Solutions to the twisted cocycle equation over hyperbolic systems

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Abstract

A twisted cocycle with values in a Lie group G is a cocycle that incorporates an automorphism of G. Suppose that the underlying transformation is hyperbolic. We prove that if two Hölder continuous twisted cocycles with a sufficiently high Hölder exponent assign equal 'weights' to the periodic orbits of ϕ, then they are Hölder cohomologous. This generalises a well-known theorem due to Livšic in the untwisted case. Having determined conditions for there to be a solution to the twisted cocycle equation, we consider how many other solutions there may be. When G is a torus, we determine conditions for there to be only finitely many solutions to the twisted cocycle equation.

§1 Introduction

Let ϕ by a map on a space Λ and let G be a Lie group. Denote by $C^\theta(\Lambda, G)$ the space of Hölder continuous functions $\Lambda \to G$ of exponent $\theta \in (0, 1)$. The following equivalence relation on $C^\theta(\Lambda, G)$ is important when studying the dynamics of ϕ. We say that two functions $f, g \in C^\theta(\Lambda, G)$ are *cohomologous* if the *cocycle equation*

$$f = u\phi \cdot g \cdot u^{-1} \tag{1}$$

has a solution $u \in C^\theta(\Lambda, G)$. (We shall often use a dot '$\cdot$' to denote multiplication in G for clarity.) We call such a solution u a *transfer function*. A well-studied question is the following: For suitable ϕ, determine conditions on f, g that ensure that the cocycle equation (1) admits a Hölder continuous solution u. We shall assume that ϕ is *hyperbolic*, by which we essentially mean that ϕ satisfies Smale's Axiom A and Λ is a basic set occurring in the spectral decomposition of the non-wandering set of ϕ [Bo]—we shall make this more precise in §2. In this case we have the following well-known result.

Theorem 1.1 (Livšic's periodic data criterion [Lv])
Let ϕ be hyperbolic and let G be an abelian group with a left-right invariant metric. Then a necessary and sufficient condition for $f, g \in C^\theta(\Lambda, G)$ to be cohomologous via a Hölder transfer function is that $f(\phi^{n-1}x) \cdots f(x) = g(\phi^{n-1}x) \cdots g(x)$ whenever $\phi^n x = x$.

Indeed, necessity is clear. For if u is a solution to the cocycle equation then $f(\phi^{n-1}x) \cdots f(x) = u(\phi^n x) g(\phi^{n-1}x) \cdots g(x) u(x)^{-1}$ and when G is abelian and $\phi^n x = x$, the terms involving u cancel.

The case when G is non-abelian is more subtle, and has recently been addressed independently by Parry [Pa] and Schmidt [Sc].

Theorem 1.2 ([Pa, Sc])
Let ϕ be hyperbolic and let G be a complete metric group with a left-right invariant metric. Let $f, g \in C^\theta(\Lambda, G)$. If $f(\phi^{n-1}x) \cdots f(x) = g(\phi^{n-1}x) \cdots g(x)$ whenever $\phi^n x = x$ then $f = u\phi \cdot g \cdot u^{-1}$ for some $u \in C^\theta(\Lambda, G)$.

(In fact, the existence of a left-right invariant metric is unnecessarily restrictive and, as Schmidt observes, can be replaced by the existence of a Lipschitz metric and a bounded distortion assumption on f and g; we shall discuss this in §3.)

There is a more general version still, again due to Parry and Schmidt, which deals with the case when $f(\phi^{n-1}x) \cdots f(x)$ is conjugate to $g(\phi^{n-1}x) \cdots g(x)$.

The cocycle equation (1) arises when one studies skew products over ϕ. Let $f \in C^\theta(\Lambda, G)$ and define the skew-product $\phi_f : \Lambda \times G \to \Lambda \times G : (x, y) \mapsto (\phi x, f(x)y)$. Then f and g are cohomologous precisely when the map $U(x, y) = (x, u(x)y)$ conjugates ϕ_f to ϕ_g.

A more general class of skew-product is formed by incorporating an automorphism α of G. Let $f \in C^\theta(\Lambda, G)$ and define the *twisted skew product* $\phi_{f,\alpha} : \Lambda \times G \to \Lambda \times G : (x, y) \mapsto (\phi x, f(x)\alpha(y))$. When G is compact and the spectra of the derivatives of α and ϕ satisfy a certain bunching condition, the map $\phi_{f,\alpha}$ is partially hyperbolic in the sense of [BP]. The study of such twisted skew products motivates much of this note.

The bundle map $U(x,y) = (x, u(x)y)$ conjugates $\phi_{f,\alpha}$ and $\phi_{g,\alpha}$ precisely when u solves the *twisted cocycle equation*

$$f = u\phi \cdot g \cdot \alpha u^{-1}. \tag{2}$$

In this case, we say that f and g are *α-cohomologous* .

The aim of this note is to consider similar questions to those addressed in [Pa, Sc] in the twisted case. For ease of reference, we refer to (1) as the *untwisted cocycle equation.*

Observe from (2) that if we define $f^{n,\alpha}(x) = f(\phi^{n-1}x)\alpha f(\phi^{n-2}x)\cdots\alpha^{n-1}f(x)$ then $f^{n,\alpha} = u\phi^n \cdot g^{n,\alpha} \cdot \alpha^n u^{-1}$. Our main result is the following.

Theorem 1.3
Let ϕ be hyperbolic. Let G be a connected Lie group with a left-right invariant metric and let α be an automorphism of G satisfying the spectral bunching condition above (which we state precisely in §2). Let $f, g \in C^\theta(\Lambda, G)$ be Hölder continuous with exponent θ sufficiently close to 1 (how close we make precise in §3). If $f^{n,\alpha}(x) = g^{n,\alpha}(x)$ whenever $\phi^n x = x$ then $f = u\phi \cdot g \cdot \alpha u^{-1}$ for some $u \in C^\theta(\Lambda, G)$.

Having given a condition for the existence of a solution to the twisted cocycle equation (2), it is natural to consider how many other solutions there can be. We shall deal only with the case when G is a torus. There may be infinitely many solutions, but when ϕ and α are *spectrally disjoint* (a term we define precisely in §5) we prove that there are only finitely many solutions.

Theorem 1.4
Let ϕ be hyperbolic, G a torus and α an ergodic automorphism of G. Suppose the linear map induced by ϕ on the first cohomology group $H^1(\Lambda, \mathbb{R})$ (using Čech cohomology if Λ is not a manifold) and the linear map covering α have disjoint sets of eigenvalues. Then there are at most finitely many solutions to the twisted cocycle equation (2).

