Regularity of invariant graphs over hyperbolic systems

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8$^{\text{th}}$ September, 1999

Abstract

We consider cocycles with negative Lyapunov exponents defined over a hyperbolic dynamical system. It is well-known that such systems possess invariant graphs and that under spectral assumptions these graphs have some degree of Hölder regularity. When the invariant graph has a slightly higher Hölder exponent than the a priori lower bound (even on just a set of positive measure), we show that the graph must be Lipschitz or (in the Anosov case) as smooth as the cocycle.

§1 Introduction

In this paper we consider the regularity of dynamically invariant graphs of skew-product systems with uniformly hyperbolic base dynamics and uniform or nonuniform contraction in the fibre.

Our work is related to that of Hirsch, Pugh and Shub [2], Stark [5, 6] and Kaplan, Mallet-Paret and Yorke [4], who have investigated the existence and regularity properties of invariant graphs in skew-product systems. Similar results have been proved in the setting of group-valued cocycle equations [7, 8].

Suppose $f : M \to M$ is a C^1 diffeomorphism of a C^∞ compact Riemannian manifold M and suppose that Λ is a locally maximal hyperbolic set and $f|_\Lambda$ is an Axiom A diffeomorphism which has an invariant equilibrium measure m corresponding to a Hölder continuous function. Often we shall assume that $\Lambda = M$, so that f is Anosov.

There exist constants $C > 0$, $\mu_{s,\max} \geq \mu_s > 0$ and $\mu_{u,\max} \geq \mu_u > 0$ such that on local stable manifolds $\{W^s_{\text{loc}}(x)\}$,

$$Ce^{-n\mu_{s,\max}} d(x, y) \leq d(f^n x, f^n y) \leq Ce^{-n\mu_s} d(x, y)$$

for $y \in W^s_{\text{loc}}(x)$ and

$$Ce^{-n\mu_{u,\max}} d(x, y) \leq d(f^{-n} x, f^{-n} y) \leq Ce^{-n\mu_u} d(x, y)$$

for $y \in W^u_{\text{loc}}(x)$. Thus $\mu_{s,\max}$ and μ_s correspond to the *maximum* and *minimum* rates of contraction along stable manifolds, respectively. We denote the conditional measures of m along the stable $(\{W^s_{\text{loc}}(x)\})$ and unstable $(\{W^u_{\text{loc}}(x)\})$ foliations by $m_s(x)$ and $m_u(x)$ respectively. We will write m_s, m_u when the point x is clear by context.

Let Y be a complete separable metric space and suppose that $g : \Lambda \times Y \to Y$ is a continuous mapping which is α-Hölder in the first component; that is, for each compact $W \subset Y$ there exists K_W such that

$$d(g(x,y), g(x',y)) \leq K_W d(x,x')^\alpha$$

for each $y \in W$. We shall often assume that g is C^k; in this case, we implicitly require Y to have a smooth Riemannian structure.

We form the skew-product

$$F(x,y) = (f(x), g(x,y)). \tag{1.1}$$

To simplify notation we write $g(x,y) = g(x)y$ and consider $g : \Lambda \to C(Y,Y)$ as a family of continuous mappings of Y. If for each x, $g(x)$ is assumed invertible (as we will for most of this paper) we denote $g(x)^{-1}$ by $h(x)$. We define $g^n(x)$ by $F^n(x,y) = (f^n(x), g^n(x)y)$ and similarly define (where appropriate) $h^n(x)$ by $F^{-n}(x,y) = (f^{-n}(x), h^n(x)y)$. For heuristic reasons we also write $g^n(x)y$ as $g(f^{n-1}x)g(f^{n-2}x)\cdots g(x)y$, i.e. as a composition of operators acting on Y.

We say that g is uniformly contracting if there exists $\lambda_+ < 0$ and $c \geq 0$ such that

$$d(g^n(x)y, g^n(x)y') \leq ce^{n\lambda_+} d(y,y') \tag{1.2}$$

for all $x \in X$. Take any continuous function $v : \Lambda \to Y$. Then it is easy to check that $g^n(f^{-n}x)v(f^{-n}x)$ converges uniformly to a continuous function ϕ. Moreover, the graph of ϕ is invariant under the skew product F; equivalently ϕ solves the functional equation

$$\phi(fx) = g(x)\phi(x). \tag{1.3}$$

Under the additional assumption that g is α-Hölder for some $0 < \alpha \leq 1$ then the following result provides a degree of regularity for ϕ.

Proposition 1.1 ([2, 5, 6]) *Suppose g is α-Hölder in the first coordinate. Then ϕ is uniformly Hölder on Λ with exponent γ, for any $0 < \gamma \leq \alpha$, such that $\gamma < -\lambda_+/\mu_{s,\max}$. In particular if $\lambda_+ < -\mu_{s,\max}$ and $\alpha = 1$, then ϕ is Lipschitz.*

Remark 1.2 Proposition 1.1 does not need the hyperbolicity assumption on f; we merely need to know the maximum amount of expansion by f^{-1}.

