Stable ergodic properties of cocycles over hyperbolic attractors

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Abstract

We consider a hyperbolic flow ϕ^t defined on an attracting basic set Λ. A map from the first (Čech) cohomology group of Λ into the dynamic cohomology group is constructed. This map is used to discuss the stable ergodicity and mixing of compact Lie group extensions and velocity changes of ϕ^t.

§1 Introduction

In [PS] it is conjectured that 'a little hyperbolicity goes a long way in guaranteeing stable ergodic behaviour'. By *stably ergodic* we mean the existence of an open dense set of suitable perturbations for which the perturbed dynamical system is ergodic. In particular, in [PS] conditions are given for a volume preserving partially hyperbolic diffeomorphism to be stably ergodic in the class of all volume preserving diffeomorphisms. An example of a partially hyperbolic diffeomorphism is given by a *group extension* or *skew product* of a hyperbolic diffeomorphism. These are maps of the form $\phi_f(x, y) = (\phi x, f(x)y)$ on $\Lambda \times G$ where ϕ is a hyperbolic diffeomorphism, G is a compact Lie group and $f : \Lambda \to G$ is smooth. One can ask how the ergodic properties of ϕ_f depend on f. This has been considered by Adler, Kitchens and Shub [AKS] and by Parry and his co-authors [P2, PP1, FP]. When G is a torus, it is proved in [PP1] that such skew products are stably ergodic and stably mixing. Analogous results when G is a compact connected Lie group are proved in [FP].

In this paper we consider the case of continuous time. We prove:

Theorem A
Let ϕ^t be a hyperbolic flow on a manifold Λ with vector field X_ϕ and preserving a suitable measure μ. Let G be a compact connected Lie group and let $\mathcal{F} : \Lambda \times \mathbb{R} \to G$ be a map such that $\mathcal{F}^{t+s}(x) = \mathcal{F}^t(\phi^s x)\mathcal{F}^s(x)$. Define a flow on

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$\Lambda \times G$ by $\phi^t_{\mathcal{F}}(x,y) = (\phi^t x, \mathcal{F}^t(x)y)$. Then there is an open dense set of $\mathcal{F}$ (in an appropriate topology) for which $\phi^t_{\mathcal{F}}$ is ergodic or mixing.

The flow $\phi^t_{\mathcal{F}}$ is the continuous time analogue of ϕ_f. If $\phi^t_{\mathcal{F}}$ is weak-mixing then, by a result in [R], it is Bernoulli. For brevity, we say such a flow is *mixing*. We also prove the existence of an open dense set of mixing skew products.

Observe that $\phi^t_{\mathcal{F}}$ is a flow on $\Lambda \times G$, a trivial G-bundle over Λ, and that $\phi^t_{\mathcal{F}}$ commutes with the (right) G-action on $\Lambda \times G$. More generally, we could consider a manifold $\hat{\Lambda}$ with a free G-action. Suppose we have a flow $\hat{\phi}^t$ on $\hat{\Lambda}$ that commutes with the G-action and is such that the quotient flow ϕ^t on $\hat{\Lambda}/G$ is hyperbolic. In this setting, we prove the existence of open dense sets of ergodic and mixing G-equivariant flows that project to ϕ^t. An example of such a flow $\hat{\phi}^t$ is given by the frame bundle flow on a compact manifold with negative sectional curvatures; in §4 we briefly study the relevance of Theorem A to such flows.

Skew products are intimately connected with *dynamic cocycles*. We call a map $\mathcal{F} : \Lambda \times \mathbb{R} \to G$ such that $\mathcal{F}^{t+s}(x) = \mathcal{F}^t(\phi^s x)\mathcal{F}^s(x)$ a *dynamic cocycle*. Two dynamic cocycles $\mathcal{F}_1, \mathcal{F}_2$ are *cohomologous* if $\mathcal{F}^t_1(x) = u(\phi^t x)\mathcal{F}^t_2(x)u(x)^{-1}$ for some $u : \Lambda \to G$. We call the resulting space of equivalence classes the *dynamic cohomology* $\mathcal{H}^1(\phi; G)$. Observe that cohomologous cocycles define conjugate skew products: if $\mathcal{F}_1$ is cohomologous to $\mathcal{F}_2$ then $(x,y) \mapsto (x, u(x)y)$ conjugates $\phi^t_{\mathcal{F}_1}$ to $\phi^t_{\mathcal{F}_2}$.

To prove Theorem A we consider separately the cases when G is a torus and G is semi-simple (any compact connected Lie group is locally the product of such groups). The semi-simple case in continuous time is essentially the same as the semi-simple case in discrete time, considered in [FP]; for this reason we merely sketch the arguments. The toral case is more interesting. To prove stable ergodicity of a toral skew product of a hyperbolic flow, we first construct a homomorphism ι from the first cohomology group $H^1(\Lambda; \mathbb{Z})$ of Λ into the dynamic cohomology $\mathcal{H}^1(\phi; \mathbb{R})$. We show that $\iota(H^1(\Lambda; \mathbb{R}))$ has infinite index in $\mathcal{H}^1(\phi; \mathbb{R})$ and this forms the basis of our analysis: the non-ergodic or unstably ergodic skew products are associated to dynamic cocycles in $\iota(H^1(\Lambda; \mathbb{R}))$. In particular, we precisely identify those cocycles defining stably ergodic skew products in terms of the cohomology of Λ.

The homomorphism ι is defined and discussed in §3 and we believe it to be of independent interest. For example, we can also use ι to study velocity changes of hyperbolic flows. Recall that if ϕ^t is a hyperbolic flow then we can obtain a new hyperbolic flow ψ^t by keeping the same oriented orbits as ϕ^t but traversing them at different speeds. We can write $\phi^t(x) = \psi^{k^t(x)}(x)$ and the flow condition ensures that $k \in \mathcal{H}^1(\phi; \mathbb{R})$. Let $k(x) = \lim_{t \to 0} k^t(x)/t > 0$. If μ is an ergodic probability for ϕ^t then the measure with density $k\,d\mu$ is ergodic for ϕ^t. However $k\,d\mu$ need not be mixing. We prove:

Theorem B
The set of (normalised) velocity changes of ϕ^t for which the resulting flow ψ^t is mixing contains an open dense set.

Indeed, the only way a hyperbolic flow can fail to be mixing is if it is the constant suspension of a hyperbolic diffeomorphism [Pl, B3]. With this in mind, our result is related to one of Bowen's. In [B4], Bowen proved that there is an open dense set of mixing suspensions of an Anosov diffeomorphism of an infranilmanifold.

(In fact, Bowen's proof is valid for suspensions of a hyperbolic diffeomorphism ϕ on M with the condition that 1 is not an eigenvalue of ϕ^* on $H^1(M;\mathbb{Z})$. Whether this is always the case is an important open problem.) Our result is more general in that it applies to suspensions of any hyperbolic diffeomorphism.

This note is organised as follows. In §2 we quickly describe some background on hyperbolic flows. In §3 we construct the map ι. In §4 we discuss skew products, with particular emphasis on the case of a torus. In §5 we discuss velocity changes.

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§2 Hyperbolic preliminaries

Hyperbolic flows. Let M be a smooth compact Riemannian manifold. A flow $\phi^t : M \to M$ is said to be C^1 if the vector field $x \mapsto X_\phi(x) : M \to T_xM$ is C^1. A closed ϕ^t-invariant set Λ is called a *basic set* if:

1. the tangent bundle of M restricted to Λ splits continuously into a Whitney sum $T_\Lambda M = E^0 \oplus E^s \oplus E^u$ of $d\phi^t$-invariant sub-bundles such that
   - there exist constants $C > 0$, $0 < \lambda < 1$ such that
   $$\begin{array}{c} \|d\phi^t(v)\| \le C\lambda^t\|v\| \text{ for all } t \ge 0, \text{ if } v \in E^s_x, \\ \|d\phi^{-t}(v)\| \le C\lambda^t\|v\| \text{ for all } t \ge 0, \text{ if } v \in E^u_x, \end{array}$$
   - E^0 is one-dimensional and tangential to the flow direction;
2. Λ contains a dense orbit;
3. there exists an open neighbourhood $U \supset \Lambda$ such that $\bigcap_{t=-\infty}^{\infty} \phi^t U = \Lambda$;
4. Λ has no fixed points and is not a single orbit.