This note is organised as follows. In §2 we make precise the definitions of hyperbolicity and partial hyperbolicity and discuss properties of such dynamical systems that will prove useful. Theorem 3.1 is proved in §3. In §4 we provide an application of Theorem 3.1 to a class of cocycles not covered in [Pa, Sc]. Finally, we prove Theorem 5.2 and some related results in §5.

§2 Preliminaries

§2.1 Cocycles

Let us briefly introduce some notation and terminology that will prove useful. All functions below are assumed to be Hölder continuous. Let ϕ be a map and let G be a group. A function $f : \Lambda \to G$ determines an *(untwisted) cocycle* f^n by defining $f^n(x) = f(\phi^{n-1}x) \cdots f(x)$, $f^{-n}(x) = f^n(\phi^{-n}x)^{-1}$ for $n > 0$ and $f^0(x) = e$, the group identity. Then f^n satisfies the *cocycle identity* $f^{n+m}(x) = f^n(\phi^m x) f^m(x)$. Two cocycles f^n, g^n are *cohomologous* if $f^n(x) = u(\phi^n x) g^n(x) u(x)^{-1}$ for some $u : \Lambda \to G$. The notion of a cocycle is related to the notion of a skew product by observing that $\phi_f^n(x, y) = (\phi^n x, f^n(x) y)$.

We now define a *twisted cocycle* in analogy with the untwisted case. Let α be an automorphism of G. A function $f : \Lambda \to G$ determines a twisted cocycle by setting $f^{n,\alpha}(x) = f(\phi^{n-1}x) \alpha f(\phi^{n-2}x) \cdots \alpha^{n-1} f(x)$, $f^{-n,\alpha}(x) = \alpha^{-n} f^{n,\alpha}(\phi^{-n}x)^{-1}$ for $n > 0$ and $f^{0,\alpha}(x) = e$. We have the *twisted cocycle identity* $f^{n+m,\alpha}(x) = f^{n,\alpha}(\phi^m x) \alpha^n f^{m,\alpha}(x)$. Two twisted cocycles $f^{n,\alpha}, g^{n,\alpha}$ are said to be *α-cohomologous* if $f^{n,\alpha}(x) = u(\phi^n x) g^{n,\alpha}(x) \alpha^n u(x)^{-1}$ for some u. If f is α-cohomologous to the function $g(x) = e$ then we call f an *α-coboundary*. With $\phi_{f,\alpha}$ as in §1, we have $\phi_{f,\alpha}^n(x, y) = (\phi^n x, f^{n,\alpha}(x) \alpha^n(y))$.

§2.2 Hyperbolic and partially hyperbolic dynamics

Let M be a manifold and Φ a smooth diffeomorphism of M. Let $\Lambda \subset M$ be a Φ-invariant subset. We call $\Phi : \Lambda \to \Lambda$ *partially hyperbolic* if there exists a continuous splitting of the tangent bundle of M restricted to Λ into three $d\Phi$-invariant sub-bundles $T_x M = E_x^s \oplus E_x^c \oplus E_x^u$ such that for $n \geq 0$:

1. $\|d\Phi^n v\| \leq C_1 \lambda^n \|v\|$ if $v \in E_x^s$,
2. $\|d\Phi^n v\| \geq C_2 \lambda^{-n} \|v\|$ if $v \in E_x^u$,
3. $C_3 \mu^{-n} \|v\| \leq \|d\Phi^n v\| \leq C_4 \mu^n \|v\|$ if $v \in E_x^c$,

for constants $C_1, C_2, C_3, C_4 > 0$, $\lambda \in (0, 1)$, $\mu > 1$ such that $\mu\lambda < 1$. Thus $d\Phi$ contracts vectors in E_x^s more strongly than it contracts vectors in E_x^c and expands vectors in E_x^u more strongly than it expands vectors in E_x^c. We call E^s, E^c, E^u the *stable, centre* and *unstable* sub-bundles, respectively.

Observe that 2. above is equivalent to

$$\|d\Phi^{-n}v\| \leq C_2^{-1}\lambda^n\|v\|$$

for $v \in E_x^u$, $n \geq 0$. We refer to λ as the *hyperbolic rate of contraction*.

A map $\phi : \Lambda \to \Lambda$ is said to be *hyperbolic* if it is partially hyperbolic with trivial centre bundle. We shall further assume that Λ is locally maximal, *i.e.* there exists an open set $U \supset \Lambda$ such that $\Lambda = \bigcap_{n=-\infty}^{\infty} \phi^{-n}U$, and that ϕ is topologically mixing on Λ. For example, a (transitive) Anosov diffeomorphism is hyperbolic; in this case, $\Lambda = M$.

Let us give a method for constructing partially hyperbolic maps from hyperbolic ones. Let $\phi : \Lambda \to \Lambda$ be hyperbolic with hyperbolic rate of contraction $\lambda \in (0,1)$. Let G be a compact connected Lie group. Let $f : \Lambda \to G$. Let α be an automorphism of G and denote by $d\alpha$ the corresponding automorphism of the Lie algebra of G. Let $\mu_0 = \max\{r(d\alpha), r(d\alpha^{-1})\}$ (here and throughout, $r(A)$ denotes the spectral radius of the linear operator A). Then if $\mu_0\lambda < 1$, the twisted skew-product $\phi_{f,\alpha}$ is partially hyperbolic [BP]. Furthermore, we may take $\mu = \mu_0 + \delta$ for any sufficiently small $\delta > 0$ in the definition of partially hyperbolic.

Let us recall some useful facts about hyperbolic dynamical systems.

Stable and unstable manifolds. Let $z \in \Lambda$. For suitably small $\varepsilon > 0$ we define the *local stable* and *unstable manifolds* through z to be

$$\begin{aligned} W_\varepsilon^s(z) &= \{x \in \Lambda \mid d(\phi^n x, \phi^n z) \leq \varepsilon \text{ for all } n \geq 0\}, \\ W_\varepsilon^u(z) &= \{x \in \Lambda \mid d(\phi^{-n} x, \phi^{-n} z) \leq \varepsilon \text{ for all } n \geq 0\}, \end{aligned}$$

respectively. (Note that they are subsets of Λ.) We have the following estimates. There exists a constant $C_1(\phi) > 0$ such that

$$\begin{aligned} d(\phi^n x, \phi^n z) &\leq C_1(\phi)\lambda^n d(x,z) \text{ for } n \geq 0, x \in W_\varepsilon^s(z), \\ d(\phi^{-n} x, \phi^{-n} z) &\leq C_1(\phi)\lambda^n d(x,z) \text{ for } n \geq 0, x \in W_\varepsilon^u(z). \end{aligned} \tag{3}$$

If ε is sufficiently small then there exists $\delta > 0$ such that if $d(x,y) < \delta$ then $W_\varepsilon^u(x) \cap W_\varepsilon^s(y)$ is a single point. We denote this point by $\langle x, y\rangle$.

