as $n \to \infty$. The result follows easily.

We prove 2. Note that

$$\begin{aligned}
\rho(F^n(z,x),e) &= \rho(\alpha^{-n} f^n(x), \alpha^{-n} f^n(z)) \\
&\leq \sum_{j=0}^{n-1} \rho(\alpha^{-(1+j)} f(\phi^j x), \alpha^{-(1+j)} f(\phi^j z))
\end{aligned}$$

(using the triangle inequality n times and the left-right invariance of ρ). Hence,

$$\begin{aligned}
\rho(F^n(z,x),e) &\leq C_\delta \sum_{j=0}^{n-1} (1+\delta)^{1+j} \rho(f(\phi^j x), f(\phi^j z)) \\
&\leq \text{Const.} \sum_{j=0}^{\infty} ((1+\delta)\lambda^\theta)^j d(x,z)^\theta
\end{aligned}$$

where the constant is independent of x and z. By the assumption on δ, this sum is finite and the result follows. ❑

We can now complete the proof of Theorem A for hyperbolic diffeomorphisms using the method in [PP].

Proof of Theorem A (for a hyperbolic diffeomorphism). Suppose f and g are as above and u is a measurable function such that $f = u\phi \cdot g \cdot \alpha u^{-1}$ a.e. Let $\pi : \Sigma \to \Lambda$ be a semi-conjugacy as described in Proposition 2.1. Let $\hat{f} = f\pi$, $\hat{g} = g\pi$ and $\hat{u} = u\pi$. Then $\hat{f}, \hat{g} \in F_{\theta\lambda}(G)$ and $\hat{u}$ is measurable. Moreover, $\hat{f} = \hat{u}\sigma \cdot \hat{g} \cdot \alpha \hat{u}^{-1}$ a.e. (with respect to a Hölder equilibrium state for σ). By Theorem A in the case of a shift of finite type, we can find a Hölder continuous $\hat{u}' \in F_{\theta\lambda}(G)$ such that $\hat{u}' = \hat{u}$ a.e. and $\hat{f} = \hat{u}'\sigma \cdot \hat{g} \cdot \alpha \hat{u}'^{-1}$ everywhere. We show that $\hat{u}' = u'\pi$ for some $u' : \Lambda \to G \in F_\theta(G)$.

Recall from §2 that a point $x \in \Lambda$ is called *good* if it has a unique preimage under π. Let $x, y \in \Lambda$ be good periodic points. Suppose $\pi\hat{x} = x$, $\pi\hat{y} = y$. Define u' on the orbit of x by $u'(\phi^n x) = \hat{u}'(\sigma^n x)$, and similarly define u' on the orbit of y. Thus we have defined u' on a dense subset of Λ. Suppose x and y are sufficiently close so that $z = \langle x, y \rangle$ is defined. Then z

is good and we may define u' on the orbit of z as above. When u' is defined, clearly we have $f = u'\phi \cdot g \cdot \alpha u'^{-1}$. We claim that u' extends to a Hölder continuous function defined on Λ. It then follows that the cocycle equation holds everywhere for u'. Moreover, $u' = u$ a.e. as π is one-to-one on a set of full measure. Thus Theorem A is proved.

It remains to check that u' does indeed extend to Λ. We claim the following holds:

$$F(z,x) = u'(z)G(z,x)u'(x)^{-1}. \tag{19}$$

To see this, note that

$$\begin{aligned} F^n(z,x) &= \alpha^{-n}(f^n(z)^{-1}f^n(x)) \\ &= u'(z)\alpha^{-n}(g^n(z)^{-1}u'(\phi^n z)^{-1}u'(\phi^n x)g^n(x))u'(x)^{-1}. \end{aligned} \tag{20}$$

Moreover,

$$\begin{aligned} &\rho(G^n(z,x), \alpha^{-n}(g^n(z)^{-1}u'(\phi^n z)^{-1}u'(\phi^n x)g^n(x))) \\ &= \rho(\alpha^{-n}(g^n(z)^{-1}g^n(x)), \alpha^{-n}(g^n(z)^{-1}u'(\phi^n z)^{-1}u'(\phi^n x)g^n(x))) \\ &\le C_\delta(1+\delta)^n \rho(u'(\phi^n z), u'(\phi^n x)) \\ &= C_\delta(1+\delta)^n \rho(\hat{u}'(\sigma^n z), \hat{u}'(\sigma^n x)) \\ &\le C_\delta(1+\delta)^n |\hat{u}'|_{\theta\lambda} \lambda^{\theta n} \end{aligned}$$

and this tends to zero as $n \to \infty$ by the assumption on δ. Letting $n \to \infty$ in (20) yields (19).

