Stable ergodicity of skew products of one-dimensional hyperbolic flows

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Abstract

We consider hyperbolic flows on one dimensional basic sets. Any such flow is conjugate to a suspension of a shift of finite type. We consider compact Lie group skew-products of such symbolic flows and prove that they are stably ergodic and stably mixing, within certain naturally defined function spaces.

§1 Introduction

In a recent sequence of papers [AKS, P, FP, W2], skew products over a hyperbolic base were considered. In discrete time, these are maps of the form $\phi_f(x,y) = (\phi x, f(x)y)$ on $\Lambda \times G$, where ϕ is a hyperbolic diffeomorphism of Λ, G is a compact connected Lie group and $f : \Lambda \to G$ is, say, smooth. If μ is a ϕ-invariant measure, then $\mu \times d\mathrm{Haar}$ is ϕ_f-invariant. One can ask whether ϕ_f is ergodic with respect to $\mu \times d\mathrm{Haar}$. In [FP] it is proved that, for a large class of hyperbolic diffeomorphisms ϕ, there is an open dense set of f (in an appropriate topology) for which the map ϕ_f is ergodic; we say that ϕ_f is *stably ergodic*.

The case of continuous time is considered in [W2]. A skew product of a hyperbolic flow $\phi^t : \Lambda \to \Lambda$ is a flow of the following form. Let $\mathcal{F} : \Lambda \times \mathbb{R} \to G$ be a map such that $\mathcal{F}^{t+s}(x) = \mathcal{F}^t(\phi^s x)\mathcal{F}^s(x)$ (such a map is called a *cocycle*). Define the *skew product flow* $\phi^t_{\mathcal{F}} = (\phi^t x, \mathcal{F}^t(x)y)$ on $\Lambda \times G$. Note that a cocycle $\mathcal{F}$ determines and is determined by the Lie algebra valued function $f(x) = \lim_{t\to 0} \mathcal{F}^t(x)/t$.

Let ϕ^t be a hyperbolic flow restricted to a basic set Λ (see [KH] or [W2] for a precise definition). Flows of the form $\phi^t_{\mathcal{F}}$ were considered in [W2] under the hypothesis that the first Čech cohomology group $H^1(\Lambda;\mathbb{Z})$ of Λ has finite rank (this is the case when ϕ^t is Anosov, or more generally, an attractor). It was proved that the set of cocycles $\mathcal{F}$ for which the skew product $\phi^t_{\mathcal{F}}$ is ergodic

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or mixing contains an open dense set in the spaces of Hölder, C^1 and smooth functions, as appropriate.

The simplest case when $H^1(\Lambda;\mathbb{Z})$ does not have finite rank is when Λ is one dimensional. It is well-known that such a flow must be conjugate to a suspension of a shift of finite type [B]. In this note, we consider skew products over a suspension of a shift of finite type $\sigma_r^t : \widetilde{\Sigma}_r \to \widetilde{\Sigma}_r$ and prove that there exist open dense sets of ergodic and mixing skew products (Theorem 3.1). To be able to state such a result more precisely, we need to consider an appropriate function space on the suspension space $\widetilde{\Sigma}_r$. Such function spaces do not appear to have been considered in detail in the literature.

The aim of this note is two-fold: firstly, to define and examine the relationships between various natural space of Hölder functions on $\widetilde{\Sigma}_r$ and secondly, to prove the stable ergodicity of skew products within these spaces.

§2 Suspensions of shifts of finite type

We discuss suspensions of shifts of finite type and three notions of 'Hölder continuous' functions. We first recall some facts about shifts of finite type.

Shifts of finite type. Let A be an aperiodic $k \times k$ matrix with entries from $\{0,1\}$. Define

$$\Sigma = \{(x_j)_{j=-\infty}^{\infty} \in \{1,\ldots,k\}^{\mathbb{Z}} \mid A_{x_j,x_{j+1}} = 1 \text{ for } j \in \mathbb{Z}\}.$$

Then Σ is invariant under the shift σ defined by $(\sigma x)_j = x_{j+1}$.

For each $0 < \theta < 1$ define a metric d^θ on Σ by $d^\theta(x,y) = \theta^{n(x,y)}$ where $n(x,y) = \sup\{n \mid x_j = y_j,\ |i| \le n\}$.