We define the *global stable* and *unstable manifolds* through $z \in \Lambda$ to be:

$$W^s(z) = \bigcup_{n\geq 0} \phi^{-n} W_\varepsilon^s(\phi^n z), \quad W^u(z) = \bigcup_{n \geq 0} \phi^n W_\varepsilon^s(\phi^{-n} z),$$

respectively. We have the estimates

$$\begin{aligned} d(\phi^n x, \phi^n z) &\le C(x,z)\lambda^n \text{ for } n \ge 0, x \in W^s(z), \\ d(\phi^{-n} x, \phi^{-n} z) &\le C(x,z)\lambda^n \text{ for } n \ge 0, x \in W^u(z). \end{aligned} \tag{4}$$

Note that, in contrast to the local case, the constant $C(x,z)$ depends on x and z.

Let $z \in \Lambda$ be a fixed point. The intersection $W(z) = W^s(z) \cap W^u(z)$ is called the *homoclinic set* of z and is dense in Λ. We have

$$d(\phi^n x, z) \le C(x,z)\lambda^{|n|} \tag{5}$$

for $n \in \mathbb{Z}$ and $x \in W(z)$.

Anosov's closing lemma. Suppose x is an almost closed orbit: $\phi^n x$ is close to x for some $n \neq 1$. Anosov's closing lemma guarantees the existence of a periodic point approximating the orbit segment $x, \ldots, \phi^{n-1}x, \phi^n x$ exponentially well. More precisely,

Theorem 2.1 ([An])
There exist constants $C_2(\phi)$, $\varepsilon > 0$ such that if $x \in \Lambda$ satisfies $d(\phi^n x, x) < \varepsilon$ then there exists a periodic point $w = \phi^n w \in \Lambda$ such that for $j = 0, \ldots, n-1$, we have

$$d(\phi^j x, \phi^j w) \le C_2(\phi)\lambda^{\min\{j, n-j\}} d(\phi^n x, x).$$

§2.3 Automorphisms of Lie groups

Let G be a connected Lie group with a left-right invariant Riemannian metric ρ (such a metric exists, for example, if G is compact or abelian). Let α be an automorphism of G. We shall need an estimate of how fast α can expand distances. Let $d\alpha$ denote the derivate of α acting on the Lie algebra $\mathfrak{g}$ of G. Let $\mu_0 = \max\{r(d\alpha), r(d\alpha^{-1})\}$. Then if $\mu > \mu_0$, the eigenvalues of $d\alpha$ lie strictly inside the annulus centred at the origin in $\mathbb{C}$ with radii μ^{-1}, μ.

The following is well-known; we include a sketch of the proof for completeness.

Proposition 2.2
Let G, α and μ be as above. Then there exists a constant $C(\alpha) > 0$ such that $\rho(\alpha^n g, \alpha^n h) \le C(\alpha)\mu^{|n|}\rho(g,h)$ for all $g, h \in G$, $n \in \mathbb{Z}$.

Proof (sketch). Let $\|\cdot\|_g$ denote the Finsler corresponding to the left-right invariant Riemannian metric ρ on the tangent space at $g \in G$. Define a new norm on $T_eG = \mathfrak{g}$ by $|||X|||_e = \sum_{n=0}^{\infty} \|d\alpha^n X\|_e/\mu^n$. One can easily see that by the n^{th} root test, the spectral radius formula and the assumption on μ, this is a well-defined norm. Let $C^+(\alpha) = \sum_{n=0}^{\infty} \|d\alpha^n\|/\mu^n < \infty$. Then it is easy to see that for $X \in \mathfrak{g}$ and $n > 0$, $\|d\alpha^n X\|_e \le |||d\alpha^n X|||_e \le C^+(\alpha)\mu^n\|X\|_e$. Hence for $X \in T_gG$ and $n > 0$,

$$\|d\alpha^n X\|_{\alpha^n(g)} \le C^+(\alpha)\mu^n\|X\|_g. \tag{6}$$

If $g, h \in G$ and γ is a smooth path between them then $\alpha^n\gamma$ is a smooth path between $\alpha^n g, \alpha^n h$. Letting $l(\gamma)$ denote the length of γ, it is easy to check from (6), that $l(\alpha^n\gamma) \le C^+(\alpha)\mu^n l(\gamma)$. Taking the infimum over all such γ proves the proposition for $n > 0$ and a similar argument yields the same for $n < 0$. ❑

§3 A periodic data criterion for twisted cocycles

In this section, we shall mainly be dealing with twisted cocycles. Therefore we shall simplify slightly the notation introduced in §2 and write f^n for $f^{n,\alpha}$.

The following is the main result of this section.

Theorem 3.1
Let ϕ be a hyperbolic map with hyperbolic rate of contraction $\lambda \in (0,1)$ and fixed point $z = \phi z$. Let G be a connected Lie group possessing a left-right invariant Riemannian metric. Let α be an automorphism of G and let $\mu_0 = \max\{r(d\alpha), r(d\alpha^{-1})\}$. Suppose $\mu_0\lambda < 1$ and let $\theta_{\text{crit}} = \log\mu_0/\log\lambda^{-1} \in (0,1)$.

Suppose $f, g \in C^\theta(\Lambda, G)$, $\theta > \theta_{\text{crit}}$, satisfies $f^n(x) = g^n(x)$ whenever $\phi^n x = x$. Then there exists $u \in C^\theta(\Lambda, G)$ such that $f(x) = u(\phi x)g(x)\alpha u(x)^{-1}$.

Remarks.

1. Observe that we are assuming that $\mu_0\lambda < 1$, *i.e.* $\phi_{f,\alpha}$ and $\phi_{f,\alpha}$ are partially hyperbolic (when G is compact).

2. If $\theta > \theta_{\text{crit}}$ then $\mu_0\lambda^\theta < 1$. Throughout the course of the proof, we assume that $\mu > \mu_0$ has been chosen so that $\mu\lambda^\theta < 1$.

First of all, recall that if $f \in C^\theta(\Lambda, G)$ then $\rho(f(x), f(y)) \le Cd(x,y)^\theta$ for some constant $C > 0$. We denote by $|f|_\theta$ the smallest possible constant $C > 0$.

We begin with a definition. Let $f \in C^\theta(\Lambda, G)$ with $\theta > \theta_{\mathrm{crit}}$. For $x \in \Lambda$ and $y \in W^s(x)$ define $F_x^s(y) = \lim_{n\to\infty} \alpha^{-n}(f^n(y)^{-1} f^n(x))$.