When ϕ^t is restricted to Λ, we call ϕ^t a *hyperbolic flow*. If in 3. the open neighbourhood U satisfies
$$\bigcap_{t=0}^{\infty} \phi^t U = \Lambda,$$
then we call ϕ^t or Λ an *attractor*. Typically, Λ is not a manifold. If $\Lambda = M$ then we call ϕ^t a *(transitive) Anosov flow*. Clearly, Anosov flows are attractors.

When proving our stability results, we will need to assume that the first Čech cohomology group $H^1(\Lambda;\mathbb{Z})$ of Λ has finite rank. Recall that the rank of a free abelian group is defined to be the maximum number of integrally independent elements. We call such hyperbolic flows *cohomologically finite*. How often this occurs appears to be an open question. For Anosov flows, $H^1(\Lambda;\mathbb{Z})$ is finitely generated and so has finite rank. More generally, it is easy to see that any attractor is cohomologically finite [FP, Theorem 2.7]. However, if Λ is one-dimensional then it is conjugate to a suspension of a shift of finite type [B2] and $H^1(\Lambda, \mathbb{Z})$ is the free abelian group with countably many generators, so does not have finite rank [PT, Theorem 4.19].

Equilibrium states. For a continuous function $f : \Lambda \to \mathbb{R}$ define the *pressure* of f to be

$$P(f) = \sup\{h_m(\phi) + m(f) \mid m \text{ is a } \phi^t\text{-invariant probability}\} \tag{1}$$

(here $h_m(\phi)$ is the entropy with respect to m of the time-one map ϕ^1). If f is Hölder continuous then this supremum is achieved by a unique ϕ^t-invariant probability μ [BR]. We call μ the *equilibrium state* of f. Moreover, μ is ergodic and positive on non-empty open sets [BR]. It is not true, in general, that μ is weak-mixing. However, if μ is weak-mixing then (ϕ^t, μ) is isomorphic to a Bernoulli flow (see [BR, Remark 3.5] and the references cited there); for brevity, we say such a flow is *mixing*.

Function spaces. Let d be the metric on Λ induced from the Riemannian metric. A function $f : \Lambda \to \mathbb{R}$ is *Hölder continuous with exponent* θ, $0 < \theta < 1$, if $|\, f(x) - f(y)\,| \leq C\, d(x,y)^\theta$ for all $x, y \in \Lambda$ and some $C > 0$. We denote the space of such functions by $C^\theta(\Lambda;\mathbb{R})$ and equip it with the norm

$$\|f\|_\theta := |f|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\theta},$$

where $|\cdot|_\infty$ denotes the uniform norm. Then $C^\theta(\Lambda;\mathbb{R})$ is a Banach space.

When $\Lambda = M$ (i.e. the Anosov case), we can also speak of C^r functions for $r \geq 1$. Denote the space of such functions by $C^r(\Lambda;\mathbb{R})$ (or $C^r(\Lambda;K)$ for circle-valued functions). We shall mostly be concerned with the cases $r = 1$ and $r = \infty$. In the C^1 case, we use the norm

$$\|f\|_{C^1} := |f|_\infty + \sup_{x \in \Lambda} \|df(x)\|.$$

Then $C^1(\Lambda;\mathbb{R})$ is a Banach space and $C^1(\Lambda;K)$ is complete. The topology on $C^\infty(\Lambda;\mathbb{R})$ and $C^\infty(\Lambda;K)$ is given by convergence in the C^r-norm for each $r \in \mathbb{N}$. These spaces are not Banach spaces, although they are complete.

For convenience we shall often use C^θ to denote either Hölder continuous functions of exponent $\theta \in (0,1)$, C^1 functions or smooth functions. (If we refer to C^r functions for $r = 2, 3, \ldots$ then we shall mention this explicitly.) When $\theta = 1$ or ∞, we assume without mention that the flow is Anosov. When $\theta = \infty$, we always assume that the flow is itself smooth.

Toral skew products. We shall be interested in toral skew products of ϕ^t of the following form. Let K denote the unit circle in $\mathbb{C}$ and K^n the n-torus. For $1 \leq i \leq n$, let $f_i \in C^\theta(\Lambda;\mathbb{R})$ and denote $f = (f_1, \ldots, f_n)$. Define the flow $\phi_f^t : \Lambda \times K^n \to \Lambda \times K^n$ by

$$\phi_f^t(x, y_1, \ldots, y_n) = \left(\phi^t x, e^{2\pi i \int_0^t f_1(\phi^u x)\, du} y_1, \ldots, e^{2\pi i \int_0^t f_n(\phi^u x)\, du} y_n\right).$$

Then ϕ_f^t has vector field $(x, y_1, \ldots, y_n) \mapsto (X_\phi(x), f_1(x), \ldots, f_n(x))$. Note that ϕ_f^t is not in general C^1, merely Hölder continuous; however, it will be C^1 when all the f_i are C^1 and smooth when all the f_i are smooth.

Let μ be a Hölder equilibrium state and let ν denote the normalised Haar measure on K^n. Then $\mu \times \nu$ is ϕ_f^t-invariant. Moreover, if $\mu \times \nu$ is ergodic, then it is the only invariant measure that projects to m [KN] (this is true for *any* group extension of *any* flow). When we discuss the ergodic properties of ϕ_f^t, we mean with respect to $\mu \times \nu$.

Velocity changes. We shall also be interested in C^1 velocity changes of ϕ^t. The details of this construction are well-known and we summarise it in the C^1 Anosov case here. We also consider smooth velocity changes of smooth flows. (For hyperbolic flows when Λ is not a manifold, our methods only allow us to consider Hölder velocity changes. The resulting flows are only Hölder continuous.)

Let $C^1(\Lambda;\mathbb{R}^+)$ denote the space of strictly positive C^1 functions. Let $k : \Lambda \to \mathbb{R}$ be such a function, let $l(x)k(x) = 1$ and define

$$k^t(x) = \int_0^t k(\phi^u x)\,du.$$

Then k is a C^1 ϕ-cocycle (see §3). Moreover, for each x, the map $t \mapsto k^t(x)$ is strictly increasing and fixes 0. Hence, there is a well-defined map $l^t(x)$ defined by $t = k^{l^t(x)}(x)$. Define $\psi_k^t(x) := \phi^{l^t(x)}(x)$. Then ψ_k^t is a flow; this amounts to checking that l is a ψ_k^t-cocycle and this is easy from the definitions. We call ψ_k^t a *flow obtained by a velocity change of* ϕ^t. Geometrically, ψ_k^t has the same oriented orbits as ϕ^t, but the orbits are traversed at different speeds.

One can invert this process, i.e. velocity change ψ_k^t by multiplying speeds by l to recover ϕ^t. In particular, if X_ϕ, X_{ψ_k} denote the vector fields generating ϕ^t, ψ_k^t respectively, then $X_{\psi_k}(x) = l(x)X_\phi(x)$. Hence ψ_k^t is C^1 when k and ϕ^t are. Moreover, one can show [AS] that if the tangent bundle of Λ has a hyperbolic splitting with respect to ϕ^t, then it has a hyperbolic splitting with respect to ψ_k^t. Clearly ϕ^t and ψ_k^t have the same orbits. Hence, if ϕ^t is hyperbolic, then so is ψ_k^t.

§3 Dynamic versus Bruschlinsky cohomology

We construct the homomorphism ι from $H^1(\Lambda;\mathbb{Z})$ into the dynamic cohomology. In general Λ is not a manifold and so we use Čech cohomology to calculate $H^1(\Lambda;\mathbb{Z})$. A particularly useful presentation of the first Čech cohomology group is given below.

Bruschlinsky cohomology. Let $C(\Lambda;K)$ denote the group of continuous circle-valued function on Λ with pointwise multiplication. let $B(\Lambda;K)$ denote the subgroup of all null-homotopic functions. The *Bruschlinsky group* is defined to be the quotient $C(\Lambda;K)/B(\Lambda;K)$. It is known [H] that this is isomorphic to the first Čech cohomology group $H^1(\Lambda;\mathbb{Z})$ of Λ.

A function $w : \Lambda \to \mathbb{C}$ is said to be ϕ-differentiable if

$$\lim_{t\to 0} \frac{w(\phi^t x) - w(x)}{t} = w'_\phi(x)$$

exists for each $x \in \Lambda$. We say that w is C^θ *ϕ-differentiable* if w is C^θ and w'_ϕ exists and is C^θ.

In [S], Schwartzman and Kakutani proved that any cohomology class in $H^1(\Lambda,\mathbb{Z})$ has a representative that is *continuously* differentiable along ϕ. We will need the following strengthening of this result.

