Livšic regularity theorems for twisted cocycle equations over hyperbolic systems

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Abstract

Let ϕ be a hyperbolic map. We consider cocycle equations of the form $f = u\phi \cdot g \cdot \alpha u^{-1}$ with f, g, u taking values in a compact connected Lie group G, α an automorphism of G and f, g Hölder continuous. When the eigenvalues of the derivative of α have modulus one, we prove that any measurable solution u has a Hölder continuous version. When f, g are C^k then u may be taken to be $C^{k-1+\varepsilon}$ for any $\varepsilon \in (0, 1)$.

§1 Introduction

Let ϕ be a diffeomorphism of a compact Riemannian manifold M and let $\Lambda \subset M$ be a closed ϕ-invariant subset. Let G be a compact connected Lie group and let α be an automorphism of G. For $f : \Lambda \to G$ define a skew product $\phi_f : \Lambda \times G \to \Lambda \times G$ by $\phi_f(x, y) = (\phi(x), f(x)\alpha(y))$. We call f a *twisted cocycle* or *α-cocycle* .

We say that two twisted cocycles f, g are *α-cohomologous* if the map U : $(x, y) = (x, u(x)y)$, $u : \Lambda \to G$, conjugates ϕ_f and ϕ_g: $\phi_f U = U\phi_g$. This is equivalent to u satisfying the *twisted cocycle equation*

$$f(x) = u(\phi x) \cdot g(x) \cdot \alpha u(x)^{-1} \tag{1}$$

(we often use a dot '$\cdot$' to denote multiplication in G for clarity). It is usually important to specify the class of function to which u belongs. Indeed, the main aim of this note is to give conditions on ϕ and α for when the existence of a measurable solution to (1) (holding a.e. with respect to a suitable measure) implies the existence of a more regular solution. We call such a result a *regularity theorem*.

Throughout, we shall assume that ϕ is hyperbolic. By hyperbolic we essentially mean that ϕ can be represented as an Axiom A diffeomorphism restricted to its non-wandering set (we give a precise definition in §2); examples include (transitive) Anosov diffeomorphisms and shifts of finite type.

The classical case is when α is the identity (in which case we drop the prefix α-). In this case, the twisted cocycle equation (1) becomes the *cocycle equation*:

$$f = u\phi \cdot g \cdot u^{-1} \text{ a.e.} \tag{2}$$

Suppose f, g are Hölder continuous and we have a measurable solution u to (2). Is u necessarily equal a.e. to a continuous function? This was first considered by Livšic [Lv] where it is proved to be the case for abelian Lie groups, under the assumptions that ϕ is C^2 Anosov and the measure is smooth. For compact Lie group valued cocycles satisfying the cocycle equation (2), Parry and Pollicott proved the following:

Theorem 1.1 ([PP])
Let ϕ be a hyperbolic diffeomorphism and G a compact Lie group. Suppose $f, g : \Lambda \to G$ are Hölder continuous and satisfy $f = u\phi \cdot g \cdot u^{-1}$ a.e. (where the measure is the equilibrium state corresponding to a Hölder continuous function) for some measurable $u : \Lambda \to G$. Then $u = u'$ a.e. for a Hölder continuous u' and $f = u'\phi \cdot g \cdot u'^{-1}$ everywhere.

(Schmidt [Sc] has proved a similar theorem with the assumption that f, g satisfy a 'bounded distortion' condition.)

Now suppose that Λ is a manifold (i.e. ϕ is Anosov), f, g are C^r ($r = 1, 2, \dots, \infty, \omega$) and u is a continuous solution to the untwisted cocycle equation (2). Does u possess some degree of differentiability? This problem has a long history [Lv, GK, LMM, J1, HK, NT1, Ll1] culminating in the following:

Theorem 1.2 ([NT2])
Suppose G is a compact Lie group and $f, g : \Lambda \to G$ are C^r. Then a continuous solution u to $f = u\phi \cdot g \cdot u^{-1}$ is $C^{r-1+\varepsilon}$ for any $\varepsilon \in (0, 1)$.

(With some extra hypotheses [NT2] also considered some classes of non-compact Lie groups.)

Let us now consider twisted cocycles. Let G be a torus and let α be an ergodic automorphism of G. A theorem due to Lind [Li] then says that any two α-cocycles and measurably α-cohomologous. More precisely:

Theorem 1.3 ([Li])
Let ϕ be a hyperbolic diffeomorphism, G a torus and α an ergodic automorphism of G. Suppose $f, g : \Lambda \to G$ are Hölder continuous. Then there exists a measurable $u : \Lambda \to G$ such that $f = u\phi \cdot g \cdot \alpha u^{-1}$.

(In fact, Lind proved considerably more: ϕ need only be an (aperiodic) invertible measure preserving transformation (not even ergodic), G need only be compact and abelian and f, g need only be measurable.) An observation due to Lind shows that when α is a hyperbolic toral automorphism, the measurable solution guaranteed by Theorem 1.3 need not be continuous. This suggests the problem that we consider in this paper: what conditions on α ensure that the regularity theorem holds for the twisted cocycle equation (1).

We shall mostly be interested in the case when α is a unipotent automorphism of a compact connected Lie group. We say that α is *unipotent* if all the eigenvalues of the derivative $d\alpha$ on the Lie algebra of G have modulus 1. This is equivalent to the eigenvalues being roots of unity. (*Caution*: this is not the usual meaning of the term 'unipotent' which is usually only defined for abelian groups. However, it is a generalisation appropriate for Lie groups. See the remark preceding Theorem 3.1 in §3.)

Our main result is the following.