Lemma 3.2

1. *$F_x^s(y)$ exists.*

2. *If $y \in W_\varepsilon^s(x)$ then $\rho(F_x^s(y), e) \le C_1(\phi, \alpha, \lambda)|f|_\theta d(x,y)^\theta$ for some constant $C_1(\phi, \alpha, \theta)$ independent of x, y.*

Proof. We prove 2. Note that

$$\begin{aligned}
&\rho(\alpha^{-(n+1)}(f^{n+1}(y)^{-1}f^{n+1}(x)), \alpha^{-n}(f^n(y)^{-1}f^n(x))) \\
&= \rho(\alpha^{-(n+1)}(\alpha f^n(y)^{-1}f(\phi^n y)^{-1}f(\phi^n x)\alpha f^n(x)), \alpha^{-n}(f^n(y)^{-1}f^n(x))) \\
&= \rho(\alpha^{-(n+1)}f(\phi^n y), \alpha^{-(n+1)}f(\phi^n x)) \\
&\le C(\alpha)\mu^{n+1}|f|_\theta C_1(\phi)^\theta \lambda^{\theta n} d(x,y)^\theta
\end{aligned}$$

using (3). As $\mu\lambda^\theta < 1$, this expression tends to zero exponentially fast as $n \to \infty$. Hence $\alpha^{-n}(f^n(y)^{-1}f^n(x))$ is Cauchy and therefore converges. It is easy to check that

$$\rho(F_x^s(y), e) \le \frac{C(\alpha)\mu C_1(\phi)^\theta}{1-\mu\lambda^\theta}|f|_\theta d(x,y)^\theta.$$

The proof of 1. is similar, using (4). ❑

Similarly, we can define for $y \in W^s(x)$, $G_x^s(y) = \lim_{n\to\infty} \alpha^{-n}(g^n(y)^{-1}g^n(x))$. For $y \in W^u(x)$ we define $F_x^u(y) = \lim_{n\to\infty} \alpha^n(f^{-n}(y)^{-1}f^{-n}(x)), G_x^u(y) = \lim_{n\to\infty} \alpha^n(g^{-n}(y)^{-1}g^{-n}(x))$. When $y \in W_\varepsilon^s(x)$ or $W_\varepsilon^u(x)$, analogous estimates to those given in Lemma 3.2 hold.

The following lemma and its corollary are the key facts in our analysis.

Lemma 3.3
Let $z = \phi z$ be a fixed point and let $x \in W(z)$. Assume the hypotheses of Theorem 3.1 hold. Then

$$\lim_{n\to\infty} \alpha^{-n}f^n(x)\alpha^n f^{-n}(x)^{-1}\alpha^n g^{-n}(x)\alpha^{-n}g^n(x)^{-1} = e.$$

Corollary 3.4
Let $z = \phi z$ be a fixed point and let $x \in W(z)$. Assume the hypotheses of Theorem 3.1 hold. Then $F_z^u(x) G_z^u(x)^{-1} = F_z^s(x) G_z^s(x)^{-1}$.

Proof of Lemma 3.3. We first prove

$$\lim_{n\to\infty} \alpha^{-n}(f^{2n}(\phi^{-n}x)g^{2n}(\phi^{-n}x)^{-1}) = e. \tag{7}$$

As $x \in W(z)$, we can choose n sufficiently large so that $\phi^{-n}x, \phi^n x$ are close enough to apply Anosov's closing lemma. Then there exists $w = \phi^{2n}w$ such that

$$\begin{aligned} d(\phi^{-n+j}x, \phi^{-n+j}w) &\leq C_2(\phi)\lambda^{\min\{j,2n-j\}} d(\phi^{-n}x, \phi^n x) \\ &= \begin{cases} C_2(\phi)\lambda^j d(\phi^{-n}x, \phi^n x), & 0 \leq j \leq n, \\ C_2(\phi)\lambda^{2n-j} d(\phi^{-n}x, \phi^n x), & n \leq j \leq 2n. \end{cases} \end{aligned}$$

By the hypotheses of Theorem 3.1, we have $\alpha^{-n}(f^{2n}(\phi^{-n}w)g^{2n}(\phi^{-n}w)^{-1}) = e$. Hence,

$$\begin{aligned} &\rho(\alpha^{-n}(f^{2n}(\phi^{-n}x)g^{2n}(\phi^{-n}x)^{-1}), e) \\ &\quad = \rho(\alpha^{-n}(f^{2n}(\phi^{-n}x)g^{2n}(\phi^{-n}x)^{-1}), \alpha^{-n}(f^{2n}(\phi^{-n}w)g^{2n}(\phi^{-n}w)^{-1})) \\ &\quad \leq \rho(\alpha^{-n}f^{2n}(\phi^{-n}x), \alpha^{-n}f^{2n}(\phi^{-n}w)) + \rho(\alpha^{-n}g^{2n}(\phi^{-n}x), \alpha^{-n}g^{2n}(\phi^{-n}w)). \end{aligned}$$

Now,

$$\begin{aligned} &\rho(\alpha^{-n}f^{2n}(\phi^{-n}x), \alpha^{-n}f^{2n}(\phi^{-n}w)) \\ &\quad \leq \sum_{j=0}^{2n-1} \rho(\alpha^{n-1-j}f(\phi^{-n+j}x), \alpha^{n-1-j}f(\phi^{-n+j}w)) \\ &\quad = \sum_{j=0}^{n-1} \rho(\alpha^{n-1-j}f(\phi^{-n+j}x), \alpha^{n-1-j}f(\phi^{-n+j}w)) \\ &\qquad + \sum_{j=0}^{n-1} \rho(\alpha^{j-n}f(\phi^{n-1-j}x), \alpha^{j-n}f(\phi^{n-1-j}w)) \\ &\quad \leq \sum_{j=0}^{n-1} C(\alpha)\mu^{n-1-j}|f|_\theta C_2(\phi)^\theta \lambda^{\theta j} d(\phi^{-n}x, \phi^n x)^\theta \\ &\qquad + \sum_{j=0}^{n-1} C(\alpha)\mu^{n-j}|f|_\theta C_2(\phi)^\theta \lambda^{\theta(j+1)} d(\phi^{-n}x, \phi^n x)^\theta \\ &\quad \leq C(\alpha)\frac{\mu^{-1}+\lambda^\theta}{1-\mu^{-1}\lambda^\theta} C_2(\phi)^\theta |f|_\theta \mu^n d(\phi^{-n}x, \phi^n x)^\theta, \end{aligned}$$

and similarly for $\rho(\alpha^{-n}g^{2n}(\phi^{-n}x), \alpha^{-n}g^{2n}(\phi^{-n}w))$. Hence,

$$\begin{aligned}&\rho(\alpha^{-n}(f^{2n}(\phi^{-n}x)g^{2n}(\phi^{-n}x)^{-1}), e)\\ &\quad\le\ C_2(\phi,\alpha,\theta)(|f|_\theta + |g|_\theta)\mu^n d(\phi^{-n}x, \phi^n x)^\theta.\end{aligned}$$