Define $\psi = \phi^{-1}$, $\beta = \alpha^{-1}$, $f' = \beta f \psi$ and $g' = \beta g \psi$. Then $f' = \beta u \cdot g' \cdot (u\psi)^{-1}$ a.e. Define

$$f'^n = \beta^{n-1} f' \cdot \beta^{n-2} f'\psi \cdots f'\psi^{n-1}$$

and let

$$F'(y,z) = \lim_{n\to\infty} \beta^{-n}(f'^n(y)f'^n(z)^{-1});$$

this makes sense as $z \in W^u_\varepsilon(y)$. Then all of the above estimates on F and G continue to hold for F' and G'. In particular,

$$F'(y,z) = u'(y)G'(y,z)u'(z)^{-1} \tag{21}$$

Combining (21) with (19) gives

$$u'(y) = F'(y,z)F(z,x)u'(x)G(z,x)^{-1}G'(y,z)^{-1}.$$

By Lemma 5.3.2 (and its analogues for F' and G') and the left-right invariance of ρ, we have

$$\begin{aligned}\rho(u'(x),u'(y)) &\le \rho(F(z,x),e)+\rho(G(z,x),e)\\ &\quad +\rho(F'(y,z),e)+\rho(G'(y,z),e)\\ &\le \text{Const.}\,(d(x,z)^{\theta}+d(y,z)^{\theta})\end{aligned}$$

for some constant independent of x, y, z. By Proposition 19.1.1 in [KH], u' is Hölder continuous of exponent θ on the dense set of 'good' periodic orbits. Hence u' extends (uniquely) to a function $u' \in F_\theta(G)$. This completes the proof of Theorem A. ❏

§6 Higher regularity results

In this section we prove Theorem B. Let ϕ be a C^k (transitive) Anosov diffeomorphism M. Let $f, g : M \to G$ be C^k and such that $f = u\phi \cdot g \cdot \alpha u^{-1}$ a.e. (with respect to a Hölder equilibrium state) where α is unipotent. By Theorem A, we may assume $f = u\phi \cdot g \cdot \alpha u^{-1}$ everywhere and u is Lipschitz.

In [NT] the analogue of Theorem B is proved in the case where $\alpha = \text{id}$. In fact, Niţică and Török consider cocycles taking values in the diffeomorphism group of a compact manifold. As our cocycles take values in a compact Lie group, the proof simplifies considerably.

We will need the following result. The case when $1 \le k \le \infty$ is due to Journé [Jo] and the analytic case is due to de la Llave [Ll].

Proposition 6.1

Let ϕ be a C^k Anosov diffeomorphism on M ($1 \le k \le \infty$ or $k = \omega$). Let $f : M \to U \subset \mathbb{R}^n$ be uniformly C^k on stable and unstable manifolds. Then f is $C^{k-1+\varepsilon}$ for any $\varepsilon \in (0,1)$.

Hence it is sufficient to prove that u is C^k along stable manifolds.

We will need the notion of a partially hyperbolic dynamical system. Recall that a diffeomorphism $\Phi : M \to M$ is *partially hyperbolic* if the tangent bundle TM has a continuous splitting

$$TM = E^s \oplus E^c \oplus E^u$$

into $d\Phi$-invariant sub-bundles for which there exist constants $C > 0$, $0 < \lambda < 1 \leq \mu$ such that $\lambda\mu < 1$ and

$$\begin{aligned} \|d\Phi^n|_{E^s}\| &\leq C_1\lambda^n, \\ \|d\Phi^{-n}|_{E^u}\| &\leq C_2\lambda^n, \\ C_3\mu^{-n} \leq \|d\Phi^n|_{E^c}\| &\leq C_4\mu^n, \end{aligned}$$

for all $n > 0$ and some constants $C_1, \ldots, C_4 > 0$. Hence $d\Phi$ expands E^u more sharply than it expands E^c and contracts E^s more sharply than it contracts E^c.

The following proposition lists the properties of partially hyperbolic diffeomorphisms that we shall need. The case when $1 \leq k \leq \infty$ follows from Theorem 5.5 in [HPS] and the analytic case can be found in [NT].

Proposition 6.2
Let Φ be a C^k partially hyperbolic diffeomorphism.