Theorem A
Let ϕ be a hyperbolic map. Let α be a unipotent automorphism of a compact connected Lie group G. Let f, g be Hölder continuous. Suppose there exists a measurable solution $u : \Lambda \to G$ to $f = u\phi \cdot g \cdot \alpha u^{-1}$ a.e. (where the measure is the equilibrium state corresponding to a Hölder continuous function). Then there exists a Hölder continuous u such that $u' = u$ a.e. and $f = u'\phi \cdot g \cdot \alpha u'^{-1}$ everywhere.

The condition on α is optimal in the following sense. Let α be a non-unipotent automorphism of G. Then, after replacing α and G by locally isomorphic copies, there exists a hyperbolic ϕ, Hölder f, g and a measurable u such that $f = u\phi \cdot g \cdot \alpha u^{-1}$ but u does not have a continuous version.

Now suppose ϕ is a transitive Anosov diffeomorphism and f, g are C^r. Suppose u is a continuous solution to $f = u\phi \cdot g \cdot \alpha u^{-1}$. Using techniques used by Niţică and Török in the untwisted case we prove:

Theorem B
Let ϕ be a C^k (transitive) Anosov diffeomorphism ($1 \leq k \leq \infty$ or $k = \omega$) on a compact manifold M. Let α be a unipotent automorphism of a compact connected Lie group G. Let $f, g : M \to G$ be C^r. Suppose $f = u\phi \cdot g \cdot \alpha u^{-1}$ a.e. (with respect to a Hölder equilibrium state) for a measurable u. Then there exists a continuous u' such that $u = u'$ a.e. and $f = u'\phi \cdot g \cdot \alpha u'^{-1}$ everywhere. Moreover u' is $C^{r-1+\varepsilon}$ for any $\varepsilon \in (0,1)$. (When $r = 1, \infty, \omega$ we define $r - 1 + \varepsilon = 1, \infty, \omega$ respectively, for any $\varepsilon \in (0,1)$.)

This paper is organised as follows. In §2, we briefly review some definitions. In §3, we consider the case of a torus and prove the regularity theorem for a unipotent toral automorphism. In §4, we consider automorphisms of compact semi-simple Lie groups. In §5, we 'glue' the results of §§3,4 together to obtain Theorem A. We prove Theorem B in §6. In §7 we show that Theorem 1.1 can fail for a class of non-compact Lie groups.

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§2 Preliminaries

Notation. The following notation will be useful in the sequel. For $f : \Lambda \to G$ define

$$\begin{aligned} f^n(x) &= f(\phi^{n-1}x) \cdot \alpha f(\phi^{n-2}x) \cdots \alpha^{n-1} f(x) \\ f^{-n}(x) &= \alpha^{-1} f(\phi^{-n}x)^{-1} \cdot \alpha^{-2} f(\phi^{-n+1}x)^{-1} \cdots \alpha^{-n} f(\phi^{-1})^{-1} \end{aligned}$$

for $n > 0$ and $f(x, 0) = e$, the group identity, so that $\phi_f^n(x, y) = (\phi^n x, f^n(x)\alpha^n(y))$. Clearly

$$f^{n+m}(x) = f^n(\phi^m x) \cdot \alpha^n f^m(x). \tag{3}$$

Hyperbolic dynamics. Let M be a smooth compact Riemannian manifold and let ϕ be a C^1 diffeomorphism of M. We call a compact ϕ-invariant subset $\Lambda \subset M$ a *basic set* and ϕ restricted to Λ a *hyperbolic diffeomorphism* if

1. the tangent bundle restricted to Λ can be written as a continuous Whitney sum $E^s \oplus E^u$ of $d\phi$-invariant sub-bundles for which there exist constants $C > 0, \lambda \in (0,1)$ such that

$$\|d\phi^n|_{E^s}\| \leq C\lambda^n,\ n \geq 0,$$
$$\|d\phi^{-n}|_{E^u}\| \leq C\lambda^n,\ n \geq 0;$$

2. the periodic orbits in Λ are dense in Λ and Λ is not a single orbit;

3. there is a dense orbit in Λ;

4. there is an open neighbourhood $U \supset \Lambda$ such that $\Lambda = \bigcap_{n=-\infty}^{\infty} \phi^n U$.

Thus in defining a basic set, we are merely abstracting the properties of the basic sets appearing in Smale's spectral decomposition of the non-wandering set of an Axiom A diffeomorphism. In particular (see [KH] for example) by replacing ϕ by ϕ^n, for some $n > 0$, there is no loss in assuming that $\phi|_\Lambda$ is topologically mixing.

Give Λ the metric d induced from the Riemannian metric on M. For a metric space (Y, ρ) and $\theta \in (0,1)$, let $C^\theta(Y)$ denote the space of Hölder continuous functions $\Lambda \to Y$ of exponent θ; i.e. those $f : \Lambda \to Y$ that satisfy $\rho(f(x), f(y)) \leq Cd(x,y)^\theta$. We denote the least possible such C by $|f|_\theta$. It is easy to see that if $\theta_1 \leq \theta_2$ then $C^{\theta_1} \supset C^{\theta_2}$.

Let $f \in C^\theta(\mathbb{R})$. As μ ranges over all ϕ-invariant probability measures, the quantity $h_\mu(\phi) + \int f\, d\mu$ (h_μ denotes entropy) is maximised by a unique ϕ-invariant measure. We call this the equilibrium state of f and denote it by m_f or m. Equilibrium states corresponding to Hölder continuous functions enjoy various nice properties. For example, they assign positive measure to non-empty open sets and have no atoms.

A standard, and very useful, result allows us to view a hyperbolic diffeomorphism as a quotient of a shift of finite type in a particularly nice way. Before describing this, let us first discuss shifts of finite type.