Using the elementary inequality $(a+b)^\theta \le a^\theta + b^\theta$ for $\theta \in (0,1)$ and $a, b \ge 0$ we have $d(\phi^{-n}x, \phi^n x)^\theta \le d(\phi^{-n}x, z)^\theta + d(\phi^n x, z)^\theta \le 2C(x,z)^\theta \lambda^{\theta n}$ by (5). Hence,

$$\begin{aligned}&\rho(\alpha^{-n}(f^{2n}(\phi^{-n}x)g^{2n}(\phi^{-n}x)^{-1}, e))\\ &\quad\le\ 2C(\phi,\alpha,\theta)C(x,z)^\theta(|f|_\theta + |g|_\theta)(\mu\lambda^\theta)^n\end{aligned}$$

which tends to 0 as $n \to \infty$ as $\theta > \theta_{\mathrm{crit}}$. This proves (7).

Finally, observe from the twisted cocycle identity that $\alpha^{-n}f^{2n}(\phi^{-n}x) = \alpha^{-n}(f^n(x))f^n(\phi^{-n}x)$ from which one easily sees that

$$\alpha^{-n}(f^{2n}(\phi^{-n}x)g^{2n}(\phi^{-n}x)^{-1}) = \alpha^{-n}f^n(x)\alpha^n f^{-n}(x)^{-1}\alpha^n g^{-n}(x)\alpha^{-n}g^n(x)^{-1},$$

proving the lemma. ❏

Proof of Corollary 3.4. By Lemma 3.3, we have

$$\rho(\alpha^{-n}f^n(x)\alpha^n f^{-n}(x)^{-1}\alpha^n g^{-n}(x)\alpha^{-n}g^n(x)^{-1}, e) \to 0$$

as $n \to \infty$. As $\phi^n z = z$ we have $f^n(z)g^n(z)^{-1} = e = f^{-n}(z)g^{-n}(z)^{-1}$ by the hypotheses of Theorem 3.1. Hence,

$$\begin{aligned}\rho(&\alpha^{-n}f^n(x)\alpha^n f^{-n}(x)^{-1}\alpha^n f^{-n}(z)\alpha^n g^{-n}(z)^{-1}\alpha^n g^{-n}(x)\alpha^{-n}g^n(x)^{-1},\\ &\alpha^{-n}f^n(z)\alpha^{-n}g^n(z)^{-1})\ \to\ 0\end{aligned}$$

as $n \to \infty$. Using the left-right invariance of ρ, we have $F_z^s(x)^{-1}F_z^u(x)G_z^u(x)^{-1}G_z^s(x) = e$ and the corollary follows. ❏

We can now define a transfer function u on the dense set $W(z)$. Define, for $x \in W(z)$, $u(x) = F_z^s(x)G_z^s(x)^{-1} = F_z^u(x)G_z^u(x)^{-1}$.

Lemma 3.5
With the hypotheses of Theorem 3.1 and u defined as above, we have $f(x) = u(\phi x)g(x)\alpha u(x)^{-1}$ for $x \in W(z)$.

Proof. Note that

$$\begin{aligned} F_z^s(\phi x) &= \lim_{n\to\infty} \alpha^{-n}(f^n(\phi x)^{-1} f^n(\phi z)) \\ &= \lim_{n\to\infty} \alpha^{-n}(\alpha^n f(x) f^{n+1}(x)^{-1} f^{n+1}(z)\alpha^n f(z)^{-1}) \\ &= f(x)\alpha F_z^s(x) f(z)^{-1}. \end{aligned}$$

By hypothesis, $f(z) = g(z)$. Hence,

$$\begin{aligned} u(\phi x) &= F_z^s(\phi x) G_z^s(\phi x)^{-1} \\ &= f(x)\alpha F_z^s(x) f(z)^{-1} g(z)\alpha G_z^s(x)^{-1} g(x)^{-1} \\ &= f(x)\alpha u(x) g(x)^{-1}. \end{aligned}$$

❑

Hence u is defined and satisfies $f(x) = u(\phi x)g(x)\alpha u(x)^{-1}$ on the dense set $W(z)$. We show that $u : W(z) \to G$ is uniformly Hölder continuous. It then follows that u extends uniquely to a Hölder continuous function $u : \Lambda \to G$. Moreover, the identity $f(x) = u(\phi x)g(x)\alpha u(x)^{-1}$ persists. This proves Theorem 3.1.

Lemma 3.6
Suppose the hypotheses of Theorem 3.1 are satisfied. Then there exists a constant $C_3(\phi, \alpha, \theta) > 0$ such that if $x, y \in W(z)$ are sufficiently close then

$$\rho(u(x), u(y)) \le C_3(\phi, \lambda, \theta)(|f|_\theta + |g|_\theta) d(x, y)^\theta.$$

Proof. Let $x, y \in W(z)$ be sufficiently close so that $w = \langle x, y\rangle$ is defined. Clearly, $w \in W(z)$. Then

$$\begin{aligned} \rho(u(x), u(y)) &= \rho(F_z^u(x) G_z^u(x)^{-1}, F_z^s(y) G_z^s(y)^{-1}) \\ &= \lim_{n\to\infty} \rho(\alpha^n(f^{-n}(x)^{-1} g^{-n}(x)), \alpha^{-n}(f^n(y)^{-1} g^n(y))) \\ &\qquad \text{(as } f^{-n}(z) g^{-n}(z)^{-1} = e = f^n(z) g^n(z)^{-1}) \\ &= \lim_{n\to\infty} \rho(\alpha^{-n} f^n(y)\alpha^n f^{-n}(x)^{-1}\alpha^n g^{-n}(x)\alpha^{-n} g^n(y)^{-1}, e). \end{aligned}$$

As $w \in W(z)$, we have by Lemma 3.3,

$$\lim_{n\to\infty} \alpha^{-n} f^n(w)\alpha^n f^{-n}(w)^{-1}\alpha^n g^{-n}(w)\alpha^{-n} g^n(w)^{-1} = e.$$

Hence,

$$
\begin{aligned}
&\rho(u(x), u(y)) \\
&= \lim_{n\to\infty} \rho(\alpha^{-n} f^n(y)\alpha^n f^{-n}(x)^{-1}\alpha^n g^{-n}(x)\alpha^{-n} g^n(y)^{-1}, \\
&\qquad \alpha^{-n} f^n(w)\alpha^n f^{-n}(w)^{-1}\alpha^n g^{-n}(w)\alpha^{-n} g^n(w)^{-1}) \\
&\leq \rho(F^s_y(w), e) + \rho(F^u_x(w), e) + \rho(G^u_x(w), e) + \rho(G^s_y(w), e) \\
&\leq C_1(\phi, \alpha, \theta)(|f|_\theta + |g|_\theta)(d(x, w)^\theta + d(w, y)^\theta) \\
&\leq C_3(\phi, \alpha, \theta)(|f|_\theta + |g|_\theta) d(x, y)^\theta,
\end{aligned}
$$

by Proposition 19.1.1 in [KH]. This proves Lemma 3.6. By the discussion above, this is sufficient to prove Theorem 3.1. ❑

Remarks.