Let $F_\theta(\Sigma;\mathbb{R})$ denote the space of all functions $\Sigma \to \mathbb{R}$ that are Lipschitz continuous with respect to d^θ. Then $F_\theta(\Sigma;\mathbb{R})$ is a Banach space with respect to the norm $\|f\|_\theta = |f|_\infty + |f|_\theta$ where $|f|_\theta$ is the least Lipschitz constant of $f \in F_\theta(\Sigma;\mathbb{R})$. We shall often refer to F_θ-functions as Hölder functions of exponent θ.

Define the following subspaces of $F_\theta(\Sigma;\mathbb{R})$:

$$\begin{aligned} B_\theta &= \{u\sigma - u \mid u \in F_\theta\}, \\ V_\theta &= \mathrm{cl}\,\{v : \Sigma \to \mathbb{R} \mid v \text{ is locally constant}\}. \end{aligned}$$

(The closure of the locally constant functions is taken with respect to the F_θ topology.) If $b \in B_\theta$ then we call b a *coboundary*.

The following is proved in [PP2].

Theorem 2.1
$B_\theta + V_\theta$ is a closed subspace of $F_\theta(\Sigma;\mathbb{R})$ and $F_\theta(\Sigma;\mathbb{R})/(B_\theta + V_\theta)$ is non-separable.

For a continuous $f : \Sigma \to \mathbb{R}$ we define the *pressure* of f to be

$$P(f) = \sup\{h_m(\sigma) + m \mid m \text{ is a } \sigma\text{-invariant probability}\}. \tag{1}$$

For $f \in F_\theta(\Sigma;\mathbb{R})$ this supremum is attained by a unique σ-invariant probability μ; we call this the *equilibrium state* of f. It is well known that (σ,μ) is mixing.

Suspensions of shifts of finite type. Let $r \in F_\theta(\Sigma;\mathbb{R})$ be strictly positive. Define

$$\widetilde{\Sigma}_r = \{(x,t) \mid x \in \Sigma, 0 \le t \le r(x)\}/\sim$$

where $\sim$ identifies $(x, r(x))$ with $(\sigma x, 0)$. Then $\widetilde{\Sigma}_r$ is a compact, one-dimensional space. We define the suspended flow $\sigma_r^t : \widetilde{\Sigma}_r \to \widetilde{\Sigma}_r$ by $\sigma_r^t(x,u) = (x, u+t)$ for small t and then extend to all $t \in \mathbb{R}$ by using the identifications. Thus σ_r^t can be thought of as the flow that travels along the fibre $[0, r(x)]$ over $x \in \Sigma$ at unit speed and then 'jumps' to the fibre over σx. Alternatively, we can think of σ_r^t as the flow on the space $\tilde{\Sigma}_1$ where the fibre over x is traversed at speed $1/r(x)$. It will usually simplify our notation to consider $\widetilde{\Sigma}_r$.

We now define three possible notions of 'Hölder' continuous functions. Many of the proofs are somewhat technical and we postpone them to §4.

Hölder functions. In [BW] a metric is constructed on the constant suspension space $\widetilde{\Sigma}_1$. We modify their construction to obtain a metric on $\widetilde{\Sigma}_r$.

For each $0 \le t \le 1$ define the t^{th} *r-section* of $\widetilde{\Sigma}_r$ to be $\Sigma_{r,t} = \{(x, tr(x)) \mid x \in \Sigma\}$. Then $\widetilde{\Sigma}_r$ is the disjoint union of the $\Sigma_{t,r}$ for $0 \le t < 1$ and $\Sigma_{1,r}$ is identified with $\Sigma_{0,r}$. Let $r_0 = \inf\{r(x) \mid x \in \Sigma\}$. We define a metric d_t^θ on each $\Sigma_{t.r}$ by

$$d_t^\theta((x, tr(x)), (y, tr(y))) = r_0((1-t)\, d^\theta(x,y) + t\, d^\theta(\sigma x, \sigma y)).$$

Observe that d_0^θ and d_1^θ are compatible with the identifications between $\Sigma_{0,r}$ and $\Sigma_{1,r}$. For $0 \le t < 1$, each $\Sigma_{t,r}$ can be canonically identified with Σ and it will often simplify our notation to regard d_t^θ as a metric on Σ. It is easy to see that the d_t^θ are all pairwise equivalent. Indeed, more is true:

Lemma 2.2
There exists a constant $C > 0$ such that for all $0 \le t', t \le 1$ we have $d_{t'}^\theta(x,y) \le C\, d_t^\theta(x,y)$ for all $x, y \in \Sigma$.