Recall that a shift of finite type is defined as follows. Let A be an aperiodic $k \times k$ matrix with entries from $\{0, 1\}$. Let

$$\Sigma = \{(x_j)_{j=-\infty}^{\infty} \in \{1, \ldots, k\}^{\mathbb{Z}} \mid A_{x_j x_{j+1}} = 1\}.$$

Then Σ is a totally disconnected zero-dimensional compact metrisable space. Define the shift $\sigma : \Sigma \to \Sigma$ by $(\sigma x)_j = x_{j+1}$ and note that $\sigma(\Sigma) = \Sigma$. We call σ a *shift of finite type*.

For each $\theta \in (0,1)$ we obtain a metric on Σ by defining

$$d^\theta((x_j), (y_j)) = \theta^{n(x,y)},$$

where $n(x,y) = \sup\{|n| \mid x_j = y_j, |j| \leq n\}$. For a metric space (Y, ρ) we define $F_\theta(Y)$ to be the space of all Lipschitz continuous functions with respect to ρ and d^θ:

$$F_\theta(Y) = \{f : \Sigma \to Y \mid \rho(f(x), f(y)) \leq Cd^\theta(x,y) \text{ for some } C > 0\}.$$

Write $|f|_\theta$ for the least possible C appearing above. Often, we shall abuse this definition and speak of functions in $F_\theta(Y)$ as being Hölder continuous of exponent θ. Observe, however, that in this case, if $\theta_1 \leq \theta_2$ then $F_{\theta_1} \subset F_{\theta_2}$.

Again, if $f \in F_\theta(\mathbb{R})$ then we can define the equilibrium state m_f of f.

We first prove Theorem A in the case of a shift of finite type. To deduce Theorem A for a hyperbolic diffeomorphism we use the fact that a hyperbolic diffeomorphism is a quotient of a shift of finite type in a nice way via Markov partitions. We give a quick review of the parts of this construction that will be needed in §5.

For $x \in \Lambda$ and small $\varepsilon > 0$, define

$$W^s_\varepsilon(x) = \{y \in M \mid d(\phi^n x, \phi^n y) \leq \varepsilon,\ n \geq 0\},$$
$$W^u_\varepsilon(x) = \{y \in M \mid d(\phi^{-n} x, \phi^{-n} y) \leq \varepsilon,\ n \geq 0\}.$$

We call these the local stable and unstable manifolds through x, respectively. They are tangent to E^s and E^u, respectively. If $z \in W^s_\varepsilon(x)$ then we have the estimate $d(\phi^n x, \phi^n z) \leq C\lambda^n d(x,z)$ for $n \geq 0$; if $z \in W^u_\varepsilon(y)$ then $d(\phi^{-n} y, \phi^{-n} z) \leq C\lambda^n d(y,z)$ for $n \geq 0$. If $x, y \in \Lambda$ are sufficiently close then $\langle x, y\rangle = W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ is a single point and this point lies in Λ.

The following describes the properties of Markov partitions that we shall need in the sequel.

Proposition 2.1
Let ϕ be a hyperbolic diffeomorphism. Then there exists a shift of finite type σ and a semi-conjugacy $\pi : \Sigma \to \Lambda$ (so that $\pi\sigma = \phi\pi$) such that the following hold.

1. *π is one-to-one on a residual subset of Λ and on a set of full measure for any equilibrium state. (Call a point $x \in \Lambda$ 'good' if it has a unique pre-image under π.)*
2. *There is a dense set of good periodic points for ϕ.*
3. *If $x, y \in \Lambda$ are good periodic points such that $\langle x, y\rangle$ is defined, then $\langle x, y\rangle$ is good.*
4. *π is Lipschitz continuous with respect to d^λ. In particular, if $f : \Lambda \to G \in C^\theta(G)$ then $f\pi : \Sigma \to G \in F_{\lambda^\theta}(G)$.*

Statements 1 and 4 are well-known ([B], for example). Statements 2 and 3 are easy consequences of well-known properties of Markov partitions, see [PP].

§3 Toral automorphisms

Let $G = \mathbb{R}^k/\mathbb{Z}^k$ be the k-torus for which we use additive notation. Let $\alpha \in \mathrm{Aut}(G)$ be an automorphism.

We first consider the case when α is unipotent. In our terminology, this means that all the eigenvalues of α have modulus 1. It is well-known that this forces all the eigenvalues to be roots of unity. Hence, for some $p \geq 1$, α^p has 1 as its only eigenvalue. For definiteness, we fix p to be the least such $p \geq 1$ with this property. It follows that for some $n \geq 1$,

$$(\alpha^p - \mathrm{id})^n G = \{e\}; \tag{4}$$

here, id is the identity map on G and e is the group identity. We call the least positive integer n for which (4) holds the *unipotency order* of α^p.

Remark. Observe that (4) is equivalent to $\bigcap_{n=0}^{\infty}(\alpha^p - \mathrm{id})^n G = \{e\}$. This is the usual definition of what it means for α^p to be a unipotent automorphism of the abelian group G. Thus our definition of unipotent can be viewed as a generalisation of the usual definition to the case of a non-abelian Lie group. Note also that we say α is unipotent whereas one would usually say that some power of α is unipotent.

We prove the following.

Theorem 3.1
Let σ be a shift of finite type equipped with a Hölder equilibrium state. Let α be a unipotent toral automorphism. If $f \in F_\theta(G)$ and satisfies $f = u\sigma - \alpha u$ a.e. for some measurable $u : \Lambda \to G$ then there exists $u' \in F_\theta(G)$ such that $u' = u$ a.e. and $f = u'\sigma - \alpha u'$ everywhere.