1. Instead of considering a Lie group with a left-right invariant metric, we could have considered a Lie group possessing a Lipschitz metric, as in Schmidt [Sc]. A Riemannian metric ρ is called a *Lipschitz metric* if for each $h \in G$ there is a constant $d(h) \geq 1$ such that for all $g, g' \in G$,

   $$d(h)^{-1}\rho(g, g') \leq \left\{ \begin{array}{c} \rho(gh, g'h) \\ \rho(hg, hg') \end{array} \right\} \leq d(h)\rho(g, g').$$

   For example, $Gl(n, \mathbb{R})$ has a Lipschitz metric, but no left-right invariant metric.

   Let $D(h)$ denote the smallest possible $d(h)$. We say a twisted cocycle f^n has *bounded distortion* if $\sup\{D(\alpha^{-n} f^n(x)) \mid n \in \mathbb{Z}, x \in \Lambda\} < \infty$.

   The results in this section continue to hold if G admits a Lipschitz metric and both f and g have bounded distortion; the proofs above go through with only minor changes.

2. Suppose ϕ is a C^k Anosov diffeomorphism for some $k = 1, 2, \ldots, \infty, \omega$ (so that Λ is a manifold). Suppose $f, g \in C^r$ for some $1 \leq r \leq k$. If $f^n(x) = g^n(x)$ whenever $\phi^n x = x$ then f, g are α-cohomologous via a transfer function $u \in C^{r-\varepsilon}$ for any small $\varepsilon > 0$. (We define $r - \varepsilon = r$ when $r = 1, \infty, \omega$.) To see this, observe that the above proof ensures that such a u exists and is Lipschitz. The arguments in [Wa, §6] (which are derived from corresponding arguments in the untwisted case due to Niţică and Török [NT]) then show that u is $C^{r-\varepsilon}$.

3. There is of course an analogue of the above for shifts of finite type. Let $\Sigma \subset S^{\mathbb{Z}}$, S a finite set, be a shift of finite type (see [KH]). For each $\theta \in (0,1)$, define a metric $d^\theta(x,y) = \theta^{n(x,y)}$ where $n(x,y) = \sup\{n \mid x_i = y_i \text{ for } |i| \leq n,\ x = (x_i), y = (y_i) \in \Sigma\}$. Let $F_\theta(\Sigma, G)$ denote the functions $\Sigma \to G$ that are Lipschitz with respect to d^θ. Let $\theta_{\text{crit}} = \mu_0^{-1}$. Then the conclusion of Theorem 3.1 holds for $f, g \in F_\theta(\Sigma, G)$, provided $\theta < \theta_{\text{crit}}$.

4. When α is an ergodic automorphism of a torus, Lind [Li] proved that *any* two functions f, g are α-cohomologous via a measurable transfer function u. However, as Lind observes, u need not be continuous. An example of this phenomena when $\mu_0 \lambda < 1$ can be found in [Wa].

It is perhaps interesting to contrast our proof with those of Livšic, Parry and Schmidt. Livšic's argument (which is only valid when G is abelian) is to first define a solution u on the orbit of a transitive point and then to check that u is Hölder on this dense set. This method quickly breaks down in the twisted case as α can expand distances exponentially fast, leading to certain estimates blowing up. Our method is more closely related to those of Parry and Schmidt. Schmidt first defines a solution u on the homoclinic equivalence relation and then checks that u is Hölder on this dense set. However, Schmidt introduces an auxiliary equivalence relation and an auxiliary cocycle defined on this relation, a construction that is not possible for twisted cocycles. (Indeed, Schmidt introduces techniques that handle both the 'periodic data' theorems and 'regularity' theorems. A regularity theorem is one that ensures that a solution to the cocycle equation that *a priori* is only known to be measurable must in fact be Hölder. As is discussed in [Wa], the presence of a regularity theorem for twisted cocycles necessarily requires some very strong hypotheses on the automorphism.) Parry's method is, as in this note, to first define u on the homoclinic set of a reference fixed point, and then to check that u is Hölder on this dense set.

It was mentioned in §1 that Parry [Pa] and Schmidt [Sc] have a generalisation of Theorem 1.2 for when $\phi^n x = x$ implies $f(\phi^{n-1}x) \cdots f(x)$ and $g(\phi^{n-1}x) \cdots g(x)$ are conjugate. This takes the form:

Theorem 3.7 ([Pa, Sc])
Let ϕ be hyperbolic and let G be a locally compact group with a left-right invariant metric. Let $f, g \in C^\theta(\Lambda, G)$ and suppose the skew product ϕ_f is topo-

logically transitive. If $f(\phi^{n-1}x)\cdots f(x)$ is conjugate to $g(\phi^{n-1}x)\cdots g(x)$ whenever $\phi^n x = x$ then there exists an automorphism β of G such that βf is cohomologous to g.

There is a partial version of this for twisted cocycles.

Proposition 3.8
Let ϕ, λ, G, α and θ be as in Theorem 3.1. Suppose there exists $v \in C^\theta(\Lambda, G)$ such that $f^n(x) = v(x)g^n(x)\alpha^n v(x)^{-1}$ whenever $\phi^n(x) = x$. Then f is α-cohomologous to g.

Proof. Apply Theorem 3.1 to $g'(x) = v(\phi x)g(x)\alpha v(x)^{-1}$. ❑

Let us say that two elements $g, g' \in G$ are *α-conjugate* if $g = hg'\alpha(h)^{-1}$ for some $h \in G$. It is tempting to think that there may be a twisted analogue of Theorem 3.7 with the hypothesis that $f^n(x)$ is α-conjugate to $g^n(x)$ whenever $\phi^n x = x$. This is not the case, as the following example shows.