Let $\tilde{x}, \tilde{y} \in \widetilde{\Sigma}_r$. A *path* from $\tilde{x}$ to $\tilde{y}$ is a finite sequence of points $\tilde{x} = w_0, w_1, \ldots, w_{n-1}, w_n = \tilde{y}$ such that for each i either:

1. w_i, w_{i+1} lie in the same r-section—in this case, set $|w_i - w_{i+1}| = d_t^\theta(w_i, w_{i+1})$ if $w_i, w_{i+1} \in \Sigma_{t,r}$; or

2. w_i, w_{i+1} lie in the same σ_r^t-orbit—in this case, set $|w_i - w_{i+1}| = \inf\{|\tau| \mid \sigma_r^\tau w_i = w_{i+1}\}$.

It could happen that both possibilities occur: w_i, w_{i+1} lie in the same r-section and the same σ_r^t-orbit. In this case, we always take the first definition of $|w_i - w_{i+1}|$ (indeed, it is readily checked that $d_t^\theta(w_i, w_{i+1}) \le \inf\{|\tau| \mid \sigma_r^\tau w_i = w_{i+1}\}$ and this is the reason for introducing the factor r_0 in the definition of d_t^θ). If w is a path from $\tilde{x}$ to $\tilde{y}$ then we set $|w| = \sum_{i=0}^{n-1} |w_i - w_{i+1}|$ and define $d_r^\theta(\tilde{x}, \tilde{y}) = \inf\{|w| \mid w \text{ is a path from } \tilde{x} \text{ to } \tilde{y}\}$. Then d_r^θ is a metric and gives $\widetilde{\Sigma}_r$ the correct topology.

The following lemma says that the time-t map σ_r^t is Lipschitz continuous on $\Sigma \times \{0\}$.

Lemma 2.3
For each $t \geq 0$ there exists a constant $C(t)$ such that $d_r^\theta(\sigma_r^t(x,0), \sigma_r^t(y,0)) \leq C(t)\, d^\theta(x,y)$. for all $x, y \in \Sigma$.

Let $F_\theta(\widetilde{\Sigma}_r; \mathbb{R})$ denote the space of functions $\widetilde{\Sigma}_r \to \mathbb{R}$ that are Lipschitz continuous with respect to d_r^θ. Equip $F_\theta(\widetilde{\Sigma}_r; \mathbb{R})$ with the norm $\|f\|_\theta := |f|_\infty + |f|_\theta$ where $|f|_\theta = \sup_{\tilde{x} \neq \tilde{y}} |f(\tilde{x}) - f(\tilde{y})|/d_r^\theta(\tilde{x}, \tilde{y})$ is the least Lipschitz constant of $f \in F_\theta(\widetilde{\Sigma}_r; \mathbb{R})$. Then $F_\theta(\widetilde{\Sigma}_r; \mathbb{R})$ is a Banach space.

One should note that $F_\theta(\widetilde{\Sigma}_r; \mathbb{R})$ is very sensitive on the metric d_r^θ. If one changes d_r^θ (even just on σ_r^t orbits) then one may change $F_\theta(\widetilde{\Sigma}_r; \mathbb{R})$.

Sectionally Hölder functions. Our second space of Hölder functions is the non-constant suspension analogue of one considered in [Rue]. For a continuous $f : \widetilde{\Sigma}_r \to \mathbb{R}$ and $t \geq 0$ define

$$C_f(t) = \sup_{0 \leq u \leq t} \sup_{x,y \in \Sigma, x \neq y} \frac{|f\sigma_r^u(x,0) - f\sigma_r^u(y,0)|}{d^\theta(x,y)}.$$

Let $F_\theta^{\mathrm{S}}(\widetilde{\Sigma}_r; \mathbb{R}) = \{f : \widetilde{\Sigma}_r \to \mathbb{R} \mid f \text{ is continuous and } C_f(|r|_\infty) < \infty\}$. Then $F_\theta^{\mathrm{S}}(\widetilde{\Sigma}_r; \mathbb{R})$ is a Banach space with respect to the norm $\|f\|_\theta^{\mathrm{S}} = |f|_\infty + C_f(|r|_\infty)$. Thus $f \in F_\theta^{\mathrm{S}}(\widetilde{\Sigma}_r; \mathbb{R})$ if f restricted to each cross-section $\Sigma \times \{t\}$ is Hölder. Alternatively, we could consider functions f whose restriction to each r-section $\{(x, tr(x))\}$ is Hölder. These spaces are distinct, although they behave in the same way in the sequel and it make little difference which space we use.