Proof. Define $f^p = \sum_{j=0}^{p-1} \alpha^j f \sigma^{p-1-j}$ so that $f^p = u\phi^p - \alpha^p u$ a.e. Note that σ^p is a shift of finite type. Hence there is no loss in assuming that all the eigenvalues of α are equal to 1.

We induct on the unipotency order n of α. When $n = 1$, α is the identity and the result follows from Theorem 1.1.

Let α have unipotency order $n > 1$. Let $H = (\alpha - \mathrm{id})G$. Then H is a closed sub-torus of G (in particular, H cannot be dense in G as $\alpha - \mathrm{id}$ has zero as its only eigenvalue, hence all the eigenvectors can be taken to be rational). Clearly H is α-invariant. Moreover, the induced automorphism α_H on G/H is the identity. (To see this, observe that $\alpha g + H = g + \alpha g - g + H = g + H$.) It is also clear the H has unipotency order $n - 1$.

Let π_H denote the quotient map $G \to G/H$. Then, as $\alpha_H = \mathrm{id}$, we have $\pi_H f = (\pi_H u)\sigma - \pi_H u$ a.e. By Theorem 1.1, there exists a Hölder $v'' : \Sigma \to G/H$ such that $\pi_H u = v''$ a.e. and $\pi_H f = v''\sigma - v''$ everywhere.

As Σ is totally disconnected, choose a Hölder lift of v''; i.e. a function $v' : \Sigma \to G$ such that $\pi v' = v''$. Then $u - v' \in H$ a.e. Moreover,

$$f - v'\sigma + \alpha v' = (u - v')\sigma - \alpha(u - v') \text{ a.e.}$$

As $\pi_H(f - v'\sigma + \alpha v') = 0$, this is an H-valued twisted cocycle equation. As α restricted to H has unipotency order $n-1$, the inductive hypothesis ensures we can write $u - v' = v$ a.e. for some Hölder v. Then $u' = v + v'$ is the required Hölder solution to $f = u'\sigma - \alpha u'$ and $u = u'$ a.e. ❑

§4 Semi-simple Lie groups

We first prove the regularity theorem for an inner automorphism of a compact Lie group in the context of a shift of finite type. This follows from a simple modification to the proof of Theorem 1.1 given in [PP]. Throughout, we use the notation $U(Y)$ to denote the group of unitary operators on a Euclidean space Y. The key fact is the following:

Proposition 4.1
Let $M(d)$ denote the space of $d \times d$ matrices over $\mathbb{C}$ with the Euclidean structure induced from $\mathbb{C}^{d^2}$. Let $\Psi : \Lambda \to U(M(d)) \in F_\theta(U(M(d)))$ and let $u : \Lambda \to M(d)$ be measurable. Let σ be a shift of finite type equipped with a Hölder equilibrium

state. Suppose $u(\sigma x) = \Psi(x)u(x)$ a.e. Then there exists $u' \in F_\theta(M(d))$ such that $u' = u$ a.e. and $u'(\sigma x) = \Psi(x)u'(x)$ everywhere.

Remark. Parry and Pollicott only concluded that $u' \in F_{\theta^{\frac{1}{2}}}(M(d))$. To see that $u' \in F_\theta(M(d))$, it is sufficient to show that $u' \in F_\theta(M(d))$ along local stable and unstable 'manifolds' [KH, Prop. 19.1.1] (the local stable 'manifold' through $x \in \Sigma$ is $\{y \in \Sigma \mid x_j = y_j \text{ for } j \geq 0\}$). Let $\Psi^n(x) = \Psi(\sigma^{n-1}x)\cdots\Psi(x) \in U(M(d))$. Then

$$\Psi^n(y)(u'(x) - u'(y)) = u'(\sigma^n x) - u'(\sigma^n y) - (\Psi^n(x) - \Psi^n(y))u'(x).$$

By ergodicity and the fact that Ψ is unitary, we may assume that $\|u'\| = 1$ so that

$$\|u'(x) - u'(y)\| \leq \|u'(\sigma^n x) - u'(\sigma^n y)\| + \|\Psi^n(x) - \Psi^n(y)\|. \tag{5}$$

It is easy to see that when y belongs to the same stable 'manifold' as x, we have $\|\Psi^n(x) - \Psi^n(y)\| \leq \sum_{j=0}^{n-1} \|\Psi\|_\theta \theta^n d^\theta(x,y) \leq \text{Const.}\, d^\theta(x,y)$ for some constant independent of n. Letting $n \to \infty$ in (5) we see that $\|u'(x) - u'(y)\| \leq \text{Const.}\, d^\theta(x,y)$.

Let G be a compact Lie group. Then for some d, G is a closed subgroup of $U(d)$, the $d \times d$ unitary matrices. Let $N(G)$ denote the normaliser of G in $Gl(d)$, the general linear group. Then any $H \in N(G)$ determines an automorphism α of G by conjugation: $\alpha(A) = HAH^{-1}$.

Theorem 4.2
Let ϕ be a shift of finite type equipped with a Hölder equilibrium state. Let G be a compact Lie group and α an automorphism of G arising from an element of the normaliser. Let $f, g \in F_\theta(G)$ and suppose $f = u\sigma \cdot g \cdot \alpha u^{-1}$ a.e. for some measurable u. Then there exists $u' \in F_\theta(G)$ such that $u' = u$ a.e. and $f = u'\sigma \cdot g \cdot \alpha u'^{-1}$ everywhere.

Proof. We modify the corresponding argument given in [PP] for the case when α is the identity.