Proposition 3.1
Let $[w] \in H^1(\Lambda; \mathbb{Z})$. Let $\theta \in (0,1)$, $\theta = 1$ or $\theta = \infty$. Then there exists a C^θ $w \in [w]$ that is C^θ ϕ-differentiable.

Suppose w, w' are Hölder continuous functions and, moreover, that w is homotopic to w'. Then there exists a continuous $u : \Lambda \to \mathbb{R}$ such that $w/w' = \exp 2\pi i u$ and one can easily check that u is also Hölder. Similarly, if w, w' are C^1 or smooth, then u is C^1 or smooth, respectively. Hence

Corollary 3.2
Let $\theta \in (0,1)$, $\theta = 1$ or $\theta = \infty$. Let $C^\theta_\phi(\Lambda; K)$ denote the space of circle-valued C^θ functions that are C^θ ϕ-differentiable. Let $B^\theta_\phi(\Lambda; K)$ denote the subgroup of null-homotopic functions. Then $C^\theta_\phi(\Lambda; K)/B^\theta_\phi(\Lambda; K)$ is isomorphic to $H^1(\Lambda; \mathbb{Z})$.

Dynamic cohomology. Recall that a map $\mathcal{F} : \Lambda \times \mathbb{R} \to G$ is a *dynamic cocycle* if $\mathcal{F}^{t+s}(x) = \mathcal{F}^t(\phi^s x)\mathcal{F}^s(x)$. We say that $\mathcal{F}$ is C^θ if $f(x) = \lim_{t\to 0} \mathcal{F}^t(x)/t : \Lambda \to L(G)$, the Lie algebra of G, is C^θ.

If $\mathcal{F}^t(x) = u(\phi^t x)u(x)^{-1}$ then we call $\mathcal{F}$ a *dynamic coboundary*. The following fundamental result allows us to assume that u is as regular as $\mathcal{F}$.

Theorem 3.3 (Livšic's regularity theorem)
Let ϕ^t be a hyperbolic flow on a basic set Λ. Let μ be a Hölder equilibrium state. Let G be a compact Lie group or $\mathbb{R}^n$ and let $\mathcal{F}$ be a G-valued C^θ cocycle, $\theta \in (0,1)$. If $\mathcal{F}^t(x) = u(\phi^t x)u(x)^{-1}$ a.e. for some measurable u then there exists a Hölder $u' \in C^\theta(\Lambda; G)$ such that $u = u'$ a.e. and $\mathcal{F}^t(x) = u'(\phi^t x)u'(x)^{-1}$ everywhere. If ϕ^t is Anosov then u is C^1 is $\mathcal{F}$ is C^1, and smooth if $\mathcal{F}$ is smooth.

Remarks.

1. We call such results *Livšic regularity results* after the pioneering work in [L2]. There is now a large literature concerning such results [LMM, HK, NT2, PP2]. The above result for Hölder functions can be found [W1]. The smooth version is due to [LMM].

2. There is a version for an arbitrary connected Lie group [NPW].

3. More generally, one could assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are cohomologous via a measurable function u: $\mathcal{F}_1^t(x) = u(\phi^t x)\mathcal{F}_2^t(x)u(x)^{-1}$ a.e. and ask if u has a Hölder version. This is indeed the case for compact groups and $\mathbb{R}^n$ [W1].

4. Suppose the flow is C^r. If $\mathcal{F}$ is a C^r cocycle $(2 \le r < \infty)$ then u is $C^{r-\varepsilon}$ for any $\varepsilon \in (0,1)$ [NT1].

We shall also need the following criterion that ensures a cocycle is a coboundary.

Theorem 3.4 (Livšic's periodic data criterion [L1])
Let ϕ^t be a hyperbolic flow on a basic set Λ. Let G be a Lie group with a left-right invariant metric. Let $\mathcal{F}$ be a C^θ cocycle. Suppose $\mathcal{F}^T(x) = e$, the group identity, whenever $\phi^T x = x$. Then $\mathcal{F}^t(x) = u(\phi^t x)u(x)^{-1}$ for some $u \in C^\theta(\Lambda; G)$. If ϕ^t is Anosov then u is C^1 is $\mathcal{F}$ is C^1, and smooth if $\mathcal{F}$ is smooth.

Suppose G is abelian. Then the set of C^θ cocycles forms an abelian group under pointwise multiplication and the coboundaries form a subgroup. We denote the resulting quotient group by $\mathcal{H}^1(\phi; G)$ (we drop the dependence on θ for notational clarity).

We shall mostly be interested in cocycles taking values in $\mathbb{R}$ or K. These take the form

$$\mathcal{F}^t(x) = \int_0^t f(\phi^u x)\, du, \quad \mathcal{F}^t(x) = \exp\left(2\pi i \int_0^t f(\phi^u x)\, du\right), \tag{2}$$

respectively, for some $f : \Lambda \to \mathbb{R}$. We shall often identify f with $\mathcal{F}$ and say f *generates* $\mathcal{F}$, or even say that f is a cocycle.

Suppose $\int_0^t f(\phi^u x)\, du = u(\phi^t x) - u(x)$. Then $f = u'_\phi$ and u is C^θ ϕ-differentiable. Let $\mathcal{B} = \{u'_\phi \mid u \in C^\theta(\Lambda; \mathbb{R}) \text{ is } C^\theta\ \phi\text{-differentiable}\}$ and note that $\mathcal{B}$ is a subspace of $C^\theta(\Lambda; \mathbb{R})$. Then $f, f' \in C^\theta(\Lambda; \mathbb{R})$ generate cohomologous cocycles if and only if $f - f' \in \mathcal{B}$. Moreover, we can identify $\mathcal{H}^1(\phi; \mathbb{R})$ with $C^\theta(\Lambda; \mathbb{R})/\mathcal{B}$.

We shall also be interested in essential coboundaries. A real-valued cocycle f is an *essential coboundary* if $\int_0^t f(\phi^u x)\, du = u(\phi^t x) - u(x) + ct$ for some constant $c \in \mathbb{R}$. If f is C^θ then so is u and $f = u'_\phi + c$. Let $\mathcal{B}_0 = \{u'_\phi + c \mid u \in C^\theta(\Lambda; \mathbb{R}) \text{ is } C^\theta\ \phi\text{-differentiable and } c \in \mathbb{R}\}$. The essential coboundaries form a subgroup of the group of coboundaries and we denote the quotient by $\mathcal{H}^1_0(\phi; \mathbb{R})$. Again, we can identify $\mathcal{H}^1_0(\phi; \mathbb{R})$ with $C^\theta(\Lambda; \mathbb{R})/\mathcal{B}_0$.

Cohomological inclusions. For the remainder of this section, we assume that $H^1(\Lambda; \mathbb{Z})$ has finite rank. If $H^1(\Lambda; \mathbb{Z})$ is trivial then all of the results below go through with only minor changes, so we shall assume that $H^1(\Lambda; \mathbb{Z}) \neq 0$.

Let $w : \Lambda \to K$ be continuous. Then for each t, $w\phi^t$ is homotopic to w. Hence there exists a (unique) continuous function $\mathcal{G} : \Lambda \times \mathbb{R} \to \mathbb{R}$ such that

$$\frac{w(\phi^t x)}{w(x)} = \exp 2\pi i \mathcal{G}(x, t).$$

It is easy to see that $\mathcal{G}$ is a real-valued cocycle.

Suppose w is ϕ-differentiable. Then

$$g(x) = \lim_{t \to 0} \frac{\mathcal{G}(x,t)}{t}$$

exists and

$$\frac{w(\phi^t x)}{w(x)} = \exp 2\pi i \int_0^t g(\phi^u x)\, du. \tag{3}$$

If w is C^θ, C^1 or smoothly ϕ-differentiable then so is g. Conversely, if g is C^θ, C^1, or smooth then w is C^θ, C^1 or smoothly ϕ-differentiable, respectively; this follows from Theorem 3.4.

Thus to every circle-valued function w we can associate a real-valued cocycle $\mathcal{G}$, generated by g.

Suppose w' is homotopic to w and is C^θ differentiable. Then $w' = w \exp 2\pi i u$ for some $u \in C^\theta(\Lambda; \mathbb{R})$. Let g' denote the real-valued cocycle associated to w' by the above discussion. Then

$$\exp 2\pi i \int_0^t (g' - g)(\phi^u x)\, du = \frac{w'/w(\phi^t x)}{w'/w(x)} = \exp 2\pi i (u(\phi^t x) - u(x)).$$

Hence, u'_ϕ exists and $u'_\phi \in \mathcal{B}$. Moreover, $g' - g = u'_\phi$.