Assume that $\lambda_- \in (-\infty, \lambda_+)$ is the minimum rate of contraction for g, i.e. there exists $C > 0$ such that for all $n \in \mathbb{N}$

$$Ce^{n\lambda_-} d(y, y') \leq d(g^n(x)y, g^n(x)y'). \tag{1.4}$$

(Note that this implies $g(x)$ is invertible for each x.) The map F is said to be *partially hyperbolic* if

$$-\mu_s < \lambda_- < \lambda_+ < \mu_u.$$

We will assume from now on that our skew-product is partially hyperbolic. In the case of uniform contraction the system satisfies

$$-\mu_s < \lambda_- < \lambda_+ < 0 < \mu_u.$$

The main result of this paper states that in our setup Proposition 1.1 is generically optimal. To be more precise we introduce some useful definitions.

Let $H^\alpha_{C,d} := \{x \in \Lambda \mid d(\phi(x), \phi(x')) \leq Cd(x, x')^\alpha$ if $d(x, x') \leq d\}$. The graph ϕ is uniformly C^α if $\Lambda = H^\alpha_{C,d}$ for some C, d. We say that ϕ is *nowhere* C^α with respect to m if $m(H^\alpha_{C,d}) = 0$ for every C, d.

We are now ready to state our main results in the case of uniform contraction. We assume that $F(x, y) = (f(x), g(x)y)$ is a partially hyperbolic system as described above. We say that a function ψ is C^{k-} if it is C^{k-1} and the $(k-1)^{\text{st}}$ derivative is C^γ for any $\gamma \in (0, 1)$ (with the conventions that $C^{1-} = C^1$ and $C^{\infty-} = C^\infty$).

Theorem 1.3 *Let $f : M \to M$ be a C^k Anosov diffeomorphism of a C^∞ compact Riemannian manifold M. Suppose that $(x, y) \mapsto g(x, y)$ is C^k and $F(x, y) = (f(x), g(x)y) : M \times Y \mapsto M \times Y$ is partially hyperbolic. Suppose that ϕ is a continuous solution to the functional equation*

$$g(x)\phi(x) = \phi(fx).$$

Then for any $0 < \gamma \leq 1$ such that $\gamma > -\lambda_-/\mu_s$ either

(i) ϕ is nowhere C^γ, or

(ii) ϕ is C^{k-}.

Moreover, generically (i) occurs.

Remark 1.4 Under the conditions of the theorem, the graph of ϕ when restricted to local unstable manifolds is always C^k. Note also that ϕ is γ'-Hölder for any γ' satisfying $\gamma' < -\lambda_+/\mu_{s,\max}$.

Remark 1.5 A similar result is proved in [1]. They consider the regularity of the sub-bundles E^s, E^u in the Anosov splitting of the tangent bundle of M. These sub-bundles are always Hölder and, generically, this degree of regularity is nowhere exceeded.

Remark 1.6 A similar result holds if we assume only that f is Axiom A. Specifically, if F is partially hyperbolic and, for each $x \in \Lambda$, $g(x)$ is bi-Lipschitz then the analogue of Theorem 1.3 holds if property (ii) states that ϕ is Lipschitz.

We remark that Kaplan et al. [4] consider a family of skew-products over hyperbolic toral automorphisms with affine cocycles. Specifically, they study the skew-product $F : T^k \times \mathbb{R} \to T^k \times \mathbb{R} : (x, y) \mapsto (Ax, p(x) + \lambda y)$ where T^k denotes the k-dimensional torus, $A : T^k \to T^k$ is a linear hyperbolic toral automorphism, $\lambda \in \mathbb{R}$ and $p : T^k \to \mathbb{R}$ is a sufficiently smooth function of period 1 in each coordinate. When $k = 2$, A has eigenvalues μ, μ^{-1} with $|\mu| < 1$. Provided that $\mu < \lambda < 1$, the system is uniformly contracting and partially hyperbolic so that the attracting invariant graph is continuous, indeed Hölder. The following dichotomy holds.

Proposition 1.7 ([4]) *Let A be a hyperbolic automorphism of T^2 with eigenvalues μ, μ^{-1}, $|\mu| < 1$. Let $p : T^2 \to \mathbb{R}$ be C^3 and suppose that $\lambda \in (\mu, 1)$. Then either:*

(i) ϕ is nowhere differentiable and $\dim_B(\text{graph}(\phi)) = 3 - (\log \lambda / \log \mu)$; or,

(ii) ϕ is smooth (in which case $\dim_B(\text{graph}(\phi)) = 2$).

Dimension estimates in our setup will appear in a forthcoming paper [9].

§1.1 Non-uniform contraction

We now discuss our results in the context in which g contracts nonuniformly. We still assume that there exist constants $C, c > 0$, $\lambda_- < \lambda_+$, such that for all $n \in N$ equation 1.2 and equation 1.4 hold but $\lambda_+ > 0$ is allowed. In this setting we are not guaranteed the existence of a continuous invariant graph (or indeed an invariant graph at all) and so we will consider the regularity properties of invariant measurable graphs.

One context in which there always exists a measurable attracting invariant graph is that of "contraction on average".