Integrally Hölder functions. Perhaps the most natural space of Hölder functions is the following. If $f : \widetilde{\Sigma}_r \to \mathbb{R}$ is continuous, define

$$\mathcal{I}f(x) = \int_0^{r(x)} f\sigma_r^u(x,0)\, du : \Sigma \to \mathbb{R}. \tag{2}$$

Let $F_\theta^{\mathrm{I}}(\widetilde{\Sigma}_r; \mathbb{R})$ denote the space of continuous functions for which $\mathcal{I}f \in F_\theta(\Sigma; \mathbb{R})$. Then $F_\theta^{\mathrm{I}}(\widetilde{\Sigma}_r; \mathbb{R})$ is a Banach space with respect to the norm $\|f\|_\theta^{\mathrm{I}} = |f|_\infty + \|\mathcal{I}f\|_\theta$.

Clearly the map $\mathcal{I} : F_\theta^{\mathrm{I}}(\widetilde{\Sigma}_r; \mathbb{R}) \to F_\theta(\Sigma; \mathbb{R})$ defined by (2) is a continuous linear map.

The relationships between the spaces defined above is as follows.

Proposition 2.4
There exist dense continuous strict inclusions

$$F_\theta(\widetilde{\Sigma}_r; \mathbb{R}) \hookrightarrow F_\theta^{\mathrm{S}}(\widetilde{\Sigma}_r; \mathbb{R}) \hookrightarrow F_\theta^{\mathrm{I}}(\widetilde{\Sigma}_r; \mathbb{R}).$$

Moreover $\mathcal{I} : F_\theta(\widetilde{\Sigma}_r; \mathbb{R}) \to F_\theta(\Sigma; \mathbb{R})$ is onto.

Equilibrium states. Finally, we define equilibrium states for σ_r^t. They can be defined in analogy with (1) but we can also relate them to the underlying shift of finite type as follows. Let $f \in F_\theta^{\mathrm{I}}(\widetilde{\Sigma}_r; \mathbb{R})$. Then the pressure of f is the unique real number $P(f)$ such that $P(\mathcal{I}f - P(f)r) = 0$ [PP1, for example]. Moreover, the (unique) equilibrium state μ of f is given by $\mu = m \times \nu / m(r)$ where m is the equilibrium state of $\mathcal{I}f - P(f)r$ and ν is Lebesgue measure on the fibres $[0, r(x)]$. The flow σ_r^t is ergodic with respect to μ, but not necessarily mixing.

§3 Toral skew products of symbolic flows

Let K denote the unit circle in $\mathbb{C}$ and let $G = K^n$ be the n-dimensional torus. Let F_θ denote any of $F_\theta(\widetilde{\Sigma}_r;\mathbb{R})$, $F_\theta^{\mathrm{S}}(\widetilde{\Sigma}_r;\mathbb{R})$, $F_\theta^{\mathrm{I}}(\widetilde{\Sigma}_r;\mathbb{R})$. If $f_1,\ldots,f_n \in F_\theta$ then we have a toral skew product $\sigma_{r,f}^t$

$$\sigma_{r,f}^t(\tilde{x}, y_1, \ldots, y_n) = \left(\sigma_r^t(\tilde{x}), e^{2\pi i \int_0^t f_1(\sigma_r^u \tilde{x})\,du} y_1, \ldots, e^{2\pi i \int_0^t f_n(\sigma_r^u \tilde{x})\,du} y_n\right). \quad (3)$$

If $\sigma_{r,f}^t$ is weak mixing then it is automatically Bernoulli [Rud]; for brevity we say $\sigma_{r,f}^t$ is *mixing*. We prove the following.

Theorem 3.1
Let μ be a Hölder equilibrium state for σ_r^t. Let F_θ denote any of $F_\theta(\widetilde{\Sigma}_r;\mathbb{R})$, $F_\theta^{\mathrm{S}}(\widetilde{\Sigma}_r;\mathbb{R})$, $F_\theta^{\mathrm{I}}(\widetilde{\Sigma}_r;\mathbb{R})$. Then the set of n-tuples $(f_1,\ldots,f_n) \in F_\theta^n$ for which (3) is ergodic with respect to $\mu \times d\mathrm{Haar}$ contains an open dense set. If σ_r^t is mixing then there is an open dense set of mixing skew products.

The following is a well-known criterion for the ergodicity of $\sigma_{r,f}^t$ [KN].