Equip $M(d)$ with the inner product $\langle A, A' \rangle = \text{trace}\, AA'^*$ ($*$ denotes the Hermitian conjugate). Define $\Psi : \Lambda \to U(M(d))$ by $\Psi(x)A = f(x)HAH^{-1}g(x)^{-1}$. Clearly Ψ is linear. To see that Ψ is unitary, observe that

$$\begin{aligned} \langle \Psi(x)A, \Psi(x)A' \rangle &= \text{trace}\, f(x)HAH^{-1}g(x)^{-1}g(x)(H^{-1})^* A'^* H^* f(x)^{-1} \\ &= \text{trace}\, AA'^* \\ &= \langle A, A' \rangle. \end{aligned}$$

Finally, if $f = u\sigma \cdot g \cdot \alpha u^{-1}$ a.e. then $u(\sigma x) = f(x)Hu(x)H^{-1}g(x)^{-1}$ a.e. $= \Psi(x)u(x)$ a.e. and Proposition 4.1 applies. ❏

Corollary 4.3
The regularity theorem holds for an inner automorphism of a compact Lie group.

We now consider automorphisms of compact connected semi-simple Lie groups. Our first observation is that *all* automorphisms of such groups are unipotent (in the sense defined above). This fact is surely known, but we have been unable to locate a reference.

Proposition 4.4
Let α be an automorphism of a compact connected semi-simple Lie group G. Then the eigenvalues of $d\alpha$ are roots of unity.

Proof. It is is well-known ([Ti], for example) that the inner automorphisms have finite index in the automorphism group of a compact semi-simple Lie group. Choose n such that α^n is inner. It is clear that the eigenvalues of $d\alpha^n$ have modulus 1; this follows from the existence of a bi-invariant metric. We prove that the eigenvalues are roots of unity. A standard result allows us to assume that, for some d, G is a closed subgroup of $O(d)$, the $d\times d$ orthogonal matrices (previously we had embedded G in a unitary group $U(d)$). Let $\mathfrak{g}$ be the Lie algebra of G and note that it is a sub-algebra of $\mathfrak{o}(d) = \{X \in M(d) \mid X^T + X = 0\}$.

Suppose $\alpha(A) = HAH^{-1}$ for some $H \in G$. Then for $X \in \mathfrak{g}$ we have $d\alpha(X) = HXH^{-1}$. Suppose μ is an eigenvalue of $d\alpha$ with eigenvector $X \neq 0$. Then $\mu X = d\alpha(X) = HXH^{-1}$. Thus, when viewed as matrices, μX and X are similar. Therefore they have the same eigenvalues. We need the following easily-proved:

Sub-lemma. *If $X \in \mathfrak{o}(d)$ has zero as its only eigenvalue then $X = 0$.*

Hence multiplying the spectrum of X by μ merely permutes the eigenvalues of X. As this is a finite set containing a non-zero entry, some power of μ is equal to 1. Hence μ is a root of unity. ❑

We prove the regularity theorem for automorphisms of compact connected semi-simple Lie groups.

Theorem 4.5
Let σ be a shift of finite type equipped with a Hölder equilibrium state. Let G be a compact connected semi-simple Lie group and α an automorphism of G. Let $f, g \in F_\theta(G)$ and suppose $f = u\phi \cdot g \cdot \alpha u^{-1}$ a.e. for some measurable $u : \Lambda \to G$. Then there exists $u' \in F_\theta(G)$ such that $u' = u$ a.e. and $f = u'\phi \cdot g \cdot \alpha u'^{-1}$ everywhere.

Proof. Choose $n > 0$ such that α^n is inner. Then $f^n = u\phi^n \cdot g^n \cdot \alpha^n u^{-1}$ a.e. As σ^n is a shift of finite type, Corollary 4.3 applies. ❑

§5 Unipotent automorphisms of compact connected Lie groups

We now prove Theorem A. Our strategy is to first prove it for a shift of finite type, using the structure theory for compact connected Lie groups. For a hyperbolic diffeomorphism, we use Markov partitions to push down solutions from a symbolic cover.

Let G be a compact connected Lie group. Let $\hat{A}$ denote the identity component of the centre of G; hence $\hat{A}$ is a torus. Let $\hat{S}$ denote the product of the simple normal subgroups of G (of which there are only finitely many); hence $\hat{S}$ is semi-simple. Then a standard result allows us to write

$$G = \frac{\hat{A} \times \hat{S}}{D}$$

where D is a discrete normal subgroup contained in the centre of $\hat{A} \times \hat{S}$. Let $\pi_D : \hat{A} \times \hat{S} \to G$ denote the natural projection. Let α be an automorphism of G.

Then α lifts to an automorphism $\hat{\alpha}$ of $\hat{A} \times \hat{S}$ such that $\pi_D \hat{\alpha} = \alpha \pi_D$. Moreover, $\hat{A}$ and $\hat{S}$ are $\hat{\alpha}$-invariant.

Let $A = \pi_D \hat{A}$, $S = \pi_D \hat{S}$. Then A is a torus and S is semi-simple. The properties of A and S that are important for our work are listed below.

Lemma 5.1

1. *A and S are α-invariant normal subgroups of G.*
2. *$A \cap S$ is a discrete α-invariant normal subgroup of G.*
3. *$G/A \cap S \cong A/A \cap S \times S/A \cap S$.*
4. *$A/A \cap S$ is a torus and $S/A \cap S$ is semi-simple.*

By Lemma 5.1.2, there is a well-defined quotient automorphism on $G/A \cap S$. We continue to denote this automorphism by α. By Lemma 5.1.1, both $A/A \cap S$ and $S/A \cap S$ are α-invariant.