Thus to each element of $H^1(\Lambda;\mathbb{Z})$ we can associate a real-valued dynamic cocycle modulo a coboundary, i.e. an element of $\mathcal{H}^1(\phi;\mathbb{R})$. Hence we have a well-defined map

$$\iota : H^1(\Lambda;\mathbb{Z}) \to \mathcal{H}^1(\phi;\mathbb{R}) : [w] \mapsto g + \mathcal{B}.$$

Proposition 3.5
The map ι is a group homomorphism. Indeed, ι is an isomorphism onto its image.

Proof. Let $w, w' \in C^\theta(\Lambda; K)$ be C^θ ϕ-differentiable. Let g, g' denote the real-valued cocycles associated to w, w', respectively. Then

$$\begin{aligned}
\frac{(ww')(\phi^t x)}{(ww')(x)} &= \frac{w(\phi^t x)}{w(x)}\frac{w'(\phi^t x)}{w'(x)} \\
&= \exp\left(2\pi i \int_0^t G(\phi^u x)\, du\right) \exp\left(2\pi i \int_0^t G'(\phi^u x)\, du\right) \\
&= \exp 2\pi i \int_0^t (G + G')(\phi^u x)\, du,
\end{aligned}$$

and it follows that $\iota([ww']) = \iota([w]) + \iota([w'])$.

To see that ι is injective, suppose that $\iota([w]) = 0 \bmod \mathcal{B}$. Then

$$\frac{w(\phi^t x)}{w(x)} = \exp 2\pi i \int_0^t u'_\phi(\phi^u x)\, du = \frac{\exp 2\pi i u(\phi^t x)}{\exp 2\pi i u(x)}$$

so that $w \exp(-2\pi i u)$ is ϕ-invariant, thus constant by ergodicity. Hence w is null-homotopic and $[w] = 0 \in H^1(\Lambda;\mathbb{Z})$. ❏

Corollary 3.6
The following sequence is exact:

$$0 \to H^1(\Lambda;\mathbb{Z}) \xrightarrow{\iota} \mathcal{H}^1(\phi;\mathbb{R}) \xrightarrow{\pi} \mathcal{H}^1(\phi;K) \to 0.$$

Proof. We use the realisation of $H^1(\Lambda;\mathbb{Z})$ given by Corollary 3.2.

We first define the map π. Any $f \in C^\theta(\Lambda;\mathbb{R})$ determines both a real ϕ-cocycle and a circle-valued ϕ-cocycle by (2). Let π be the map that takes the former to the latter. Then π is clearly a homomorphism. Moreover, if f generates a real-valued coboundary, then πf generates a circle-valued coboundary so that the induced map on the dynamic cohomology groups is well-defined.

By Proposition 3.5, ι is injective and it is clear that π is onto. Let $w \in [w] \in H^1(\Lambda;\mathbb{Z})$. Then $w(\phi^t x)/w(x) = \exp 2\pi i \int_0^t \iota(w)(\phi^u x)\, du$ and it follows that $\iota([w]) \in \ker \pi$. Similarly, $\ker \pi \subset \operatorname{im} \iota$. ❏

There is also a well-defined map

$$\iota_0 : H^1(\Lambda;\mathbb{Z}) \to \mathcal{H}^1_0(\phi;\mathbb{R}).$$

Proposition 3.7
If ϕ^t is mixing then ι_0 is an isomorphism onto its image.

Proof. The same argument as in the proof of Proposition 3.5 shows that ι_0 is a homomorphism. If $\iota_0([w]) = 0 \bmod \mathcal{B}_0$ then

$$\begin{aligned} \frac{w(\phi^t x)}{w(x)} &= \exp 2\pi i(\int_0^t c + u'_\phi(\phi^u x)\, du) \\ &= \exp 2\pi i c t \frac{\exp 2\pi i u(\phi^t x)}{\exp 2\pi i u(x)}, \end{aligned}$$

for some $c \in \mathbb{R}$ and $u : \Lambda \to \mathbb{R}$. Hence $w \exp(-2\pi i u)$ has $2\pi i c$ as an eigenfrequency. As ϕ^t is mixing, $c = 0$ and $w \exp(-2\pi i u)$ is constant. Hence $[w] = 0 \in H^1(\Lambda; \mathbb{Z})$. ❏

Remark. We can also define K-valued essential coboundaries to be functions of the form

$$\frac{c\, w(\phi^t x)}{w(x)}$$

for some $c \in K$. These form a subgroup and we denote the group of K-valued cocycles modulo essential coboundaries by $\mathcal{H}_0^1(\phi; K)$. When ϕ is mixing there is an exact sequence

$$0 \to H^1(\Lambda; \mathbb{Z}) \stackrel{\iota_0}{\to} \mathcal{H}_0^1(\phi; \mathbb{R}) \stackrel{\pi}{\to} \mathcal{H}_0^1(\phi; K) \to 0.$$

Let R denote either $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$ and consider $\iota(H^1(\Lambda; \mathbb{Z})) \otimes R \subset \mathcal{H}^1(\phi; \mathbb{R})$. Then we can write

$$\iota(H^1(\Lambda; \mathbb{Z})) \otimes R = (\mathcal{Z}_R + \mathcal{B})/\mathcal{B}$$

for some $\mathcal{Z}_R \subset C^\theta(\Lambda; \mathbb{R})$. (We suppress the dependence of $\mathcal{Z}_R$ on θ for notational clarity.) Note that $\iota(H^1(\Lambda; \mathbb{Z})) = (\mathcal{Z}_\mathbb{Z} + \mathcal{B})/\mathcal{B}$.

For the essential cohomology groups, one can easily check that $\iota_0(H^1(\Lambda; \mathbb{Z})) \otimes R \subset \mathcal{H}_0^1(\phi; \mathbb{R})$ and

$$\iota_0(H^1(\Lambda; \mathbb{Z})) \otimes R = (\mathcal{Z}_R + \mathcal{B}_0)/\mathcal{B}_0.$$

Throughout, we urge the reader to take careful note of the suffixes on the $\mathcal{Z}$'s.

Notice that $\mathcal{Z}_\mathbb{R} + \mathcal{B}$ and $\mathcal{Z}_\mathbb{R} + \mathcal{B}_0$ are linear subspaces of $C^\theta(\Lambda; \mathbb{R})$. (Indeed, they are just the linear spans of all possible g occurring in (3).) We shall see that 'bad' ergodic behaviour corresponds to dynamic cocycles in $(\mathcal{Z}_\mathbb{R} + \mathcal{B})/\mathcal{B}$ and $(\mathcal{Z}_\mathbb{R} + \mathcal{B}_0)/\mathcal{B}_0$. We now show that such 'bad' cocycles correspond to a very small subspace of $\mathcal{H}^1(\phi; \mathbb{R})$ and $\mathcal{H}_0^1(\phi; \mathbb{R})$, respectively.

Proposition 3.8

1. *$\mathcal{B}$ is a closed subspace of $C^\theta(\Lambda; \mathbb{R})$.*
2. *$\mathcal{Z}_\mathbb{R} + \mathcal{B}$ is a closed subspace of $C^\theta(\Lambda; \mathbb{R})$.*
3. *$\mathcal{Z}_\mathbb{R} + \mathcal{B}$ has infinite codimension in $C^\theta(\Lambda; \mathbb{R})$.*

Proof. 1. Let $b_n = (u_n)'_\phi \in \mathcal{B}$ be such that $b_n \to b$ in the C^θ topology. Suppose $\phi^T x = x$; then clearly $\int_0^T b_n(\phi^u x)\, du = 0$. By Theorem 3.4, to show that $b \in \mathcal{B}$, it suffices to prove that $\int_0^T b(\phi^u x)\, du = 0$. This follows from:

$$\left| \int_0^T b(\phi^u x)\, du - \int_0^T b_n(\phi^u x)\, du \right| \leq \int_0^T |b(\phi^u x) - b_n(\phi^u x)|\, du$$
$$\leq T|b - b_n|_\infty.$$

As the uniform norm is dominated by the Hölder norm, as well as the C^r norm for each $r \in \mathbb{N}$, we see that $|b - b_n|_\infty \to 0$ as $n \to \infty$. Hence $\int_0^T b(\phi^u x)\, du = 0$.