1. *The distributions E^s, E^u are integrable. We call the resulting manifolds the stable and unstable manifolds, respectively. They are C^k submanifolds and depend continuously on x in the C^k topology on embeddings.*

2. *The stable manifold $W^s(x)$ through x is characterised as follows. We have $y \in W^s(x)$ if and only if for some (or any) ν with $\lambda < \nu < \mu^{-1}$ we have*
$$\frac{d(\Phi^n x, \Phi^n y)}{\nu^n} \to 0$$
as $n \to \infty$. A similar criterion holds for unstable manifolds.

As in the Introduction, we define for $f : M \to G$ the diffeomorphism $\phi_f : M \times G \to M \times G : (x, y) \mapsto (\phi x, f(x)\alpha(y))$. By using Theorem 2.2 of [BP] and the unipotency assumption on α we have:

Proposition 6.3
Let $f : M \to G$ be C^k. Then ϕ_f is a C^k partially hyperbolic diffeomorphism and we can take $\mu = 1 + \delta$ in the definition of partial hyperbolicity, for any small $\delta > 0$. Moreover, if $W^s_f(x_0, y_0)$ denotes the stable manifold of ϕ_f through $(x_0, y_0) \in M \times G$ then $\pi W^s_f(x_0, y_0) = W^s(x_0)$, the stable manifold of ϕ through x. Hence we can write

$$W^s_f(x_0, y_0) = \{(x, \gamma^f_{(x_0,y_0)}(x)) \mid x \in W^s(x_0)\}$$

where $\gamma^f_{(x_0,y_0)} : M \to G$ is a C^k function whose first k derivatives depend continuously on (x_0, y_0).

Proof of Theorem B. Define $U : M \times G \to M \times G : (x, y) \mapsto (x, u(x)y)$. Then clearly U is Lipschitz and $U\phi_g = \phi_f U$.

We claim that $UW^s_g(x_0, y_0) \subset W^s_f(U(x_0, y_0))$. To see this, let $(x, y) \in W^s_g(x_0, y_0)$. Then if $(1 + \delta)\lambda < \nu < 1$ we have

$$\begin{aligned} \frac{d(\phi^n_f U(x, y), \phi^n_f(x_0, y_0))}{\nu^n} &= \frac{d(U\phi^n_g(x, y), U\phi^n_g(x_0, y_0))}{\nu^n} \\ &\leq |U|_{\mathrm{Lip}} \frac{d(\phi^n_g(x, y), \phi^n_g(x_0, y_0))}{\nu^n} \\ &\to 0 \end{aligned}$$

as $n \to \infty$. By Proposition 6.2, $U(x, y) \in W^s_f(U(x_0, y_0))$.

Take $(x, y) \in W^s_g(x_0, y_0)$. Then

$$(x, y) = (x, \gamma^g_{(x_0,y_0)}(x)). \tag{22}$$

Moreover, as $U(x, y) \in W^s_f(U(x_0, y_0))$, we can write

$$(x, u(x)y) = (x, \gamma^f_{U(x_0,y_0)}(x)). \tag{23}$$

Combining (22) and (23) we see that

$$u(x) = \gamma^f_{U(x_0,y_0)}(x)\gamma^g_{(x_0,y_0)}(x)^{-1}$$

and the right-hand side is uniformly C^k along stable manifolds. Similarly, u is uniformly C^k along unstable manifolds. As is remarked above, this suffices to prove the theorem. ❑

Remark. Let α be an automorphism of G such that the spectrum of $d\alpha$ is contained in the annulus $\{z \in \mathbb{C} \mid \lambda < |z| < \lambda^{-1}\}$. (We say such an α is *slow with respect to* ϕ.) Let f, g be C^k functions and suppose $f = u\phi \cdot g \cdot \alpha u^{-1}$ everywhere for some continuous u. Then the above proof shows that u is $C^{k-1+\varepsilon}$ for any $\varepsilon \in (0,1)$.

§7 Applications

Livšic regularity fails for non-compact non-connected Lie groups. We give an example to show that Theorem 1.1 fails if G is a non-compact non-connected Lie group. It should be noted that Theorem 1.1 does hold for trivial cocycles (*i.e.* f such that $f = u\phi \cdot u^{-1}$ a.e.) taking values in a Lie subgroup of $Gl(n, \mathbb{C})$, provided the cobounding function u and its inverse u^{-1} are both essentially bounded [NP].