Remark. As both D and $A \cap S$ are discrete, the groups $\hat{A} \times \hat{S}$, G and $G/A \cap S$ are all locally isomorphic. Hence, the eigenvalues of $d\alpha$ are the same, irrespective of whether α is viewed as an automorphism of $\hat{A} \times \hat{S}$, G or $G/A \cap S$.

Proof of Theorem A (for a shift of finite type). Let $\sigma : \Sigma \to \Sigma$ be a shift of finite type. We assume

$$f = u\sigma \cdot g \cdot \alpha u^{-1} \text{ a.e.} \tag{6}$$

where $f, g \in F_\theta(G)$ and $u : \Sigma \to G$ is measurable. Apply the quotient map $\pi_{A\cap S} : G \to G/A \cap S$ to (6) to obtain

$$\pi_{A\cap S} f = (\pi_{A\cap S} u)\sigma \cdot \pi_{A\cap S} g \cdot \alpha(\pi_{A\cap S} u)^{-1} \text{ a.e.}$$

By Lemma 5.1.3, this is an equation holding in the direct product of a torus and a semi-simple Lie group. Observe that α is the direct product of a unipotent toral automorphism and an automorphism of a semi-simple Lie group. By Theorems 3.1 and 4.5, there exists $v \in F_\theta(G/A \cap S)$ such that

$$\begin{aligned} \pi_{A\cap S} f &= v\sigma \cdot \pi_{A\cap S} g \cdot \alpha v^{-1} \text{ everywhere} \\ v &= \pi_{A\cap S} u \text{ a.e.} \end{aligned}$$

As Σ is totally disconnected, we can find $w \in F_\theta(G)$ such that $\pi_{A\cap S} w = v$. By construction, $\pi_{A\cap S} w = \pi_{A\cap S} u$ a.e. Hence, there exists a measurable $p : \Sigma \to A \cap S$ such that $p \cdot w = u$ a.e.

We claim that p has a continuous version. Observe that, as $A \cap S$ is a discrete (hence finite) α-invariant subgroup of G, some power of α restricted to $A \cap S$ is the identity. Suppose $\alpha^n|_{A\cap S} = \text{id}$. As $f^n = u\sigma^n \cdot g^n \cdot \alpha^n u^{-1}$ a.e. we have

$$\begin{aligned} f^n &= p\sigma^n \cdot w\sigma^n \cdot g^n \cdot \alpha^n w^{-1} \cdot \alpha^n p^{-1} \text{ a.e.} \\ &= p\sigma^n \cdot w\sigma^n \cdot g^n \cdot \alpha^n w^{-1} \cdot p^{-1} \text{ a.e.} \end{aligned}$$

As f^n, $w\sigma^n \cdot g^n \cdot \alpha^n w^{-1} \in F_\theta(G)$ and σ^n is a shift of finite type, we can use Theorem 1.1 to find $p' \in F_\theta(A \cap S)$ such that $p' = p$ a.e. and the above equation holds everywhere for p'.

Define $u = p' \cdot w$. Then $u' \in F_\theta(G)$, $u' = u$ a.e. and $f = u'\sigma \cdot g \cdot \alpha u'^{-1}$ everywhere. ❑

We now want to prove Theorem A for hyperbolic diffeomorphisms. We will need the fact that α^n expands distances at an arbitrarily small exponential rate. The following is well-known:

Proposition 5.2
Let ρ be a Riemannian metric on G. Let $\delta > 0$. Then there exists $C_\delta > 0$ such that for all $n \in \mathbb{Z}$ and for all $g, h \in G$, $\rho(\alpha^n g, \alpha^n h) \leq C_\delta (1+\delta)^{|n|} \rho(g,h)$.

Proof (sketch). Let $\|\cdot\|_e$ denote the norm on the tangent space at $e \in G$, identified with the Lie algebra of G. It follows from the spectral radius formula that $\|d\alpha^n X\|_e \leq C_\delta^+ (1+\delta)^n \|X\|_e$ for $n \geq 0$, for some constant $C_\delta^+ > 0$. From the definition of the Riemannian metric, it is easy to check that $\rho(\alpha^n g, \alpha^n h) \leq C_\delta^+ (1+\delta)^n \rho(g,h)$ for $n > 0$ and a similar construction yields the same for $n < 0$. ❑

Before proving Theorem A we need some preliminary estimates.

We are nearly in a position to prove Theorem A for hyperbolic diffeomorphisms. We first prove some preliminary estimates.

Let $f, g \in C^\theta(G)$. Recall that λ is the exponential rate of contraction and expansion of ϕ on stable and unstable manifolds. Choose $\delta > 0$ such that

$$(1+\delta)\lambda^\theta < 1; \tag{7}$$

this is possible as both $\lambda, \theta \in (0,1)$.

For $z \in W_\varepsilon^s(x)$ and $n \geq 0$, define $F^n(z,x) = \alpha^{-n}(f^n(z)^{-1} f^n(x))$. Similarly, define g^n and G^n.

Let ρ denote a bi-invariant metric on G.

Lemma 5.3
Let $z \in W_\varepsilon^s(x)$. Then

1. *$F(z,x) = \lim_{n\to\infty} F^n(z,x)$ exists;*
2. *$\rho(F(z,x), e) \leq \mathrm{Const.}\, d(x,z)^\theta$, where the constant is independent of x, z.*

Proof. We prove 1. Using the identity $f^{n+1}(x) = f(\phi^n x) \cdot \alpha f^n(x)$, Proposition 5.2 and the bi-invariance of the metric ρ on G, it is a simple matter to estimate

$$\rho(F^{n+1}(z,x), F^n(z,x)) \leq \mathrm{Const.}\, ((1+\delta)\lambda^\theta)^n d(x,z)^\theta$$

where the constant is independent of x, z and n. As $(1+\delta)\lambda^\theta < 1$, this estimate tends to zero exponentially fast as $n \to \infty$, and the result follows.