Suppose that

$$\rho(x) = \lim_{n\to\infty} \sup_{y,y' \in Y} \frac{1}{n} \log \frac{d(g^n(x)y, g^n(x)y')}{d(y, y')} \tag{1.5}$$

exists for m a.e. $x \in \Lambda$. This condition is satisfied if, for example, the functions $\log^+ ||g(x)||, \log^+ ||g(x)^{-1}|| \in L^1(\Lambda, m)$ where

$$||g(x)|| := \sup_{y,y' \in Y} \frac{d(g(x)y, g(x)y')}{d(y, y')} \tag{1.6}$$

(and $||g(x)^{-1}||$ is defined similarly). In this context we will say that the skew-product *contracts on average* if there exists $\rho < 0$ such that

$$\rho(x) \leq \rho < 0 \tag{1.7}$$

for m a.e. $x \in \Lambda$. As a consequence of contraction on average we have the following result of Stark [6], which applies in a more general setting:

Proposition 1.8 ([6]) *Suppose X is a compact metric space, $f : X \to X$ a homeomorphism, m an f-invariant measure, Y a complete metric space and $g : X \times Y \to Y$ is continuous and satisfies (1.7). Then there exists an f-invariant set $\Lambda \subset X$ such that $m(\Lambda) = 1$ and a measurable function $\phi : \Lambda \to Y$ such that the graph of ϕ is invariant and attracting under (f, g).*

Note that in this case the graph $\phi : \Lambda \to Y$ is measurable and satisfies the functional equation (1.3) for m a.e. $x \in X$.

Contraction on average implies that for m a.e. $x \in \Lambda$, given $\epsilon > 0$ there exists a measurable function $c : \Lambda \to \mathbb{R}$ such that

$$d(g^n(x)y, g^n(x)y') \leq c(x)e^{n(\rho+\epsilon)}d(y, y'). \tag{1.8}$$

Stark has the following regularity results in the contraction on average setting:

Proposition 1.9 ([6]) *Suppose further that f is a $C^{1+\alpha}$ diffeomorphism, $\rho < 0$, $c(x) \in L^1(\Lambda, m)$ and for a point $y_0 \in Y$ there exist constants β, K_1, K_2 such that*

$$(g(x)y, g(x')y) \leq (K_1 + K_2 d(y, y_0)^\beta)d(x, x')^\alpha. \tag{1.9}$$

Then for any $0 < \gamma \leq \alpha$ such that $\gamma < -\rho/\mu_{s,\max}$ we can construct a nested sequence Λ_k, $k \in \mathbb{N}$ of compact sets such that $m(\Lambda_k) \to 1$ and $\phi|_{\Lambda_k}$ is Hölder of exponent γ.

We now assume that f is a C^k Anosov diffeomorphism, equipped with an ergodic equilibrium measure m corresponding to a Hölder function.

We assume that $g(x)$ is invertible for each x and denote $h(x) = g(x)^{-1}$.

We will first assume that we are in the setting of Proposition 1.9 i.e. we assume that $\rho < 0$, $c(x) \in L^1(\Lambda, m)$ and for a point $y_0 \in Y$ there exist constants β, K_1, K_2 such that

$$d(g(x)y, g(x')y) \leq (K_1 + K_2 d(y, y_0)^\beta)d(x, x')$$

and

$$d(h(x)y, h(x')y) \leq (K_1 + K_2 d(y, y_0)^\beta)d(x, x')$$

(we have taken $\alpha = 1$). If we are in this setting we say that g contracts on average in the sense of Proposition 1.9.

Under the assumption of contraction of average in the sense of Proposition 1.9 we have the following generalisation of Theorem 1.3 in the non-uniform case. Assume that ϕ is a measurable invariant graph for the partially hyperbolic map F which contracts on average in the sense of Proposition 1.9.

Theorem 1.10 *For any $0 < \gamma \leq 1$ such that $\gamma > -\lambda_-/\mu_s$ either*

(i) ϕ is nowhere C^γ, or

(ii) ϕ is C^{k-}.

In fact either

(a) $\phi|_S$ is not C^γ for any set S of positive measure, or

(b) ϕ is C^{k-}.

Moreover, generically (i), (a) occur.

Remark 1.11 If we assume that $f : \Lambda \to \Lambda$ is Axiom A rather than Anosov, then the conclusions of Theorem 1.10 hold if property (ii) states that ϕ is Lipschitz.

Consider the case when ϕ is merely assumed to be an invariant measurable graph for the partially hyperbolic mapping F (i.e. we make no assumption of contraction on average) and the base is an ergodic (with respect to a Hölder equilibrium measure m) Anosov diffeomorphism. Then we have the following regularity results:

Theorem 1.12 *For any $0 < \gamma \leq 1$ such that $\gamma > \max\{-\lambda_-/\mu_s, \lambda_+/\mu_u\}$ either*

(i) ϕ is nowhere C^γ, or

(ii) ϕ is C^{k-}.

In fact either

(a) $\phi|_S$ is not C^γ for any set S of positive measure, or

(b) ϕ is C^{k-}.

Moreover, generically (i), (a) occur.

Remark 1.13 If we assume that $f : \Lambda \to \Lambda$ is Axiom A rather than Anosov and $g(x)$ is bi-Lipschitz, then the conclusions of Theorem 1.12 hold if property (ii) states that ϕ is Lipschitz.

§2 Proof of Theorem 1.3

We discuss the regularity of ϕ along the stable and unstable manifolds of f independently.