2. It is sufficient to show that $(\mathcal{Z}_\mathbb{R} + \mathcal{B})/\mathcal{B}$ is finite dimensional, hence closed. Let β denote the rank of $H^1(\Lambda; \mathbb{Z})$ and let $g_1 + \mathcal{B}, \ldots, g_\beta + \mathcal{B}$ be a maximal set of integrally independent elements in $(\mathcal{Z}_\mathbb{Z} + \mathcal{B})/\mathcal{B} = \iota(H^1(\Lambda; \mathbb{Z}))$. It is easy to see that any $g + \mathcal{B} \in (\mathcal{Z}_\mathbb{Z} + \mathcal{B})/\mathcal{B}$ is a rational linear combination of the $g_i + \mathcal{B}$ and it follows that any $g + \mathcal{B} \in (\mathcal{Z}_\mathbb{R} + \mathcal{B})/\mathcal{B}$ is a real linear combination of the $g_i + \mathcal{B}$. Hence $\dim (\mathcal{Z}_\mathbb{R} + \mathcal{B})/\mathcal{B} \leq \beta$.

3. As $(\mathcal{Z}_\mathbb{R} + \mathcal{B})/\mathcal{B}$ is finite dimensional, to show $\mathcal{Z}_\mathbb{R} + \mathcal{B}$ has infinite codimension in $C^\theta(\Lambda; \mathbb{R})$ it is sufficient to prove that $\mathcal{B}$ has infinite codimension in $C^\theta(\Lambda; \mathbb{R})$.

Any function $u'_\phi \in \mathcal{B}$ has zero integral with respect to an invariant measure m. This follows from the fact that

$$\int u'_\phi\, dm = \lim_{t \to \infty} \frac{1}{t} \int u\phi^t - u\, dm = 0,$$

by dominated convergence and the ϕ^t-invariance of m. Hence any invariant measure on Λ determines a linear functional on $C^\theta(\Lambda; \mathbb{R})/\mathcal{B}$. If the dimension of $C^\theta(\Lambda; \mathbb{R})/\mathcal{B}$ was finite, then its dual space would also be finite dimensional. However, as there are infinitely many periodic orbits, the invariant measures supported on periodic orbits generate an infinite dimensional space of linear functionals. ❑

Let us consider $\mathcal{Z}_\mathbb{R} + \mathcal{B}_0$. First observe that it is easy to modify Theorem 3.4 to give a criterion for essential cohomological triviality in terms of periodic orbits. This allows us to conclude, in exactly the same way as above, that $\mathcal{B}_0$ is closed in $C^\theta(\Lambda; \mathbb{R})$. Also, observe that $\mathcal{B}$ has codimension 1 in $\mathcal{B}_0$. Hence

Proposition 3.9
The subspace $\mathcal{B}_0$ is closed in $C^\theta(\Lambda; \mathbb{R})$. Furthermore, $\mathcal{Z}_\mathbb{R} + \mathcal{B}_0$ is closed in $C^\theta(\Lambda; \mathbb{R})$ and has infinite codimension.

We shall need some results on functions lying in 'general position'. The method of proof is surely well-known, but see, in particular, [PP1] where similar results have been used in similar circumstances.

Proposition 3.10
Let $n \geq 1$ be arbitrary. Let $(f_1, \ldots, f_n) \in C^\theta(\Lambda; \mathbb{R})^n$. Then there exists $(f'_1, \ldots, f'_n) \in C^\theta(\Lambda; \mathbb{R})^n$ arbitrarily close to $(f_1, \ldots, f_n)$ such that $(f'_1, \ldots, f'_n)$ is linearly independent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}$.

Proof. This is a well-known general position argument, using the fact that $\mathcal{Z}_\mathbb{R} + \mathcal{B}$ has infinite codimension in $C^\theta(\Lambda; \mathbb{R})$. ❑

Remark. The proposition above remains true with $\mathcal{B}$ replaced by $\mathcal{B}_0$, using the observation that the former has codimension 1 in the latter.

Proposition 3.11
Let $n \geq 1$ be arbitrary. Then the set of n-tuples $(f_1, \ldots, f_n) \in C^\theta(\Lambda; \mathbb{R})^n$ that are linearly independent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}$ forms an open set in $C^\theta(\Lambda; \mathbb{R})^n$. The same remains true of n-tuples in $C^\theta(\Lambda; \mathbb{R})^n$ that are linearly dependent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}_0$.

Proof. We show that the linearly dependent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}$ n-tuples form a closed set. Suppose $f_m = (f_{1,m}, \ldots, f_{n,m}) \in C^\theta(\Lambda; \mathbb{R})^n$ are linearly dependent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}$. Suppose $f_m \to f \in C^\theta(\Lambda; \mathbb{R})^n$. Then there exist $\lambda_{i,j} \in \mathbb{R}$ (not all zero) such that

$$\lambda_{1,m} f_{1,m} + \cdots + \lambda_{n,m} f_{n,m} = 0 \bmod \mathcal{Z}_\mathbb{R} + \mathcal{B}. \tag{4}$$

If necessary, we can re-order the indices and assume that $\lambda_{1,m} \neq 0$ and $|\lambda_{1,m}| \geq |\lambda_{i,m}|$, $2 \leq i \leq n$, for infinitely many m. Dividing (4) by $\lambda_{1,m}$ for such m, we obtain:

$$f_{1,m} + \lambda'_{2,m} f_{2,m} + \cdots + \lambda'_{n,m} f_{n,m} = 0 \bmod \mathcal{Z}_\mathbb{R} + \mathcal{B},$$

and $|\lambda'_{i,m}| \leq 1$ for $2 \leq i \leq n$. By taking a subsequence, we can assume that $\lambda'_{i,m} \to \lambda'_i$ as $m \to \infty$, for each j. Hence (writing $f = (f_1, \ldots, f_n)$),

$$f_1 + \lambda'_2 f_2 + \cdots + \lambda'_n f_n = 0 \bmod \mathcal{Z}_\mathbb{R} + \mathcal{B},$$

as $\mathcal{Z}_\mathbb{R} + \mathcal{B}$ is closed. Thus $f_1, \ldots, f_n$ are linearly dependent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}$. ❏

One can say more about the relationship between $H^1(\Lambda; \mathbb{Z})$ and the various dynamical cohomology groups, but we postpone this to the following sections where their context makes them clearer.

§4 Stable ergodicity of group extensions

Let G be a compact Lie group. Recall from §1 that each G-valued cocycle determines a skew product flow on $\Lambda \times G$. We show that the set of cocycles for which the extension flow is ergodic or mixing contains an open dense set. We concentrate on the case when G is a torus.

Toral extensions. Recall that a toral skew-product is determined by an n-tuple $f = (f_1, \ldots, f_n) \in C^\theta(\Lambda; \mathbb{R}^n)$:

$$\phi_f^t(x, y_1, \ldots, y_n) = \left(\phi^t x, e^{2\pi i \int_0^t f_1(\phi^u x)\, du} y_1, \ldots, e^{2\pi i \int_0^t f_n(\phi^u x)\, du} y_n\right). \tag{5}$$

Then f determines an element of $\mathcal{H}^1(\phi; K^n)$. If f' is cohomologous to f then ϕ_f^t is metrically and topologically conjugate to $\phi_{f'}^t$ via a map of the form $(x, y_1, \ldots, y_n) \mapsto (x, u_1(x) y_1, \ldots, u_n(x) y_n)$. In particular, the ergodic properties of ϕ_f^t depend only on the dynamic cohomology class of f.

We shall need the following well-known criterion for ϕ_f^t to be ergodic, given that ϕ^t is [KN].

Proposition 4.1
Suppose ϕ^t is ergodic with respect to μ. Then the following are equivalent:

1. *ϕ_f^t is not ergodic with respect to $\mu \times \nu$;*

2. *there exist integers $l_1, \ldots, l_n$ (not all zero) and a measurable function $w : \Lambda \to K$ such that*

$$\frac{w(\phi^t x)}{w(x)} = \exp 2\pi i \int_0^t \sum_{i=1}^n l_i f_i(\phi^u x)\, du \text{ a.e.}$$

By Theorem 3.3, the above proposition becomes:

Proposition 4.2
Let μ be a Hölder equilibrium state. Then the following are equivalent:

1. *ϕ_f^t is not ergodic with respect to $\mu \times \nu$;*

2. *there exist integers $l_1, \ldots, l_n$ (not all zero) and $w \in C^\theta(\Lambda; K)$ such that*

$$\frac{w(\phi^t x)}{w(x)} = \exp 2\pi i \int_0^t \sum_{i=1}^n l_i f_i(\phi^u x)\, du \tag{6}$$

   everywhere.

Observe that equation (6) says that an integral combination of the f_i lies in $\iota(H^1(\Lambda; \mathbb{Z}))$.