Let α be an ergodic toral automorphism. Then the map $g \mapsto g \cdot \alpha^n(g)^{-1}$ is an endomorphism for each $n > 0$. Hence any $f \in C^\theta(\Lambda, G)$ satisfies $f^n(x) = g \cdot \alpha^n(g)^{-1}$ for some $g \in G$ whenever $\phi^n x = x$. We show that in general f cannot be written as $\beta f = u\phi \cdot \alpha u^{-1}$ for some automorphism β of G.

Take Λ and G to be the 2-dimensional torus K^2 and take ϕ and α to be the hyperbolic toral automorphisms with corresponding matrices

$$\phi = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

respectively. Note that $\mu_0 \lambda < 1$. Choose f to be the homomorphism of K^2 with matrix

$$f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $\phi_{f,\alpha}$ is topologically transitive. Suppose $u : \Lambda \to G$ is a continuous solution to $\beta f = u\phi \cdot \alpha u^{-1}$ for some automorphism β of K^2. Then u can be written $u = U \exp(2\pi i v)$ where U is a homomorphism of K^2 and $v : K^2 \to \mathbb{R}^2$. Then $\beta f = U\phi - \alpha U$ as matrices in $Gl(2, \mathbb{Z})$. It is easy to check that this equation admits no solution $U \in Gl(2, \mathbb{Z})$ and $\beta \in Sl(2, \mathbb{Z})$.

§4 Semi-direct products

Twisted cocycles with values in G can be regarded as untwisted cocycles if we embed G in a larger group as a semi-direct product. Often this larger group will be non-compact. We can then use twisted cocycles to study some classes of cocycles taking values in a non-compact group.

Let G by a Lie group and H a subgroup of the automorphism group of G. The semi-direct product $G \rtimes H$ is the Lie group with multiplication given by $(g,\beta)(g',\beta') = (g \cdot \beta(g'), \beta\beta')$, $g, g' \in G, \beta, \beta' \in H$.

Proposition 4.1
Let ϕ, λ, G and α be as in Theorem 3.1 and assume Λ is connected. Let $\tilde{G} = G \rtimes \mathbb{Z}$, identifying $\mathbb{Z}$ with $\{\alpha^n\}$. Let $\tilde{f}, \tilde{g} \in C^\theta(\Lambda, \tilde{G})$ for θ sufficiently close to 1. If $\tilde{f}^n(x) = \tilde{g}^n(x)$ whenever $\phi^n x = x$ then $\tilde{f}$ and $\tilde{g}$ are cohomologous.

Proof. Write $\tilde{f} = (f,p), \tilde{g} = (g,q)$ where p and q are integer valued, hence constant. Then $\tilde{f}^n = (f^{n,\alpha^p}, np)$ and $f \in C^\theta(\Lambda, G)$. Hence $p = q$ and $f^{n,\alpha^p}(x) = g^{n,\alpha^p}(x)$ whenever $\phi^n x = x$. If θ is sufficiently close to 1 then Theorem 3.1 applies and $f = u\phi \cdot g \cdot \alpha^p u^{-1}$ for some Hölder u. Define $\tilde{u} = (u, 0)$ so that $\tilde{f} = \tilde{u}\phi \cdot \tilde{g} \cdot \tilde{u}^{-1}$. ❑

Observe that $\tilde{G}$ is not a Lipschitz metric group and that the cocycles $\tilde{f}, \tilde{g}$ do not have bounded variation, so the results of Schmidt [Sc] do not apply.

§5 The Bruschlinsky group

Theorem 3.1 provides a sufficient condition for the existence of a solution to the twisted cocycle equation (2). It is natural to ask: how many solutions are there to (2)? It is interesting to note that (in the untwisted case) the number of solutions depends on the group G in the sense that the cocycle equation can have no solutions $u : \Lambda \to G$ but can have a solution when G is replaced by a larger group $G' \supset G$. An example of this, due to D. Holt, can be found in [Pa].

Under the hypotheses of Theorem 3.1, we have seen that the function u given on $W(z)$ by $u(x) = F_z^s(x)G_z^s(x)^{-1}$ extends to a function defined on Λ and solves the twisted cocycle equation. In a sense, all solutions are essentially of this form. More specifically, if v is a solution to $f = v\phi \cdot g \cdot \alpha v^{-1}$ then for $x \in W(z)$ we have $v(x) = F_z^s(x)v(z)G_z^s(x)^{-1}$. In particular, when

G is abelian, $v(x) = u(x)c$ for some $c \in G$ and it follows that c is a fixed point of α. When G is a torus, α is an ergodic automorphism of G and $\mu_0\lambda < 1$, we see that there are only finitely many solutions to (2).

Consider the case when G is a torus but $\mu_0\lambda \geq 1$. Then clearly there can be infinitely many solutions to $f = u\phi \cdot \alpha u^{-1}$. For example, take $\phi = \alpha$, a hyperbolic toral automorphism (so that Λ is a torus and ϕ is hyperbolic), $f(x) = e$ and $u(x) = \alpha^n(x)$ for any $n \in \mathbb{Z}$. However, when the spectra of ϕ and α are sufficiently disjoint, there can again only be finitely many solutions to (2).

Throughout the remainder of this note, we shall assume that α is an automorphism of the n-dimensional torus $G = K^n$ and look for solutions $u : \Lambda \to G$ to (2). As G is abelian, it suffices to find solutions u to $f = u\phi \cdot \alpha u^{-1}$.

We shall need the first (Čech) cohomology group $H^1(\Lambda; \mathbb{Z})$ of Λ. A particularly useful presentation of this group is the Bruschlinsky group. Let $C(\Lambda, K)$ denote the continuous functions $\Lambda \to K$ and let $B(\Lambda, K) = \{\exp(2\pi i r) \mid r : \Lambda \to \mathbb{R} \text{ is continuous}\}$ denote those that are null-homotopic. Then $C(\Lambda, K)$ is an abelian group under pointwise multiplication and $B(\Lambda, K)$ is a subgroup. The quotient $C(\Lambda, K)/B(\Lambda, K)$ is called the *Bruschlinsky group* and is naturally isomorphic to $H^1(\Lambda; \mathbb{Z})$ [Hu] (using Čech cohomology is Λ is not a manifold).

Recall that there is a natural inclusion $H^1(\Lambda; \mathbb{Z}) \subset H^1(\Lambda; \mathbb{R})$. When ϕ is Anosov (so that Λ is a manifold) or more generally when ϕ is an attractor [FP], $H^1(\Lambda; \mathbb{R})$ is a finite dimensional vector space $\mathbb{R}^b$. From now on we assume that $H^1(\Lambda; \mathbb{R})$ is finite dimensional and refer to such ϕ as being *cohomologically finite*.