§2.1 Regularity on local stable manifolds

Let $x \in \Lambda$. For each $n > 0$, define

$$\gamma_x^{s,n}(\cdot) : W^s(x) \to Y : t \mapsto h^n(t)\phi(f^n x).$$

Then

$$\begin{aligned}
&d(\gamma_x^{s,n+1}(t), \gamma_x^{s,n}(t)) \\
&= d(h^n(t)h(f^n t)g(f^n x)\phi(f^n x), h^n(t)h(f^n t)g(f^n t)\phi(f^n x)) \\
&\leq Ce^{-n\lambda_-} d(g(f^n x)\phi(f^n x), g(f^n(t))\phi(f^n x)).
\end{aligned}$$

As ϕ is continuous, hence bounded, we have

$$\begin{aligned}
d(g(f^n x)\phi(f^n x), g(f^n(t))\phi(f^n x)) &\leq Kd(f^n x, f^n t)^\alpha \\
&\leq Ke^{-n\alpha\mu_s} d(x,t)^\alpha
\end{aligned}$$

where α is the Hölder constant of $x \mapsto g(x, \cdot)$. Hence

$$d(\gamma_x^{s,n+1}(t), \gamma_x^{s,n}(t)) \leq Ce^{-n(\lambda_- + \alpha\mu_s)} d(x,t)^\alpha$$

for some constant $C > 0$, independent of x, t and n. Hence if $\alpha > -\lambda_-/\mu_s$, $\gamma_x^{s,n}(t)$ is Cauchy by the partial hyperbolicity assumption and therefore converges. Denote the limit by $\gamma_x^s(t)$.

Lemma 2.1 *The graph of $\gamma_x^s(\cdot)$ is F-invariant in the sense that $F(t, \gamma_x^s(t)) = (f(t), \gamma_{f(x)}^s(ft))$.*

Proof. As $F(t, \gamma_x^s(t)) = (f(t), g(t)\gamma_x^s(t))$ the claim follows from the fact that

$$\begin{aligned}
g(t)\gamma_x^s(t) &= \lim_{n\to\infty} g(t)h^n(t)g^n(x)\phi(x) \\
&= \lim_{n\to\infty} g(t)h(t)h^{n-1}(ft)g^{n-1}(fx)g(x)\phi(x) \\
&= \gamma_{fx}^s(ft).
\end{aligned}$$

□

The following quantity shall prove useful. For $t \in W_{\mathrm{loc}}^s(x)$, define

$$s_j(t) = d(h(t)\cdots h(f^{j-1}t)h(f^j t)\phi(f^j x), h(t)\cdots h(f^{j-1}t)h(f^j x)\phi(f^j x)).$$

Lemma 2.2 *The maps $t \mapsto \gamma_x^s(t) : W^s_{\mathrm{loc}} \to Y$ are uniformly Lipschitz.*

Proof. Observe that

$$\begin{aligned} d(\gamma_x^{s,n}(t), \gamma_x^{s,n}(x)) &= d(h^n(t)\phi(f^n x), h^n(x)\phi(f^n x)) \\ &\leq \sum_{j=0}^{n-1} d(h(t)\cdots h(f^{j-1}t)h(f^j t)h(f^{j+1}x)\cdots h(f^{n-1}x)\phi(f^n x), \\ &\qquad h(t)\cdots h(f^{j-1}t)h(f^j x)h(f^{j+1}x)\cdots h(f^{n-1}x)\phi(f^n x)) \\ &= \sum_{j=0}^{n-1} s_j(t). \end{aligned}$$

We estimate

$$\begin{aligned} s_j(t) &\leq Ce^{-\lambda_- j} d(h(f^j t, f^j x)) \\ &\leq Ce^{-\lambda_- j} d(f^j t, f^j x) \text{ as } h \text{ is Lipschitz in the first coordinate} \\ &\leq Ce^{-j(\lambda_- + \mu_s)} d(t, x). \end{aligned}$$

By the partial hyperbolicity assumption, we see that

$$d(\gamma_x^{s,n}(t), \gamma_x^{s,n}(x)) \leq \sum_{j=0}^{n-1} s_j(t) \leq \sum_{j=0}^{\infty} s_j(t) \leq Cd(t,x).$$

Letting $n \to \infty$ gives the result. □

We now show that if ϕ is Lipschitz on a set of positive measure then it is globally Lipschitz. For each $x \in \Lambda$ and $t \in W^s_{\mathrm{loc}}(x)$ define

$$\begin{aligned} \Delta(x,t) &= d(\phi(t), \gamma_x^s(t)) \\ \Delta(x) &= \sup_{t \in W^s_{\mathrm{loc}}(x)} \Delta(x,t). \end{aligned}$$

The functions $\Delta(x,t)$ and $\Delta(x)$ will act as obstructions to the regularity of ϕ along the local stable manifold through x. Note that $x \mapsto \Delta(x)$ is continuous.

Lemma 2.3 *If ϕ is C^γ on a set of positive measure, where $\gamma > -\lambda_-/\mu_s$, then it is globally uniformly Lipschitz on local stable manifolds.*

Proof. Suppose $m(H^\gamma_{C,d}(\phi)) > 0$ for some C, d. Then, by Poincaré recurrence, there exists a dense orbit $\{f^n x'\}$ that visits $H^\gamma_{C,d}(\phi)$ infinitely often. Let $n_j \uparrow \infty$ be such that $f^{n_j} x' \in H^\gamma_{C,d}(\phi)$. Then for each n_j

$$\begin{aligned} d(\phi(t), \gamma_{x'}^s(t)) &= d(h^{n_j}(t)\phi(f^{n_j} t), h^{n_j}(t)\phi(f^{n_j} x')) \\ &\leq Ce^{-n_j(\lambda_- + \gamma\mu_s)} d(x', t)^\gamma. \end{aligned}$$