Proposition 3.2
The following are equivalent:

1. *$\sigma_{r,f}^t$ is not ergodic;*
2. *there exist integers $l_1,\ldots,l_n$ (not all zero) and a measurable function $w : \widetilde{\Sigma}_r \to K$ such that*

$$\frac{w(\sigma_r^t \tilde{x})}{w(\tilde{x})} = \exp 2\pi i \int_0^t \sum_{i=1}^n l_i f_i(\sigma_r^u \tilde{x})\,du \text{ a.e.}$$

We shall need the following regularity theorem, which is valid in far more general circumstances [W1].

Theorem 3.3 (Livšic regularity theorem for symbolic flows)
Let μ be a Hölder equilibrium state for σ_r^t and let $f \in F_\theta$. Suppose $\tilde{w} : \widetilde{\Sigma}_r \to K$ is a measurable solution to the cohomological equation

$$\frac{\tilde{w}\sigma_r^t(\tilde{x})}{\tilde{w}(\tilde{x})} = \exp 2\pi i \int_0^t f\sigma_r^u(\tilde{x})\,du \text{ a.e.}$$

Then there exists a unique continuous $\tilde{w}'$ such that $\tilde{w}' = \tilde{w}$ a.e.,

$$\frac{\tilde{w}'\sigma_r^t(\tilde{x})}{\tilde{w}'(\tilde{x})} = \exp 2\pi i \int_0^t f\sigma_r^u(\tilde{x})\,du$$

everywhere and $\tilde{w}'(\cdot,0) \in F_\theta(\Sigma;K)$.

The following criterion is the key to our analysis,

Proposition 3.4
If $(\mathcal{I}f_1,\ldots,\mathcal{I}f_n)$ is linearly independent mod $B_\theta + V_\theta$ then $\sigma_{r,f}^t$ is ergodic.

Proof. If $\sigma_{r,f}^t$ is not ergodic then there exist integers $l_1, \ldots, l_n$ (not all zero) and a measurable $\tilde{w} : \widetilde{\Sigma}_r \to K$ such that

$$\frac{\tilde{w}\sigma_r^t(\tilde{x})}{\tilde{w}(\tilde{x})} = \exp 2\pi i \int_0^t \sum_{i=1}^n l_i f_i \sigma_r^u(\tilde{x})\, du \text{ a.e. } [m]. \tag{4}$$

By Theorem 3.3, $\tilde{w}$ is continuous, (4) holds everywhere and $x \mapsto \tilde{w}(x, 0) \in F_\theta(\Sigma; K)$. As Σ is zero-dimensional, we can write $\tilde{w}(x, 0) = \exp 2\pi i W(x)$ for some $W \in F_\theta(\Sigma; \mathbb{R})$. Setting $t = r(x)$ it follows that $\exp 2\pi i\,(W(\sigma x) - W(x)) = \exp 2\pi i \sum_{i=1}^n l_i \mathcal{I} f_i(x)$. Hence, $W\sigma - W + V = \sum_{i=1}^n l_i \mathcal{I} f_i$ for some continuous integer-valued, hence locally constant, $V \in V_\theta$. Thus $(\mathcal{I}f_1, \ldots, \mathcal{I}f_n)$ is integrally dependent, hence linearly dependent, mod $B_\theta + V_\theta$. ❏

We now show that the condition in Proposition 3.4 is an open dense condition on the n-tuples $(f_1, \ldots, f_n)$.

Proposition 3.5
The set of n-tuples $(f_1, \ldots, f_n) \in F_\theta$ for which $(\mathcal{I}f_1, \ldots, \mathcal{I}f_n)$ is linearly independent mod $B_\theta + V_\theta$ is an open set.

Proof. We prove that the set of n-tuples $(f_1, \ldots, f_n) \in F_\theta$ with $(\mathcal{I}f_1, \ldots, \mathcal{I}f_n)$ linearly dependent mod $B_\theta + V_\theta$ is closed. Suppose $f_m = (f_{1,m}, \ldots, f_{n,m}) \in F_\theta$ is such that $f_m \to f = (f_1, \ldots, f_n)$ in F_θ^n and, for each m, $(\mathcal{I}f_{1,m}, \ldots, \mathcal{I}f_{n,m})$ is linearly dependent mod $B_\theta + V_\theta$. Then there exist real numbers $\lambda_{1,m}, \ldots, \lambda_{n,m}$, not all zero, such that

$$\lambda_{1,m}\mathcal{I}f_{1,m} + \cdots + \lambda_{n,m}\mathcal{I}f_{n,m} = 0 \bmod B_\theta + V_\theta.$$