We prove 2. Note that

$$\rho(F^n(z,x), e) \leq \sum_{j=0}^{n-1} \rho(F^{j+1}(z,x), F^j(z,x)) \leq \mathrm{Const.}\, ((1+\delta)\lambda^\theta)^n d(x,z)^\theta$$

by the above. Letting $n \to \infty$ and observing that by (7) the sum is finite, the result follows. ❑

Proof of Theorem A (for a hyperbolic diffeomorphism). Suppose $f, g \in C^\theta(G)$ are as above and u is a measurable function such that $f = u\phi \cdot g \cdot \alpha u^{-1}$ a.e. Let $\pi : \Sigma \to \Lambda$ be a semi-conjugacy as described in Proposition 2.1. Let $\hat{f} = f\pi$, $\hat{g} = g\pi$ and $\hat{u} = u\pi$. Then $\hat{f}, \hat{g} \in F_{\lambda^\theta}(G)$ and $\hat{u}$ is measurable. Moreover, $\hat{f} = \hat{u}\sigma \cdot \hat{g} \cdot \alpha\hat{u}^{-1}$ a.e. (with respect to a Hölder equilibrium state for σ). By Theorem A in the case of a shift of finite type, we can find a Hölder continuous $\hat{u}' \in F_{\lambda^\theta}(G)$ such that $\hat{u}' = \hat{u}$ a.e. and $\hat{f} = \hat{u}'\sigma \cdot \hat{g} \cdot \alpha\hat{u}'^{-1}$ everywhere. We show that $\hat{u}' = u'\pi$ for some $u' : \Lambda \to G \in C^\theta(G)$.

Recall from §2 that a point $x \in \Lambda$ is called *good* if it has a unique pre-image under π. Let $x, y \in \Lambda$ be good periodic points. Suppose $\pi\hat{x} = x$, $\pi\hat{y} = y$. Define u' on the orbit of x by $u'(\phi^n x) = \hat{u}'(\sigma^n x)$, and similarly define u' on the orbit of y. Thus we have defined u' on a dense subset of Λ. Suppose x and y are sufficiently close so that $z = \langle x, y\rangle$ is defined. Then z is good and we may define u' on the orbit of z as above. When u' is defined, clearly we have $f = u'\phi \cdot g \cdot \alpha u'^{-1}$. We claim that u' extends to a Hölder continuous function defined on Λ. It then follows that the cocycle equation holds everywhere for u'. Moreover, $u' = u$ a.e. as π is one-to-one on a set of full measure. Thus Theorem A is proved.

It remains to check that u' does indeed extend to Λ. We claim the following holds:

$$F(z, x) = u'(z)G(z, x)u'(x)^{-1}. \tag{8}$$

To see this, note that

$$\begin{aligned} F^n(z, x) &= \alpha^{-n}(f^n(z)^{-1}f^n(x)) \\ &= u'(z)\alpha^{-n}(g^n(z)^{-1}u'(\phi^n z)^{-1}u'(\phi^n x)g^n(x))u'(x)^{-1}. \end{aligned} \tag{9}$$

Moreover,

$$\begin{aligned} &\rho(G^n(z, x), \alpha^{-n}(g^n(z)^{-1}u'(\phi^n z)^{-1}u'(\phi^n x)g^n(x))) \\ &= \rho(\alpha^{-n}(g^n(z)^{-1}g^n(x)), \alpha^{-n}(g^n(z)^{-1}u'(\phi^n z)^{-1}u'(\phi^n x)g^n(x))) \\ &\leq C_\delta(1+\delta)^n\rho(u'(\phi^n z), u'(\phi^n x)) \\ &= C_\delta(1+\delta)^n\rho(\hat{u}'(\sigma^n\tilde{z}), \hat{u}'(\sigma^n\tilde{x})) \\ &\leq C_\delta(1+\delta)^n|\hat{u}'|_{\lambda^\theta}\lambda^{\theta n} \end{aligned}$$

and this tends to zero as $n \to \infty$ by the assumption on δ. Letting $n \to \infty$ in (9) yields (8).

Define

$$F'(z, y) = \lim_{n\to\infty} \alpha^n(f^{-n}(z)^{-1}f^{-n}(y));$$

this makes sense as $z \in W^u_\varepsilon(y)$. Then all of the above estimates on F and G continue to hold for F' and G'. In particular,

$$F'(z, y) = u'(z)G'(z, y)u'(y)^{-1} \tag{10}$$

Combining (10) with (8) gives

$$u'(y) = F'(z, y)^{-1}F(z, x)u'(x)G(z, x)^{-1}G'(z, y).$$

By Lemma 5.3.2 (and its analogues for F' and G') and the bi-invariance of ρ, we have

$$\rho(u'(x), u'(y)) \quad \leq \quad \rho(F(z, x), e) + \rho(G(z, x), e)$$

$$
\begin{aligned}
& + \rho(F'(y,z),e) + \rho(G'(y,z),e) \\
\leq \;& \text{Const.}\,(d(x,z)^\theta + d(y,z)^\theta)
\end{aligned}
$$

for some constant independent of x, y, z. By Proposition 19.1.1 in [KH], u' is Hölder continuous of exponent θ on the dense set of 'good' periodic orbits. Hence u' extends (uniquely) to a function $u' \in C^\theta(G)$. This completes the proof of Theorem A. ❑

We give examples to show that the unipotency hypothesis is optimal in the following sense: Let α be a non-unipotent automorphism of G. Then (upto a local isomorphism in G) there exists a hyperbolic ϕ, Hölder f, g and a measurable u such that $f = u\phi \cdot g \cdot \alpha u^{-1}$ but u does not have a continuous version.