Letting $n_j \to \infty$ yields $\Delta(x') = 0$ by partial hyperbolicity and the assumption on γ. Similarly $\Delta(f^n x') = 0$ for each $n > 0$ so that $\Delta(x) = 0$ on a dense set; hence $\Delta(x) = 0$ everywhere and the lemma follows. □

We now show that if ϕ is globally Lipschitz then it is as smooth as g along the local stable manifold through $x \in \Lambda$. Note that by Lemma 2.3, if the graph of ϕ is Lipschitz then the graph of ϕ contains the graphs of the functions γ_x^s. To show that ϕ is C^k along local stable manifolds, it suffices to prove that the functions γ_x^s are C^k. To do this, we show that the graphs of the functions γ_x^s are the local stable manifolds for the partially hyperbolic (in the sense of [2]) skew product F. By [2, Theorem 5.5, page 61]), it is sufficient to show that the rate of convergence of orbits along the graph of γ_x^s is exponential at rate $-\mu_s$ (possibly replacing $-\mu_s$ by $-\mu_s + \epsilon$ for a suitably small $\epsilon > 0$).

Lemma 2.4 *There exists a constant $C > 0$ such that for all $x \in \Lambda$ and $t \in W^s_{\mathrm{loc}}(x)$, we have*

$$d(F^n(t, \gamma_x^s(t)), F^n(x, \phi(x))) \leq Ce^{-n\mu_s}.$$

Proof. Note that $F^n(t, \gamma_x^s(t)) = (f^n(t), g^n(t)\gamma_x^s(t))$ and $F^n(x, \phi(x)) = (f^n(x), \phi(f^n(x)))$. As $d(f^n(t), f^n(x)) \leq Ce^{-n\mu_s} d(x, t)$, it suffices to prove that $d(g^n(t)\gamma_x^s(t), \phi(f^n(x))) \leq Ce^{-n\mu_s} d(x, t)$.

Now

$$\begin{aligned} d(g^n(t)\gamma_x^s(t), \phi(f^n x)) &= d(g^n(t)h^{n+m}(t)\phi(f^{n+m}x), h^m(f^n x)\phi(f^{n+m}x)) \\ &= d(h^m(f^n t)\phi(f^{n+m}x), h^m(f^n x)\phi(f^{n+m}x)) \\ &\leq \sum_{j=0}^{n-1} s_j(f^n t). \end{aligned}$$

Using similar arguments to those in Lemma 2.2, it is easy to check that

$$s_j(f^n t) \leq Ce^{-n\mu_s - j(\lambda_- + \mu_s)} d(t, x).$$

Hence

$$\sum_{j=0}^{n-1} s_{j,n} \leq Ce^{-n\mu_s} \sum_{j=0}^{\infty} e^{-j(\lambda_- + \mu_s)} d(t, x). \tag{2.10}$$

By partial hyperbolicity, the sum on the right-hand side of (2.10) is finite, and the lemma follows. □

§2.2 Regularity along unstable manifolds

We show that ϕ, when restricted to unstable manifolds, is always as regular as g. We emphasis that we do not need any partial hyperbolicity assumption here and we remark that the analogues of the functions $\Delta(x,t)$ and $\Delta(x)$ are always identically equal to zero.

Lemma 2.5 *Let ϕ be an invariant graph corresponding to a C^k cocycle g. Then ϕ is uniformly C^k along each unstable manifold.*

Proof. The proof is similar to (but easier than) the stable manifold case above, so we merely sketch the arguments.

Define for each $x \in \Lambda$ and $n > 0$,

$$\gamma_x^{u,n}(\cdot) : W^u_{\mathrm{loc}}(x) \to Y : t \mapsto g^n(f^{-n}t)\phi(f^{-n}x).$$

This is easily seen to be Cauchy (and we do not need the partial hyperbolicity assumption for this to hold). Hence the limit $\gamma_x^u(t) = \lim_{n\to\infty} \gamma_x^{u,n}(t)$ exists.

One can again check that the family of graphs defined by γ_x^u are Lipschitz and are invariant in the sense that $F^{-1}(t, \gamma_x^u(t)) = (f^{-1}(t), \gamma^u_{f^{-1}x}(f^{-1}t))$. A similar argument to that for the local stable manifold case allows us to conclude that $\gamma_x^u(t) = \phi(t)$ (without any partial hyperbolicity assumption). Finally, the convergence under f^{-1} in the graphs is exponential at rate $e^{-\mu_u}$. Hence, by [2], these graphs correspond to the unstable manifolds of F and are therefore C^k. □

Proof of Theorem 1.3. We have seen that if alternative (i) in Theorem 1.3 fails then ϕ is uniformly C^k along local stable and unstable manifolds. By a result of Journé [3], we can conclude that ϕ is globally C^{k-}. □

§2.3 Generically the graph is nowhere differentiable

We now show that generically the first alternative in Theorem 1.3 occurs, namely that the invariant graph is nowhere C^γ.

Recall that the function $\Delta(x,t)$ is an obstruction to the regularity of ϕ along stable manifolds. Also, recall that along unstable manifolds ϕ is always C^k.

We now show that for a generic set of cocycles $\Delta(x,t) \neq 0$; hence for this set of cocycles ϕ is not C^k and therefore nowhere differentiable.