We may assume that $\lambda_{1,m} \neq 0$ and $|\lambda_{1,m}| \geq |\lambda_{i,m}|$, $2 \leq i \leq n$, for infinitely many m. Dividing by $\lambda_{1,m}$ we have

$$\mathcal{I}f_{1,m} + \mu_{2,m}\mathcal{I}f_{2,m} + \cdots + \mu_{n,m}\mathcal{I}f_{n,m} = 0 \bmod B_\theta + V_\theta$$

and $|\mu_{i,m}| \leq 1$ for $2 \leq i \leq n$. By taking a subsequence, we can assume that $\mu_{i,m} \to \mu_i$ as $m \to \infty$. As $\mathcal{I}$ is continuous and $f_{i,m} \to f_i$, it follows that

$$\mathcal{I}f_1 + \mu_2\mathcal{I}f_2 + \cdots + \mu_m\mathcal{I}f_m = 0 \bmod B_\theta + V_\theta,$$

as $B_\theta + V_\theta$ is closed. ❏

We now show that the set of n-tuples $(f_1, \ldots, f_n)$ for which $(\mathcal{I}f_1, \ldots, \mathcal{I}f_n)$ is linearly independent mod $B_\theta + V_\theta$ is dense. Fix $(f_1, \ldots, f_n) \in F_\theta^n$ and let $V = B_\theta + V_\theta + \operatorname{span}\{\mathcal{I}f_1, \ldots, \mathcal{I}f_n\}$ so that V is a closed subspace of F_θ. As $\mathcal{I}$ is continuous, $\mathcal{I}^{-1}V$ is a closed subspace of F_θ.

Lemma 3.6
$F_\theta/\mathcal{I}^{-1}V$ is non-separable.

Proof. As $B_\theta + V_\theta$ has finite index in V, it follows from Theorem 2.1 that F_θ/V is non-separable. As $\mathcal{I}(\mathcal{I}^{-1}V) \subset V$, we have a well-defined map $\mathcal{I}_V : F_\theta/\mathcal{I}^{-1}V \to F_\theta(\Sigma; \mathbb{R})/V$. As $\mathcal{I}$ is onto, so is $\mathcal{I}_V$. The lemma follows from this, together with the fact that $\mathcal{I}_V$ is continuous. ❏

Proposition 3.7
The set of n-tuples $(f_1,\dots,f_n)\in F_\theta^n$ such that $(\mathcal{I}f_1,\dots,\mathcal{I}f_n)$ is linearly independent mod $B_\theta+V_\theta$ is dense.

Proof. Fix $(f_1,\dots,f_n)\in F_\theta^n$ and let $V=\mathcal{B}_\theta+V_\theta+\operatorname{span}\{\mathcal{I}f_1,\dots,\mathcal{I}f_n\}$. Choose small functions $\varepsilon_i\in F_\theta$ such that the ε_i are linearly independent mod $\mathcal{I}^{-1}V$ (this is possible as $F_\theta/\mathcal{I}^{-1}V$ is non-separable, hence infinite dimensional). Let $f_i'=f_i+\varepsilon_i$. Then f_i' is $\|\cdot\|_\theta^\sim$-close to f_i. Moreover, $(\mathcal{I}f_1',\dots,\mathcal{I}f_n')$ is linearly independent mod $B_\theta+V_\theta$. For if $\sum_{i=1}^n\lambda_i\,\mathcal{I}f_1'=0 \bmod B_\theta+V_\theta$ then $\sum\lambda_i\,\mathcal{I}\varepsilon_i=\mathcal{I}(\sum\lambda_i\varepsilon_i)=0 \bmod V$. Hence $\sum\lambda_i\varepsilon_i=0 \bmod \mathcal{I}^{-1}V$ and it follows that $\lambda_i=0$ for $1\le i\le n$. ❑

Proof of Theorem 3.1. By Proposition 3.4, the set of n-tuples $f=(f_1,\dots,f_n)$ for which $\sigma_{r,f}^t$ is ergodic contains the n-tuples for which $(\mathcal{I}f_1,\dots,\mathcal{I}f_n)$ is linearly independent mod $B_\theta+V_\theta$. By Proposition 3.5, this set is open. By Proposition 3.7, it is also dense.