There is an analogue of Proposition 4.1 giving a criterion for ϕ_f^t to be mixing [KN]. Again, combining this with Theorem 3.3, we have

Proposition 4.3
Suppose ϕ^t is mixing with respect to the Hölder equilibrium state μ. Then the following are equivalent:

1. *ϕ_f^t is not mixing with respect to $\mu \times \nu$;*

2. *there exist integers $l_1, \ldots, l_n$ (not all zero), a constant $0 \neq c \in \mathbb{R}$ and $w \in C^\theta(\Lambda; K)$ such that*

$$\frac{w(\phi^t x)}{w(x)} = \exp 2\pi i \int_0^t \left(c + \sum_{i=1}^n l_i f_i(\phi^u x) \right) du \tag{7}$$

   everywhere.

We first show that the non-ergodic extensions correspond to the cocycles in $\mathcal{Z}_{\mathbb{Q}} + \mathcal{B}$. We again urge the reader to take careful note of the suffixes on the $\mathcal{Z}$'s.

Proposition 4.4
The following are equivalent:

1. *ϕ_f^t is not ergodic;*

2. *$f_1, \ldots, f_n$ are rationally dependent mod $\mathcal{Z}_{\mathbb{Q}} + \mathcal{B}$.*

In particular, if $f_1, \ldots, f_n$ are linearly independent mod $\mathcal{Z}_{\mathbb{R}} + \mathcal{B}$ then ϕ_f^t is ergodic.

Proof. Suppose ϕ_f^t is not ergodic so that (6) holds for suitable integers $l_1, \ldots, l_n$, not all zero, and some $w \in C^\theta(\Lambda; K)$. Applying ι to (6), we obtain

$$l_1 f_1 + \cdots + l_n f_n = \iota(w) + u'_\phi, \tag{8}$$

for some $u'_\phi \in \mathcal{B}$. Hence, $f_1, \ldots, f_n$ are integrally dependent mod $\mathcal{Z}_\mathbb{Z} + \mathcal{B}$, hence rationally dependent mod $\mathcal{Z}_\mathbb{Q} + \mathcal{B}$, hence linearly dependent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}$.

Conversely, if $f_1, \ldots, f_n$ are rationally dependent mod $\mathcal{Z}_\mathbb{Q} + \mathcal{B}$ then, by multiplying by a common denominator, they are integrally dependent mod $\mathcal{Z}_\mathbb{Z} + \mathcal{B}$. Hence (8) holds for suitable integers $l_1, \ldots, l_n$ (not all zero), $w : \Lambda \to K$ and $u'_\phi \in \mathcal{B}$. It follows that

$$\frac{\exp(2\pi i u(\phi^t x)) w(\phi^t x)}{\exp(2\pi i u(x)) w(x)} = \exp 2\pi i \int_0^t \sum_{i=1}^n l_i f_i(\phi^u x)\, du,$$

whence ϕ_f^t is not ergodic. ❑

Remark. Observe that if the $f_1, \ldots, f_n$ are rationally dependent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}$ (as opposed to $\mathcal{Z}_\mathbb{Q} + \mathcal{B}$), then we cannot conclude that the associated extension ϕ_f^t is not ergodic.

By applying ι to (7), we have the corresponding result for mixing extensions:

Proposition 4.5
The following are equivalent:

1. ϕ_f^t *is not mixing;*
2. $f_1, \ldots, f_n$ *are rationally dependent mod* $\mathcal{Z}_\mathbb{Q} + \mathcal{B}_0$.

In particular, if $f_1, \ldots, f_n$ *are linearly independent mod* $\mathcal{Z}_\mathbb{R} + \mathcal{B}_0$ *then* ϕ_f^t *is mixing.*

Again, rational dependence mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}_0$ need not imply that ϕ_f^t is not mixing. The next result shows that linear dependence mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}_0$ corresponds to 'unstable' ergodic behaviour: if the associated extension is ergodic (or mixing), then it can be perturbed by an arbitrarily small amount to a non-ergodic (or non-mixing) system.

Proposition 4.6
Let $f_1, \ldots, f_n$ *be linearly dependent mod* $\mathcal{Z}_\mathbb{R} + \mathcal{B}$. *Then there exist functions* $f'_1, \ldots, f'_n$, *arbitrarily close to* $f_1, \ldots, f_n$ *such that* $f'_1, \ldots, f'_n$ *are rationally dependent mod* $\mathcal{Z}_\mathbb{Q} + \mathcal{B}$. *In particular,* $\phi_{f'}^t$ *is not ergodic. The same is true with* $\mathcal{B}_0$ *and 'mixing' replacing* $\mathcal{B}$ *and 'ergodic,' respectively.*

Proof. Suppose $f_1, \ldots, f_n$ are linearly dependent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}$. Choose a maximal set $g_1, \ldots, g_\beta \in \mathcal{Z}_\mathbb{Z}$ that are integrally independent mod $\mathcal{B}$. Then there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ (not all zero), $\mu_1, \ldots, \mu_\beta \in \mathbb{R}$ and $u'_\phi \in \mathcal{B}$ such that

$$\lambda_1 f_1 + \cdots + \lambda_n f_n = \mu_1 g_1 + \cdots + \mu_\beta g_\beta + u'_\phi. \tag{9}$$

One can easily check that there are $f'_i, \lambda'_i \in \mathbb{Q}$ and $\mu_i \in \mathbb{Q}$ arbitrarily close to f_i, λ_i and μ_i, respectively, such that $\lambda'_1 f'_1 + \cdots + \lambda'_n f'_n = \mu'_1 g_1 + \cdots + \mu'_\beta g_\beta + u'_\phi$ so that the f'_i are rationally dependent mod $\mathcal{Z}_\mathbb{Q} + \mathcal{B}$. ❑

We restate our main theorem:

Theorem 4.7
Let ϕ^t be a cohomologically finite hyperbolic flow on a basic set Λ and let μ be an equilibrium state corresponding to a Hölder function. Let $\theta \in (0,1)$. Then the set of n-tuples $(f_1, \ldots, f_n) \in C^\theta(\Lambda; \mathbb{R})^n$ for which the skew product (5) is ergodic contains an open dense set. If ϕ^t is mixing then the set of n-tuples f for which ϕ^t_f is mixing contains an open dense set. When ϕ is Anosov, there are open dense sets of C^1 and smooth cocycles giving ergodic and mixing skew products.

Proof. The set of n-tuples in $C^\theta(\Lambda; \mathbb{R})^n$ that are linearly independent mod $\mathcal{Z}_\mathbb{R} + \mathcal{B}$ forms an open (by Proposition 3.11), dense (by Proposition 3.10) set. By Proposition 4.4, all such n-tuples correspond to ergodic extensions.
The mixing case is exactly the same, with $\mathcal{B}_0$ replacing $\mathcal{B}$. ❑

Remarks.

1. Consider the C^r case for $2 \le r < \infty$. If $f \in C^r(\Lambda; \mathbb{R})^n$ is such that ϕ^t_f is not ergodic (or mixing) then for any $\varepsilon \in (0,1)$ there is an n-tuple $f' \in C^{r-\varepsilon}(\Lambda; \mathbb{R})^n$ arbitrarily close to f in the $C^{r-\varepsilon}$ topology for which $\phi^t_{f'}$ is ergodic (or mixing).

2. For a circle extension of a hyperbolic flow on a basic set with $H^1(\Lambda; \mathbb{Z})$ trivial, there are no unstably ergodic or unstably mixing extensions. This is because all of the $\mathcal{Z}$'s drop out of the above discussion and linear dependence mod $\mathcal{B}$ coincides with rational dependence mod $\mathcal{B}$.

3. In [FP] it is observed that for toral extensions of hyperbolic diffeomorphisms, stable ergodicity implies stable mixing. This is false for hyperbolic flows. First note that if ϕ^t is mixing, then $\mathcal{Z}_\mathbb{R} + \mathcal{B}$ does not contain the constant functions. Hence the extension ϕ^t_f given by $f = (c_1, \ldots, c_n)$, where the $c_i \in \mathbb{R}$ are constants, is stably ergodic but not mixing.

4. The simplest case when ϕ^t is not cohomologically finite is when Λ is one-dimensional. In this case, we can still prove the analogue of Theorem 4.7, albeit via different methods [W2].