Recall that ϕ is the unique continuous function $\phi : \Lambda \to Y$ satisfying $g(x)\phi(x) = \phi(fx)$ for all $x \in \Lambda$. Note that for fixed $v \in Y$, $\phi(x) = \lim_{n\to\infty} g(f^{-1}x)\cdots g(f^{-n}x)v$ uniformly in n.

Thus

$$\begin{aligned}
\gamma_x^s(t) &= \lim_{n\to\infty} h^n(t)\phi(f^n x) \\
&= \lim_{n\to\infty}\lim_{m\to\infty} g(t)^{-1}\cdots g(f^{n-1}t)^{-1} g(f^{n-1}x)\cdots g(x)\cdots g(f^{-m}x)v
\end{aligned}$$

Now choose a periodic point x and a point $t' \in W^s_{\text{loc}}(x) \cap W^u_{\text{loc}}(x)$ homoclinic to x. There exists a neighbourhood U of t' such that U is disjoint from $f^n(t')$ for $n \in \mathbb{Z} \setminus \{0\}$ and the orbit of x.

We perturb g within the class of C^k diffeomorphisms (or bi-Lipschitz homeomorphisms in the Axiom A case). Denote the perturbed cocycle by $\tilde{g}$. By making a sufficiently small perturbation, we can assume that $\tilde{g}$ satisfies the partial hyperbolicity conditions. We denote the corresponding perturbed objects by $\tilde{\gamma^s_x}$, $\tilde{\phi}$ and $\tilde{\Delta}$.

Since $\tilde{g}(f^{-m}t') = g(f^{-m}t')$ for $m \geq 1$, we have $\phi(t') = \tilde{\phi}(t')$. Note that $\tilde{\gamma^s_x}(t') = \tilde{g}(t')^{-1}g(t')\gamma^s_x(t')$. Hence

$$\begin{aligned} |\tilde{\Delta}(x,t') - \Delta(x,t')| &= |d(\tilde{g}(t')^{-1}g(t')\gamma^s_x(t'), \phi(t')) - d(\gamma^s_x(t'), \phi(t'))| \\ &\leq d(\tilde{g}(t')^{-1}g(t')\gamma^s_x(t'), \gamma^s_x(t')). \end{aligned}$$

There is an open dense set of perturbations $\tilde{g}$ of g for which the right hand side is not zero, and the result follows.

§3 The case of nonuniform contraction

The proofs of Theorems 1.10 and 1.12 are similar and differ in details only. We will show first that under the conditions of Theorem 1.10 if property (i) does not hold, then the graph of ϕ has a Lipschitz version in the sense that there is a Lipschitz function ϕ' such that $\phi = \phi'$ m a.e. Once we have established this then the proof follows as in the case of uniform contraction, in fact all estimates proceed exactly as in the uniform contraction case. We will only sketch this second part of the proof.

As a straightforward consequence of the definitions of λ_-, λ_+ if $K := \max_{x \in X}\{d(g(x)y_0, y_0), d(h(x)y_0, y_0)\}$ then

$$d(g(x)y, y_0) \leq e^{\lambda_+} d(y, y_0) + K \tag{3.11}$$

and

$$d(h(x)y, y_0) \leq e^{\lambda_-} d(y, y_0) + K. \tag{3.12}$$

Let $X_0 = \{x \in M : \phi(x), \rho(x) \text{ are defined and } \rho(x) \leq \rho < 0\}$. Since $\phi(x)$ is measurable we may choose a set $L \subset M$ and constant C such that $m(L) > \frac{1}{2}$ and for all $x \in L$, $|\phi(x)| \leq C$. Denote the characteristic function of L by ξ_L, then by the ergodic theorem for m a.e. x in a full measure set $X_1 \subset M$ the partial sums

$$\sum_{j=0}^{n-1} \frac{1}{n}\xi_L(f^j x), \quad \sum_{j=0}^{n-1} \frac{1}{n}\xi_L(f^{-j} x)$$

converge to $m(L) > \frac{1}{2}$.

Define $X_2 := \bigcap_{n\in\mathbb{Z}} f^n(X_0\cap X_1)$. Note $m(X_2)=1$ and X_2 is f-invariant. As a consequence of the absolute continuity of the foliations $\{W^s_{\mathrm{loc}}, W^u_{\mathrm{loc}}\}$ (see [8]) for m a.e. y, m_s a.e. $x\in W^{u(s)}_{\mathrm{loc}}(y)$ lies in X_2.

We will, as in the uniform contraction case, prove regularity along local stable/unstable manifolds.