Let $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$ be the group of affine transformations of the n-torus $\mathbb{R}^n/\mathbb{Z}^n$. Then

$$\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n) = Sl(n, \mathbb{Z}) \ltimes \mathbb{R}^n/\mathbb{Z}^n$$

with group structure given by

$$(A, a)(B, b) = (AB, Ab + a).$$

Let $\Lambda = \mathbb{R}^n/\mathbb{Z}^n$ and let $\phi : \Lambda \to \Lambda$ be a hyperbolic toral automorphism with corresponding matrix $\alpha \in Sl(n, \mathbb{Z})$. Define $\bar{f} = (F, f)$, $\bar{g} = (G, g) : \Lambda \to Sl(n, \mathbb{Z}) \ltimes \mathbb{R}^n/\mathbb{Z}^n$ by

$$\begin{aligned} F &: \Lambda \to Sl(n, \mathbb{Z}) : x \mapsto \alpha, \quad & G &: \Lambda \to Sl(n, \mathbb{Z}) : x \mapsto \alpha, \\ f &: \Lambda \to \mathbb{R}^n/\mathbb{Z}^n : x \mapsto x, \quad & g &: \Lambda \to \mathbb{R}^n/\mathbb{Z}^n : x \mapsto 0. \end{aligned}$$

Proposition 7.1
With the above definitions, $\bar{f}$ and $\bar{g}$ are measurably cohomologous: i.e. *there exists a measurable $\bar{u} : \Lambda \to \mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$ such that $\bar{f} = \bar{u}\phi \cdot \bar{g} \cdot \bar{u}^{-1}$* a.e. *but $\bar{u}$ does not have a continuous version. Moreover, $\bar{u}$ and $\bar{u}^{-1}$ may be taken to be essentially bounded and $\bar{u}$ takes values in just one connected component of $\mathrm{Aff}(\mathbb{R}^n/\mathbb{Z}^n)$.*

Proof. Write $\bar{u} = (U, u)$ and note that $\bar{u}^{-1} = (U^{-1}, -U^{-1}u)$. Then it is easy to check that $\bar{f} = \bar{u}\phi \cdot \bar{g} \cdot \bar{u}^{-1}$ a.e. is equivalent to

$$F = U\phi \cdot G \cdot U^{-1} \text{ a.e.} \tag{24}$$

$$f = u\phi - U\phi g - U\phi \cdot G \cdot U^{-1}u \text{ a.e.} \tag{25}$$

Define $U : \Lambda \to Sl(n, \mathbb{Z})$ by $U(x) = \alpha$. Then (24) holds and (25) becomes

$$f = u\phi - \alpha u \text{ a.e.} \tag{26}$$

By Proposition 3.4, there is a measurable solution u to (26) but u does not have a continuous version. Hence $\bar{u} = (U, u)$ is measurable, has no continuous version and solves the cocycle equation a.e. Moreover, $\bar{u}$ takes values in $\{\alpha\} \times \mathbb{R}^n/\mathbb{Z}^n$, which is connected, and $\bar{u}$ is essentially bounded as $\mathbb{R}^n/\mathbb{Z}^n$ is compact. ❑

Remarks.

1. Observe that the above example can be reformulated in terms of $\mathbb{Z} \ltimes \mathbb{R}^k/\mathbb{Z}^k$. Hence the regularity theorem fails for non-compact non-connected *soluble* Lie groups.

2. Suppose G_1 and G_2 are Lie groups (not necessarily compact) such that G_1 acts smoothly on G_2 by automorphisms. Suppose that the Livšic regularity theorem (Theorem 1.1) holds for trivial cocycles taking values in G_1 and G_2. Then one can show that the regularity theorem for trivial cocycles continues to hold for the semi-direct product $G_1 \ltimes G_2$.

Criteria for ergodicity and weak-mixing of skew products. Theorem 1.1 has applications to the study of skew products of hyperbolic systems with a compact Lie group. More specifically, there are well-known criteria for a skew product of the form $(x, y) \mapsto (\phi x, f(x)y)$ (see §1) to be ergodic or weak-mixing. These criteria take the form of the non-existence of measurable solutions to certain cohomological equations. Theorem 1.1 then says that this is equivalent to the non-existence of Hölder solutions.[2]

[2] See §3 of Chapter 2.

For skew products of the form $(x, y) \mapsto (\phi x, f(x)\alpha(y))$, similar criteria continue to hold (at least in the case when G is a torus). Unfortunately, the cocycle equations are not of the form studied in this paper.

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