Non-trivial bundles and compact connected Lie groups. A toral skew product ϕ^t_f is an example of the following construction. Let G be a compact connected Lie group and let $\hat{\Lambda}$ be a space on which G acts freely. Let $\hat{\phi}^t$ be a flow on $\hat{\Lambda}$ that commutes with the G-action: $g\hat{\phi}^t(x) = \hat{\phi}^t(gx)$. We assume that the quotient flow $\phi^t = \hat{\phi}^t/G$ on $\Lambda = \hat{\Lambda}/G$ is hyperbolic. Let μ be an equilibrium state for ϕ^t and let $\hat{\mu}$ denote its Haar lift on $\hat{\Lambda}$; then $\hat{\phi}^t$ is $\hat{\mu}$-invariant. Let $\hat{\mathcal{F}} : \hat{\Lambda} \times \mathbb{R} \to G$ be a cocycle such that $\hat{\mathcal{F}}^t(gx) = g\hat{\mathcal{F}}^t(x)g^{-1}$; call such an $\hat{\mathcal{F}}$ a *G-cocycle*. Then $\hat{\phi}^t_{\hat{\mathcal{F}}}(x) = \hat{\mathcal{F}}^t(x)\hat{\phi}^t(x)$ is a G-equivariant flow and preserves $\hat{\mu}$. We topologise such cocycles by considering the $L(G)$-valued function $\hat{f}(x) = \lim_{t\to 0} \hat{\mathcal{F}}^t(x)/t$ (note that $\hat{f}(gx) = \mathrm{Ad}(g)\hat{f}(x)$).

Suppose $G = K^n$, a torus. Then $\hat{\mathcal{F}}$ can be identified with a ϕ-cocycle $\mathcal{F}$. Moreover, the implications $1 \Rightarrow 2$ in Propositions 4.2, 4.3 continue to hold (see

[KN]). We can use the above analysis to conclude that, when ϕ^t is cohomologically finite, the set of cocycles for which $\hat{\phi}^t_{\hat{\mathcal{F}}}$ is ergodic or mixing contains an open dense set.

Suppose G is a semi-simple compact connected Lie group. Skew products of such groups with a hyperbolic diffeomorphism have been considered in [FP], and the ergodic and mixing extensions were seen to form dense open sets (in the Hölder, C^1 and smooth toplogies). The arguments presented in [FP] for discrete time continue to hold for continuous time, and so we merely sketch the main ingredients of the proof.

The key fact in [FP] is that if the skew product ϕ_f is not ergodic, then $\hat{\Lambda}$ decomposes into a disjoint union of closed ϕ_f-invariant sets, each of which has the structure of an H-principle sub-bundle, for some proper closed subgroup $H < G$ [PP2]. The corresponding statement for continuous time is:

Proposition 4.8
Let $\hat{\phi}^t$ be a G-equivariant flow on the G-space $\hat{\Lambda}$ and suppose the quotient flow $\phi^t = \hat{\phi}^t/G$ is hyperbolic. Equip $\hat{\phi}^t$ with the Haar lift of a Hölder equilibrium state for ϕ^t. If $\hat{\phi}^t$ is not ergodic then there exists a closed proper subgroup $H < G$ and a closed $\hat{\phi}^t$-invariant subspace $E \subset \hat{\Lambda}$ such that:

1. *the restriction $\hat{\phi}^t|_E$ is ergodic;*
2. *if $g \in H$ then $gE = E$, if $g \notin H$ then $gE \cap E = \emptyset$;*
3. *E has the structure of a principle H-bundle.*

Proof (sketch). The case for a shift of finite type is considered in [PP2]; in this case, $\hat{\Lambda}$ is a trivial G-bundle and E is a trivial H-bundle, by zero-dimensionality. One can then prove the result for the suspension of a shift of finite type, and then for a general hyperbolic flow using symbolic dynamics. ❏

Remark. For an Anosov flow preserving a smooth measure, Proposition 4.8 can be found in [Br].

Thus to show that a G-equivariant flow is ergodic, it is sufficient to show that $H = G$.

Hence to show stable ergodicity of G-extensions, it is sufficient to show that $H = G$ is an open dense condition. This is done in discrete time by modifying the function f on small neighbourhoods of two distinct, but homoclinic, orbits and using the fact that the set of pairs of elements of G that topologically generate G is open and dense. Only minor changes are required for the case of continuous time. In particular, we do not need to assume that $H^1(\Lambda;\mathbb{Z})$ has finite rank. It is also easy to show that when G is semi-simple, ergodic extensions are mixing. We refer the reader to [FP].

A general compact connected Lie group G is locally the product of a torus and a semi-simple Lie group. In particular, we can find normal subgroups $A, S \lhd G$ such that G/A is semi-simple, G/S is a torus and $A \cap S$ is discrete. This allows us to glue the semi-simple and toral cases together (see [FP]) to obtain:

Theorem 4.9
Let G be a compact connected Lie group and let $\hat{\Lambda}$ be a G-space. Let $\hat{\phi}^t$ be a G-equivariant flow with cohomologically finite hyperbolic quotient ϕ^t. Equip $\hat{\Lambda}$ with the Haar lift of an equilibrium state for ϕ^t. Then the set of G-cocycles $\hat{\mathcal{F}}$ for which the G-equivariant flow $\hat{\phi}^t_{\hat{\mathcal{F}}}$ is ergodic or mixing contains an open dense set (using the Hölder, C^1 or smooth topologies, as appropriate).

An example of such a $\hat{\phi}^t$ is given by a frame bundle flow. Let M be a smooth compact n-dimensional Riemannian manifold with strictly negative sectional curvatures. Let $\hat{\Lambda}$ denote the space of n-dimensional oriented orthonormal frames on M. Define the *frame bundle flow* $\hat{\phi}^t$ on $\hat{\Lambda}$ by parallel transporting a frame around the geodesic on M determined by the first vector of the frame, using the Levi-Cività connection. Then $\hat{\phi}^t$ is an $\mathrm{SO}(n-1)$-extension of the geodesic flow ϕ^t on the unit tangent bundle of M. It is well-known that ϕ^t is Anosov. Constructing the flow $\hat{\phi}^t{}_{\hat{\mathcal{F}}}$ corresponds to changing the connection with which parallel transport is defined, but without changing the geodesics. Let ∇ be the Levi-Cività connection. If ∇' is another connection then $\nabla' = \nabla + A$ for some $A : TM \to \mathrm{End}(TM)$. Then ∇, ∇' have the same geodesics if $A_v(v) = 0$ for all $v \in TM$; for convenience, call such a ∇' a *geodesically equivalent connection*. Thus Theorem 4.9 has the following corollary:

Theorem 4.10
The set of geodescally equivalent connections on a compact Riemannian manifold with strictly negative sectional curvatures for which the associated frame bundle flow is ergodic or mixing contains an open dense set.

Alternatively, if ∇' is close to ∇ then ∇' is the Levi-Cività connection for a smooth Riemannian metric close to the original metric [Br]. Thus we can rederive a result in [Br]: there is an open dense set of smooth negatively curved metrics for which the frame bundle flow is mixing.

§5 Stable mixing for velocity changes

Recall from §2 that a flow ψ^t_k obtained from a velocity change of ϕ^t determines and is determined by a real-valued cocycle k. For ease of notation we shall write

$$k(x,t) = \int_0^t k(\phi^u x)\,du$$

with k strictly positive. We assume that ϕ^t is equipped with a fixed Hölder equilibrium state μ and we normalise k by requiring $\mu(k) = 1$. We have the following interpretation of cohomologous cocycles.

Proposition 5.1
Let k, k' be strictly positive. Then the following are equivalent:

1. *there exists a map $h : \Lambda \to \Lambda$ such that $h\psi^t_k = \psi^t_{k'}h$ and h maps each (oriented) orbit to itself;*
2. $k = k' \bmod \mathcal{B}$.

Proof. If 1. holds then we can write $h(x) = \psi_k^{u(x)}(x)$ for some $u : \Lambda \to \mathbb{R}^+$. Hence $\psi_k^{u(x)+t}(x) = \psi_k^{u(\psi_{k'}^t x)} \psi_{k'}^t(x)$. In the notation of §2 we have $\psi_{k'}^t(x) = \phi^{l'(x,t)}(x) = \psi_k^{k(x,l'(x,t))}(x)$ whence $u(x) + t = u(\psi_{k'}^t x) + k(x, l'(x,t))$. Let $t = k'(x,s)$ so that $s = l'(x,t)$ and $\psi_{k'}^t(x) = \psi_{k'}^{k'(x,s)}(x) = \phi^s(x)$. Hence

$$k'(x,s) + u(x) = u(\phi^s x) + k(x,s).$$

Dividing by s and letting $s \to 0$ we see that $k' = k + u'_\phi$. Hence $k = k' \bmod \mathcal{B}$.