§3.1 Regularity on local unstable manifolds

Assume $x,y\in X_2$ and $x\in W^u_{\mathrm{loc}}(y)$. For all $n\geq 0$ define x_n and y_n by $f^n x_n = x$ and $f^n y_n = y$. Then using (1.3) we have

$$\begin{aligned} d(\phi(x),\phi(y)) &= d(g(f^{n-1}x_n)\cdots g(x_n)\phi(x_n), g(f^{n-1}y_n)\cdots g(y_n)\phi(y_n)) \\ &\leq d(g(f^{n-1}x_n)\cdots g(x_n)\phi(x_n), g(f^{n-1}x_n)\cdots g(x_n)\phi(y_n)) \\ &\quad +\sum_{j=0}^{n-1} u_j \end{aligned}$$

where

$$\begin{aligned} u_j &= d(g(f^{n-1}x_n)\cdots g(f^{j+1}x_n)g(f^jx_n)g(f^{j-1}y_n)\cdots\phi(y_n), \\ &\quad g(f^{n-1}x_n)\cdots g(f^{j+1}x_n)g(f^jy_n)g(f^{j-1}y_n)\cdots\phi(y_n)) \end{aligned}$$

(defined appropriately for u_0 and u_{n-1}). Let

$$v_j = g(f^{j-1}y_n)\cdots\phi(y_n) = h(f^{-j}y)\cdots h(f^{-1}y)\phi(y).$$

Now

$$\begin{aligned} u_j &\leq (e^{\lambda}_{+})^{n-j-1} d(g(f^jx_n)v_j, g(f^jy_n)v_j) \\ &\leq (e^{\lambda}_{+})^{n-j-1}(K_1+K_2 d(v_j,y_0)^\beta) d(f^jx_n, f^jy_n) \end{aligned}$$

and by (1.9)

$$u_j \leq Ce^{(\lambda_+-\mu_u)(n-j)}(K_1+K_2d(v_j,y_0)^\beta)d(x,y).$$

Now

$$\begin{aligned} d(v_j,y_0) &= d(h(f^jy_n)\cdots h(f^{n-1}y_n)\phi(y), y_0) \\ &\leq d(h(f^jy_n)\cdots h(f^{n-1}y_n)\phi(y),\phi(y)) + d(\phi(y),y_0) \\ &\leq e^{\lambda_-(n-j)}d(\phi(y),\phi(y)) + (n-j)K + d(\phi(y),y_0) \\ &\leq (n-j)K + d(\phi(y),y_0). \end{aligned}$$

Thus

$$u_j \leq C(K_1+K_2[(n-j)K+d(\phi(y),y_0)]^\beta)e^{(\lambda_+-\mu_u)(n-j)}d(x,y)$$

where C, K, K_1 and K_2 are fixed constants. Hence, using the ratio test,

$$\sum_{j=0}^{n-1} u_j \leq \sum_{j=-0}^{\infty} u_j \leq C' d(\phi(y), y_0) d(x, y)$$

for some constant $C' > 0$.

We now consider the first term

$$u_{*n} := d(g(f^{n-1}x_n) \cdots g(x_n)\phi(x_n), g(f^{n-1}y_n) \cdots g(y_n)\phi(y_n)).$$

Because $m(L) > \frac{1}{2}$ there is a positive density of times n such that $x_n \in L$ and $y_n \in L$. At these times $\phi(x_n), \phi(y_n)$ are uniformly bounded by C. Since $\rho(x) < \rho$ by assumption of contraction on average, $\lim_{n\to\infty} u_{*n} = 0$.

Thus for m a.e. $y \in M$ and for a.e. $x \in W^u_{\mathrm{loc}}(y)$ with respect to the conditional measure m induces on $W^u_{\mathrm{loc}}(y)$,

$$d(\phi(x), \phi(y)) \leq C' d(\phi(y), y_0) d(x, y).$$

§3.2 Regularity on local stable manifolds

The argument along local stable manifolds proceeds in the same way with the obvious modifications. We define $x_n = f^n x$, $y_n = f^n y$ and consider the sum

$$\begin{aligned} d(\phi(x), \phi(y)) &= d(h(f^{(-n-1)}x_n) \cdots h(x_n)\phi(x_n), h(f^{(-n-1)}y_n) \cdots h(y_n)\phi(y_n)) \\ &\leq d(h(f^{(-n-1)}x_n) \cdots h(x_n)\phi(x_n), h(f^{(-n-1)}x_n) \cdots h(x_n)\phi(y_n)) \\ &\quad + \sum_{j=0}^{n-1} s_j \end{aligned}$$

where

$$\begin{aligned} s_j = \; & d(h(f^{-(n-1)}x_n) \cdots h(f^{-(j+1)}x_n) h(f^{-j}x_n) h(f^{-(j-1)}y_n) \cdots h(y_n)\phi(y_n), \\ & h(f^{-(n-1)}x_n) \cdots h(f^{-(j+1)}x_n) h(f^{-j}y_n) h(f^{-(j-1)}y_n) \cdots h(y_n)\phi(y_n)). \end{aligned}$$

The analysis of all terms except the first term

$$s_{*n} := d(h(f^{-(n-1)}x_n) \cdots h(x_n)\phi(x_n), h(f^{(n-1)}y_n) \cdots h(y_n)\phi(y_n))$$

proceeds exactly as before.

To deal with the first term (under the assumptions of Theorem 1.10) suppose that $m(H^\gamma_{C,d}) > 0$. Then by Poincaré recurrence $x_n, y_n \in H^\gamma_{C,d}$ for infinitely many times n, and at these times $|s_{*n}| \leq e^{n(-\lambda_- - \gamma\mu_s)}$. By assumption $-\lambda_- - \gamma\mu_s < 0$ and hence $\lim_{n\to\infty} e^{n(-\lambda_- - \gamma\mu_s)} = 0$.

Suppose that there exists a subset $M_\gamma \subset M$, $m(M_\gamma) > 0$, such that $\phi|_{M_\gamma}$ is Hölder of exponent γ where $-\lambda_- - \gamma\mu_s < 0$.