Suppose σ_r^t is mixing. Then there is a well-known analogue of Proposition 3.2: $\sigma_{r,f}^t$ is not mixing if and only if

$$\frac{w(\sigma_r^t\tilde{x})}{w(\tilde{x})}=\exp 2\pi i\int_0^t c+\sum_{i=1}^n l_i f_i(\sigma_r^u\tilde{x})\,du \text{ a.e.}$$

for a measurable $w:\widetilde{\Sigma}_r\to K$, a constant $c\neq 0$ and integers $l_1,\dots,l_n$. Applying Theorem 3.3, w is continuous. The same argument as in Proposition 3.4 shows that if $(\mathcal{I}f_1,\dots,\mathcal{I}f_n)$ is linearly independent mod $B_\theta+V_\theta+\operatorname{span}\{r\}$ then $\sigma_{r,f}^t$ is mixing. This is again an open dense condition. ❑

Remark. If σ_r^t is not mixing then there is a measurable function $w:\widetilde{\Sigma}_r\to K$ with a non-zero eigenfrequency: $w(\sigma_r^t\tilde{x})/w(\tilde{x})=\exp 2\pi ict$. By Theorem 3.3, w is continuous. Setting $t=r(x)$ and mimicking the proof of Proposition 3.4, we have $r=0 \bmod B_\theta+V_\theta$. Thus there is an open dense set of r for which σ_r^t is mixing.

More generally, we could consider a skew product defined on $\Lambda\times G$ where G is a compact connected Lie group. When G is semi-simple, the arguments for the case of discrete time apply [FP, W2] and semi-simple skew products are stably ergodic and mixing. A general compact connected Lie group is locally the product of a torus and a semi-simple Lie group and it is not hard to conclude that such skew products are also stably ergodic and mixing [FP, W2].

§4 Technicalities

It remains to prove the statements in §2.

Proof of Lemma 2.2. We prove the following: $\exists C>0$ s.t. $\forall 0\le t'\le t\le 1$, $\forall x,y\in\Sigma$, $d_{t'}^\theta(x,y)\le C\,d_t^\theta(x,y)$. The corresponding statement for $0\le t\le t'\le 1$ is proved similarly.

Observe that $\theta\,d^\theta(x,y)\le d^\theta(\sigma x,\sigma y)\le\theta^{-1}\,d^\theta(x,y)$. Then

$$\begin{aligned} d_{t'}^\theta(x,y) &= r_0((1-t')\,d^\theta(x,y)+t'\,d^\theta(\sigma x,\sigma y))\\ &\le r_0((2-t)\,d^\theta(x,y)+\theta^{-1}t'\,d^\theta(x,y)) \end{aligned}$$

$$\begin{aligned} &\leq r_0 \frac{(2-t+\theta^{-1}t)}{(\theta^{-1}(1-t)+t)}\,\theta^{-1}((1-t)\,d^\theta(x,y)+\theta t\,d^\theta(x,y)) \\ &\leq \frac{(2-t+\theta^{-1}t)}{(\theta^{-1}(1-t)+t)}\,\theta^{-1}\,d^\theta_t(x,y). \end{aligned}$$

The map $t \mapsto (2-t+\theta^{-1})/(\theta^{-1}(1-t)+t)$ is bounded above on $[0,1]$ by $\theta^{-1}+1$. Hence $d^\theta_{t'}(x,y) \leq \theta^{-1}(\theta^{-1}+1)\,d^\theta_t(x,y)$. ❑

Proof of Lemma 2.3. Let $x,y \in \Sigma$ and fix $t \geq 0$. Let n be the unique integer such that $r^n(x) \leq t < r^{n+1}(x)$ (here and throughout this proof $r^n(x) = \sum_{j=0}^{n-1} r(\sigma^j x)$). Then

$$\sigma_r^t(x,0) = \left(\sigma^n x, \frac{t-r^n(x)}{r(\sigma^n x)} r(\sigma^n x)\right) \in \Sigma_{\frac{t-r^n(x)}{r(\sigma^n x)},r}.$$

Let $\tilde{y}$ denote the point

$$\tilde{y} = \left(\sigma^n y, \frac{t-r^n(x)}{r(\sigma^n x)} r(\sigma^n y)\right) \in \Sigma_{\frac{t-r^n(x)}{r(\sigma^n x)},r}.$$

Then $\sigma_r^t(x,0)$ and $\tilde{y}$ are in the same r-section. By Lemma 2.2, there exists a constant $C_1 > 0$ such that

$$d^\theta_{\frac{t-r^n(x)}{r(\sigma^n x)}}(\sigma_r^t(x,0),\tilde{y}) \leq C_1\,d_0^\theta(\sigma^n x,\sigma^n y) \leq \frac{C_1 r_0}{\theta^n}\,d^\theta(x,y).$$