When Λ is not a manifold, $H^1(\Lambda; \mathbb{Z})$ is given by the direct limit of cohomology groups of simplicial complexes. As Λ is compact, we can choose a countable cofinal subsequence of such complexes and see that $H^1(\Lambda; \mathbb{Z})$ is a *rational* subgroup of $H^1(\Lambda; \mathbb{R}) = \mathbb{R}^b$.

The map ϕ induces an automorphism ϕ^* of $H^1(\Lambda; \mathbb{Z})$ given by $\phi^*[w] = [w\phi]$ where $w \in C(\Lambda, K)$. This automorphism extends uniquely to a linear automorphism of $H^1(\Lambda; \mathbb{R})$ which we also denote by ϕ^*.

We say that ϕ and α are *spectrally coprime* if the matrices representing ϕ^* and α have coprime characteristic polynomials. We shall denote by α the matrix in $Sl(n, \mathbb{Z})$ representing α. The basic fact in our analysis is the following easily proven lemma.

Lemma 5.1
Suppose ϕ and α are spectrally coprime. If $U \in \mathbb{R}^{n\times b}$ satisfies $U\phi^* = \alpha U$ (as $n \times b$ matrices) then $U = 0$.

Remark. In general, it is a non-trivial task to calculate the eigenvalues of ϕ^*. When ϕ is a hyperbolic toral automorphism then it is easy to check that ϕ^* has the same eigenvalues as ϕ. When ϕ is the Smale solenoid, $H^1(\Lambda, \mathbb{Z}) = \mathbb{Z}[1/2]$ and ϕ^* is given by multiplication by 2. In general, the only known fact about the spectrum of ϕ^* appears to be that the eigenvalues lie in an annulus with radii $e^{\pm h(\phi)}$ [SW] (h denotes topological entropy).

Theorem 5.2
Let ϕ be a cohomologically finite hyperbolic diffeomorphism on a connected basic set. Let α be an ergodic automorphism of K^n. Suppose ϕ and α are spectrally coprime. Then for each continuous $f : \Lambda \to G$, there are only finitely many solutions $u : \Lambda \to G$ to $f = u\phi \cdot \alpha u^{-1}$.

Proof. It is sufficient to show that there are only finitely many solutions to the equation $u\phi = \alpha u$.

Choose a basis $e_1, \ldots, e_b$ of $H^1(\Lambda; \mathbb{R})$ with $e_i \in H^1(\Lambda; \mathbb{Z})$. Let $u : \Lambda \to G$ be a continuous solution to $u\phi = \alpha u$. Write $u = (u_1, \ldots, u_n)$ with $u_i : \Lambda \to K$. Then $u_i = \exp(2\pi i r_i) e_1^{u_{i,1}} \cdots e_b^{u_{i,b}}$ for some continuous $r_i : \Lambda \to \mathbb{R}$ and integers $u_{i,j}$. Denote the matrix $(u_{i,j})$ by U.

One can easily check that the matrices determined by $u\phi$ and αu are $U\phi^*$ and αU, respectively. By Lemma 5.1, $U = 0$ so that each $u_i = \exp(2\pi i r_i)$ is null-homotopic.

Let $r = (r_1, \ldots, r_n) : \Lambda \to \mathbb{R}$. The equation $u\phi = \alpha u$ now takes the form

$$r(\phi x) = \alpha r(x) + m \tag{8}$$

in $\mathbb{R}^n$ for some $m \in \mathbb{Z}^n$. Let I denote the $n \times n$ identity matrix. Clearly, $r(x) = (I - \alpha)^{-1} m$, a constant, is a solution to (8). As α has no roots of unity as eigenvalues, $r(x) = (I - \alpha)^{-1} m$ is the only solution to (8).

Hence any solution u to $u\phi = \alpha u$ is a constant, therefore a fixed point of α, of which there are only finitely many. ❑

Recall from §3 that for a given f it may not be possible to find an automorphism β such that βf is an α-coboundary. This becomes possible if we assume α is hyperbolic and allow β to be an endomorphism (*c.f.* [Sc, Theorem 5.1]), although we may have to be content with a transfer function with a smaller Hölder exponent. The following is essentially proved in [KH, §2.9].

Lemma 5.3
Let α be a hyperbolic linear automorphism of $\mathbb{R}^n$. Then any continuous $r : \Lambda \to \mathbb{R}^n$ can be written as $r = u\phi - \alpha u$ for a (unique) continuous u. Let $C = \sup_{x\in\Lambda} \|d\phi|_{T_xM}\|$. If $r \in C^\theta(\Lambda, \mathbb{R}^n)$ and $\theta < \log \mu_0 / \log C$ then $u \in C^\theta(\Lambda, \mathbb{R}^n)$.

Proof. In [KH] an explicit formula for u is given from which the Hölder exponents follow easily. ❑

Remark. Note that $\log \mu_0 / \log C < \theta_{\text{crit}}$, so u may have a small Hölder exponent.

Proposition 5.4
Let ϕ be a cohomologically finite hyperbolic map and let α be a hyperbolic automorphism of the torus $G = K^n$. Suppose ϕ and α are spectrally co-prime. Let $f \in C^\theta(\Lambda, G)$. Then there exists an endomorphism β of G and a continuous u such that $\beta f = u\phi \cdot \alpha u^{-1}$. Moreover, $u \in C^{\theta'}(\Lambda, G)$ for any $\theta' < \log \mu_0 / \log C$.

Proof. Let F denote the matrix associated to f as in the proof of Theorem 5.2. We first solve the matrix equation $\beta F = U\phi^* - \alpha U$ where F, ϕ^*, α are known and β, U are to be found with integer coefficients. Let L denote the linear map on the space of $n \times b$ matrices given by $L(U) = U\phi^* - \alpha U$. By Lemma 5.1, L is invertible and by the discussion at the beginning of this section, L has rational entries. Let $\beta = \text{diag}(p_1, \ldots, p_n)$ for integers p_i. If the p_i are chosen to be sufficiently large then $U = L^{-1}(\beta F)$ has integer entries, $U = (u_{i,j})$.

Write $f = (f_1, \ldots, f_n)$, $f_i = \exp(2\pi i r_i) e_1^{f_{i,1}} \cdots e_b^{f_{i,b}}$ and let $r = (r_1, \ldots, r_n)$. By Lemma 5.3, $\beta r = v\phi - \alpha v$ for some $v \in C^{\theta'}(\Lambda, G)$, $\theta' < \log \mu_0 / \log C$.

Define $u_i = \exp(2\pi i v_i) e_1^{u_{i,1}} \cdots e_b^{u_{i,b}}$ so that $\beta f = u\phi \cdot \alpha u^{-1}$. ❑

Remark. Examples show that if α is an ergodic but not hyperbolic toral automorphism then Proposition 5.4 fails.

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