We will need the following lemma:

Lemma 3.1 *Suppose m is an ergodic invariant equilibrium measure for the Anosov diffeomorphism $f : M \to M$ and $m(H) > 0$. Then m a.e. y has the property that for m_s a.e. $x \in W^s_{\text{loc}}(y)$ (m_s is the conditional measure that m induces on $W^s_{\text{loc}}(y)$) there exists an infinite sequence of distinct times n such that $f^n(x), f^n(y) \in H$.*

Proof. By the absolute continuity of the stable foliation it follows that $H \cap W^s_{\text{loc}}(x)$ has positive $m_s(x)$-measure for m-a.e. $x \in \Lambda$. By [10, Cor. 10.50], $m_s(x)$-a.e. point of $W^s_{\text{loc}}(x)$ is a point of density for m_s. By the local product structure of m it follows that for m-a.e. $z \in H$,

$$\lim_{\epsilon \to 0} \frac{m_s(H \cap W^s_\epsilon(z))}{m_s(W^s_\epsilon(z))} = 1.$$

Let $\epsilon_1 < 0$. Let $\delta > 0$ be given and define

$$G_\delta := \{z \in W^s_{\epsilon_1}(y) : f^n(y), f^n(z) \in H \text{ infinitely often } \}$$

(that is for the same set of times n).

Given $\delta > 0$ there exists a subset $H_0 \subset H$, $m(H_0) > 0$, and $\epsilon_2 > 0$ such that

$$\frac{m_s(H \cap W^s_\epsilon(z))}{m_s(W^s_\epsilon(z))} > 1 - \delta$$

for all $z \in H_0$ and $\epsilon \leq \epsilon_2$. Since H_0 has positive measure, for m a.e. point y, $f^n(y) \in H_0$ for infinitely many n. Choose N so that $d(f^n(y), f^n(z)) \leq \epsilon_2$ for all $z \in W^s_\epsilon(y)$, for all $n > N$. Choose $N' > N$ such that $f^{N'}(y) \in H_0$. Since the foliation into local stable manifolds is invariant, $f^{N'} W^s_{\epsilon_1}(y) \subset W^s_{\epsilon_2}(f^{N'}(y))$. Thus letting $y' = f^{N'}(y)$

$$\frac{m_s(y')(H \cap f^{N'} W^s_{\epsilon_1}(y))}{m_s(y')(f^{N'} W^s_{\epsilon_1}(y))} > 1 - 2\delta$$

and hence $m_s(y)(G_\delta \cap W^s_{\epsilon_1}(y)) > (1 - 2\delta) m_s(y)(W^s_{\epsilon_1}(y))$. Since δ was arbitrary the lemma is proved. $\square$

Thus m a.e. y has the property that for a.e. $x \in W^s_{\text{loc}}(y)$ (with respect to the conditional measure m_s that m induces on $W^s_{\text{loc}}(y)$) there exists an infinite sequence of distinct times n_k such that $f^{n_k}(x), f^{n_k}(y) \in M_\gamma$. If $f^{n_k}(x), f^{n_k}(y) \in M_\gamma$ then $|s_{*n_k}| \leq e^{n_k(\lambda_- - \gamma\mu_s)}$. By the assumption that $\lambda_- - \gamma\mu_s < 0$, $\lim_{k \to \infty} |s_{*n_k}| = 0$.

Thus for m a.e. $y \in M$ and for a.e. $x \in W^s_{\text{loc}}(y)$ with respect to the conditional measure m_s that m induces on $W^s_{\text{loc}}(y)$,

$$d(\phi(x), \phi(y)) \leq C' d(\phi(y), y_0) d(x, y).$$

Similarly we have shown that m a.e. $y \in M$ and for a.e. $x \in W^u_{\text{loc}}(y)$ with respect to the conditional measure m_u that m induces on $W^u_{\text{loc}}(y)$,

$$d(\phi(x), \phi(y)) \leq Cd(\phi(y), y_0)d(x, y).$$

The local product structure ensures that a.e. point y has a neighbourhood $U(y)$ containing a ball of uniform diameter, and a constant $K(y)$ such that

$$d(\phi(x), \phi(x')) \leq K(y)d(x, x')$$

for m a.e. $x, x' \in U(y)$.

By compactness of M this implies that there is a Lipschitz function $\phi'(x)$ such that $\phi(x)' = \phi(x)$ m a.e.

Once we have established that ϕ is Lipschitz if property (i) of Theorem 1.10 does not hold, the proof that ϕ is C^{k-} in Theorem 1.10 proceeds exactly as in the case of uniform contraction. The proof that generically condition (i) does hold also proceeds exactly as in the case of the proof of Theorem 1.3.

The only difference in the proof of the second, stronger dichotomy (a) or (b) is that Lemma 3.1 must be used to ensure that $s_{*n_k} \to 0$ along a subsequence n_k. The proof that generically property (a) holds proceeds exactly as in the proof of Theorem 1.3.

Theorem 1.12 is proved by using the arguments of §3.2, with obvious modifications, to prove regularity of ϕ along local unstable manifolds. This is necessary since we do not have contraction on average which makes the argument establishing regularity along local unstable manifolds easier. In the case that we don't have contraction on average we require the extra condition that $\gamma > \lambda_+/\mu_u$. An obvious modification of Lemma 3.1 to conditional measures on local unstable manifolds is all that is needed to finish the proof.

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