Clearly, $\sigma_r^t(y,0)$ and $\tilde{y}$ lie in the same σ_r^t-orbit. Now $\sigma_r^t(y,0)$ is identified with (y,t) and $\tilde{y}$ is identified with

$$\left(y, \frac{(t-r^n(x))}{r(\sigma^n x)} r(\sigma^n y) + r^n(y)\right).$$

Hence, the 'vertical' distance between $\tilde{y}$ and $\sigma_r^t(y,0)$ is no more than

$$\begin{aligned} &\left|\frac{(t-r^n(x))}{r(\sigma^n x)} r(\sigma^n y) + r^n(y) - t\right| \\ \leq\ & \frac{1}{r_0}\left(|t(r(\sigma^n x)-r(\sigma^n y))| + |r^n(y)r(\sigma^n x) - r^n(x)r(\sigma^n y)|\right). \end{aligned}$$

The first term is bounded above by $|r|_\infty |r|_\theta \theta^{-n}\,d^\theta(x,y)$ as $r \in F_\theta(\Sigma;\mathbb{R})$. The second term is bounded by

$$|r^n(y)|\,|r(\sigma^n x)-r(\sigma^n y)| + |r(\sigma^n y)|\,|r^n(x)-r^n(y)|. \tag{5}$$

An easy computation shows that $|r^n(x)-r^n(y)| \leq C_2\,d^\theta(x,y)$, for some constant $C_2 > 0$. Hence

$$\begin{aligned} (5) &\leq |r|_\infty |r|_\theta \theta^{-n}\,d^\theta(x,y) + |r|_\infty C_2\,d^\theta(x,y) \\ &\leq C_3\,d^\theta(x,y). \end{aligned}$$

for some constant $C_3 > 0$. Thus the vertical distance between $\tilde{y}$ and $\sigma_r^t(y,0)$ is bounded above by $C_4\,d^\theta(x,y)$ for some $C_4 > 0$.

The sequence $\sigma_r^t(x,0), \tilde{y}, \sigma_r^t(y,0)$ is a path from $\sigma_r^t(x,0)$ to $\sigma_r^t(y,0)$ and has length at most $C_1 r_0 \theta^{-n}\,d^\theta(x,y) + C_4\,d^\theta(x,y) = C(t)\,d^\theta(x,y)$. Hence there exists a constant $C(t) > 0$ such that $d_r^\theta(\sigma_r^t(x,0),\sigma_r^t(y,0)) \leq C(t)\,d^\theta(x,y)$. ❑

Proof of Proposition 2.4. By Lemma 2.3, $C_f(|r|_\infty) \leq |f|_\theta$ so that $F_\theta(\widetilde{\Sigma}_r;\mathbb{R}) \subset F_\theta^{\mathrm{S}}(\widetilde{\Sigma}_r;\mathbb{R})$ and this inclusion is continuous.

Now

$$\begin{aligned}
|\mathcal{I}f(x) - \mathcal{I}f(y)| &= \left| \int_0^{r(x)} f\sigma_r^u(x,0)\,du - \int_0^{r(y)} f\sigma_r^u(y,0)\,du \right| \\
&\leq \int_0^{r(x)} |f\sigma_r^u(x,0) - f\sigma_r^u(y,0)|\,du + \left| \int_{r(x)}^{r(y)} |f\sigma_r^u(y,0)|\,du \right| \\
&\leq (C_f(|r|_\infty)|r|_\infty + |f|_\infty |r|_\theta) d^\theta(x,y)
\end{aligned}$$

so that $F_\theta^{\mathrm{S}}(\widetilde{\Sigma}_r;\mathbb{R}) \subset F_\theta^{\mathrm{I}}(\widetilde{\Sigma}_r;\mathbb{R})$ is continuous.

It is easy to see that both inclusions are dense.

Finally, we show that $\mathcal{I} : F_\theta(\widetilde{\Sigma}_r;\mathbb{R}) \to F_\theta(\Sigma,\mathbb{R})$ is onto. Let $f \in F_\theta(\Sigma;\mathbb{R})$ and choose a C^1 map $\tau : [0,1] \to \mathbb{R}$ such that $\tau \geq 0$, $\tau(0) = 0 = \tau(1)$ and $\int_0^1 \tau(u)\,du = 1$. Define

$$\tilde{f}(x,t) = \frac{f(x)}{r(x)} \tau\left(\frac{t}{r(x)}\right).$$

Then $\tilde{f}$ is a well-defined continuous function on $\widetilde{\Sigma}_r$ and $\mathcal{I}\tilde{f} = f$. It is easy to check from the definitions that $\tilde{f} \in F_\theta(\widetilde{\Sigma}_r;\mathbb{R})$. ❏

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