The converse follows easily by reversing the argument. ❑

The following gives a criterion in terms of the dynamic cohomology for a velocity changed flow to be mixing. Again, we urge the reader to carefully note the suffixes on the $\mathcal{Z}$'s.

Proposition 5.2
The flow ψ_k^t is not mixing if and only if, for some $0 \neq c \in \mathbb{R}$, we have $ck = 0 \bmod \mathcal{Z}_{\mathbb{Q}} + \mathcal{B}$. In particular, if $k \neq 0 \bmod \mathcal{Z}_{\mathbb{R}} + \mathcal{B}$ then ψ_k^t is mixing.

Proof. Recall that ψ_k^t is always ergodic with respect to $k\,d\mu$. Hence, ψ_k^t is not weak-mixing (equivalently, not mixing) if and only if there exists a measurable eigenfunction $w : \Lambda \to K$ corresponding to a non-zero eigenfrequency c, i.e. for each t,

$$\frac{w(\psi_k^t x)}{w(x)} = \exp 2\pi i c t \text{ a.e.}$$

This can be re-written as

$$\frac{w(\phi^s x)}{w(x)} = \exp 2\pi i \int_0^s ck(\phi^u x)\, du \text{ a.e.} \tag{10}$$

By Theorem 3.3, we may assume that w is ϕ-differentiable and that (10) holds everywhere. Hence $ck + \mathcal{B} \in \iota(H^1(\Lambda;\mathbb{Z}))$ so that $ck = 0 \bmod \mathcal{Z}_{\mathbb{Z}} + \mathcal{B}$ and, in particular, $ck = 0 \bmod \mathcal{Z}_{\mathbb{Q}} + \mathcal{B}$.

Conversely, if $ck = 0 \bmod \mathcal{Z}_{\mathbb{Q}} + \mathcal{B}$ and $c \neq 0$ then by multiplying by a suitable integer, we have (for a possibly different c) $ck = 0 \bmod \mathcal{Z}_{\mathbb{Z}} + \mathcal{B}$. Applying ι^{-1} to this equation, we recover an equation of the form (10). Hence, ψ_k^t has an eigenfunction corresponding to a non-zero eigenfrequency and thus cannot be mixing.

Finally, observe that if $k \neq 0 \bmod \mathcal{Z}_{\mathbb{R}} + \mathcal{B}$ then any non-zero multiple of k is also not congruent to $0 \bmod \mathcal{Z}_{\mathbb{R}} + \mathcal{B}$ and, in particular, is not congruent to $0 \bmod \mathcal{Z}_{\mathbb{Q}} + \mathcal{B}$. ❑

Remarks.

1. It could happen that $k = 0 \bmod \mathcal{Z}_{\mathbb{R}} + \mathcal{B}$ but no non-zero multiple of k is congruent to $0 \bmod \mathcal{Z}_{\mathbb{Q}} + \mathcal{B}$, so that ψ_k^t is mixing. One can easily show that such velocity changes are unstably mixing: there are arbitrarily small perturbations of k such that the perturbed velocity changed flow is not mixing.

2. Suppose $H^1(\Lambda;\mathbb{Z})$ is trivial. Then any velocity change of ϕ^t is mixing. This is because if $k = 0 \bmod \mathcal{B}$ then k cannot be strictly positive as it must vanish on periodic orbits.

We restate our main result:

Theorem 5.3
Let ϕ^t be a cohomologically finite hyperbolic flow on a basic set Λ and let μ be an equilibrium state corresponding to a Hölder function. Then the set of normalised strictly positive functions $k : \Lambda \to \mathbb{R}^+$ for which the associated velocity changed flow ψ_k^t is mixing (using Hölder, C^1 or smooth functions, as appropriate) contains an open dense set.

Proof. Consider the subset $\{k \in C^\theta(\Lambda; \mathbb{R}^+) \mid k \neq 0 \mod \mathcal{Z}_\mathbb{R} + \mathcal{B}\}$. By Proposition 5.2, if k lies in this set then ψ_k^t is mixing. By Proposition 3.10, this set is dense. By Proposition 3.11, it is also open. ❏

Remark. We could also have considered C^r velocity changes of C^r flows for $2 \leq r < \infty$. In this case, given $k \in C^r(\Lambda; \mathbb{R}^+)$, we can obtain an open neighbourhood of k in $C^{r-\varepsilon}$ for any $\varepsilon \in (0,1)$ of mixing velocity changes.

All velocity changes may be mixing; this happens precisely when no function in $\mathcal{Z}_\mathbb{R} + \mathcal{B}$ is strictly positive. If there is a non-mixing velocity change, then we can find a strictly positive $k \in \mathcal{Z}_\mathbb{Z} + \mathcal{B}$. Applying ι^{-1} to k, we find a $w : \Lambda \to K$ such that $w'_\phi / 2\pi i w > 0$. By a result in [S], the subset $\Lambda_w = \{x \in \Lambda \mid w(x) = 1\}$ is a global cross-section to the flow. In the Anosov case when w can be chosen to be C^1 or smooth, we recover a familiar result of Plante's [Pl]: a non-mixing Anosov flow is the constant suspension of an Anosov diffeomorphism. In the hyperbolic case, we recover Bowen's result that a non-mixing hyperbolic flow is a constant suspension of an Axiom A^* homeomorphism $\phi : \Lambda_w \to \Lambda_w$ [B1].

We can use the above to give a criterion for the existence of a cross-section in terms of the pressure functional. Observe from (1) that the pressure $P(g)$ of a continuous function $g : \Lambda \to \mathbb{R}$ depends only on the cohomology class of g. We define the *cohomological pressure* to be the map

$$P_H : H^1(\Lambda; \mathbb{Z}) \to \mathbb{R} : [w] \mapsto P(\iota[w]).$$

The following is proved in [Sh] in the Anosov case. Sharp uses the first de Rham cohomology group whereas we are using the Bruschlinsky group. Nevertheless, the proof given here is identical to that in [Sh].

Proposition 5.4
Let ϕ^t be a hyperbolic flow. Then ϕ^t has a cross-section if and only if P_H is unbounded below.

Proof. Suppose there is a cross-section. Then by the remarks above, we can find a Hölder $w : \Lambda \to K$ that is ϕ-differentiable and

$$g = \frac{1}{2\pi i} \frac{w'_\phi}{w}$$

is strictly positive. Suppose $g \geq \varepsilon > 0$. For each $n \in \mathbb{Z}$ let μ_n denote the equilibrium state corresponding to ng. Then for $n \leq 0$ we have

$$P_H([w^n]) = h_{\mu_n}(\phi) + \mu_n(nG) \leq h(\phi) + n\varepsilon \tag{11}$$

($h(\phi)$ is the topological entropy of ϕ). The right-hand side of (11) tends to $-\infty$ as $n \to -\infty$, whence P_H is unbounded below.

Conversely, if no cross-section exists then a result in [S] allows us to choose a ϕ-invariant probability measure m_0 such that $m_0(\iota[w]) = 0$ for all $[w] \in H^1(\Lambda; \mathbb{Z})$. Then

$$P_H([w]) = \sup_m \{h_m(\phi) + m(\iota[w])\} \geq h_{m_0}(\phi) \geq 0.$$

So P_H is bounded below. ❏

As in [Sh], we see that if P_H is bounded below then it is bounded below by 0.

It is interesting to consider how the dynamic cohomology groups behave as invariants of the flow. It is easy to see that if $f, f' : \Lambda \to \mathbb{R}$ generate ϕ-cohomologous cocycles then they also generate ψ_k-cohomologous cocycles. Hence $\mathcal{H}^1(\phi; \mathbb{R}) \cong \mathcal{H}^1(\psi_k; \mathbb{R})$ (and indeed for any Lie group G). However, if f, f' are *essentially* ϕ-cohomologous then it is not in general true that they are essentially ψ_k-cohomologous. Hence $\mathcal{H}^1_0(\phi; \mathbb{R})$ is not necessarily isomorphic to $\mathcal{H}^1_0(\psi_k; \mathbb{R})$. An example of this phenomena can be found in [P1]. It is known that the equilibrium state of $f \in C^\theta(\Lambda; \mathbb{R})$ depends only on the essential ϕ-cohomology class of f. In [P1] it is shown how the velocity of a hyperbolic attractor ϕ^t (with C^1 unstable distribution) can be changed to ψ_k^t so that two a priori distinct equilibrium states for ϕ^t (namely the measure of maximal entropy and the S.R.B. measure) coincide for ψ_k^t.

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