By Proposition 4.4, we may assume G is a torus. Moreover, by taking a subtorus, we may assume that α is either hyperbolic or quasi-hyperbolic.

In the hyperbolic case, we use the following example due to Lind [Li]. Take $\phi = \alpha$, so that ϕ is an Anosov toral automorphism. Take $f(x) = x$. By Theorem 1.3, $f = u\phi \cdot \alpha u^{-1}$ a.e. for some measurable u. Hence the maps ϕ_f and $\phi \times \alpha$ are metrically isomorphic. Moreover, ϕ_f and $\phi \times \alpha$ can be represented by matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ \mathrm{id} & \alpha \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},$$

respectively (here, id denotes the $k \times k$ identity matrix). If u had a continuous version then ϕ_f and $\phi \times \alpha$ would be topologically conjugate, hence, by a result of Adler and Palais [AP], algebraically conjugate. However, this is not the case as ϕ_f and $\phi \times \alpha$ have (generalised) eigenspaces of different dimensions.

We now consider the quasi-hyperbolic case, when α has an eigenvalue of modulus 1. Such an α is still ergodic, so Theorem 1.3 still applies. We give an example of an f that cannot be written $f = u\phi \cdot \alpha u^{-1}$ for a continuous u.

We use multiplicative notation. Let K denote the unit circle in $\mathbb{C}$. Let ϕ be a hyperbolic automorphism of the m-torus $\Lambda = K^m$. Let α be a quasi-hyperbolic automorphism of K^n. Recall that a K^n-valued function is null-homotopic if it lifts to $\mathbb{R}^n$. The key observation is the following:

Sub-lemma. *Suppose $f = u\phi \cdot \alpha u^{-1}$ where $f, u : K^m \to K^n$ are continuous and ϕ, α have disjoint spectra. If f is null-homotopic then u is null-homotopic.*

Suppose f is null-homotopic and $f = u\phi \cdot \alpha u^{-1}$ for some continuous u. As u is null-homotopic, we can lift this equation to $\mathbb{R}^n$ to obtain (after possibly subtracting an integer)

$$F = U\phi - \alpha U \tag{11}$$

where F is a lift of f, U is a lift of u and α is viewed as a linear map on $\mathbb{R}^n$. Conversely, if $F, U : \Lambda \to \mathbb{R}^n$ are continuous and satisfy (11) then we can obtain a continuous solution $u = e^{2\pi i U}$ to $\exp(2\pi i F) = u\phi \cdot \alpha u^{-1}$. Therefore, it is sufficient to choose a Hölder $F : \Lambda \to \mathbb{R}^n$ such that (11) does not have a continuous solution U.

Let $\mu = e^{2\pi i\theta}$ be an eigenvalue of α of unit modulus. By diagonalising (11) and considering the eigenspace of μ, we obtain a solution to

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} U_1\phi \\ U_2\phi \end{pmatrix} - R(\theta) \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \tag{12}$$

where $F_i, U_i : \Lambda \to \mathbb{R}$ and $R(\theta)$ denotes the rotation of $\mathbb{R}^2$ by θ. Alternatively, (12) can be viewed as an equation in $\mathbb{C}$:

$$\hat{F} = \hat{U}\phi - \mu\hat{U}. \tag{13}$$

Let $\eta \in \hat{\Lambda}$ be an arbitrary non-zero character of Λ. Define $\hat{F} = \eta : \Lambda \to \mathbb{C}$ and observe that we can recover a Hölder continuous (indeed analytic) function $F : \Lambda \to \mathbb{R}^2/\mathbb{Z}^2$ by taking real and imaginary parts. Write $\hat{U} = \sum_{\gamma\in\hat{\Lambda}} u_\gamma \gamma$ where $u_\gamma \in \mathbb{C}$ and $\sum_{\gamma\in\hat{\Lambda}} |u_\gamma|^2 < \infty$. Then (13) becomes

$$\eta = \sum_{\gamma\in\hat{\Lambda}} u_{\gamma\phi^{-1}}\gamma - \mu \sum_{\gamma\in\hat{\Lambda}} \gamma$$

so that

$$u_{\gamma\phi^{-1}} = \begin{cases} \mu u_\gamma & \gamma \neq \eta, \\ 1 + \mu u_\gamma & \gamma = \eta \end{cases} \tag{14}$$

Hence $u_{\eta\phi^{-p}} = \mu^{p-1}(1 + \mu u_\eta)$ for $p > 0$ so that $|u_{\eta\phi^{-p}}| = |1 + \mu u_\eta|$. As ϕ is hyperbolic, and in particular ergodic, all the $\eta\phi^{-p}$ are distinct. Hence, $u_\eta = -\mu^{-1}$. However, it also follows from (14) that for $p \geq 0$, $u_\eta = \mu^p u_{\eta\phi^p}$, whence $|u_{\eta\phi^p}| = |u_\eta|$ for $p \geq 0$. Hence $u_\eta = 0$, a contradiction.

§6 Higher regularity results

In this section we prove Theorem B. Let ϕ be a C^r (transitive) Anosov diffeomorphism M. Let $f, g : M \to G$ be C^r and such that $f = u\phi \cdot g \cdot \alpha u^{-1}$ a.e. (with respect to a Hölder equilibrium state) where α is unipotent. By Theorem A, we may assume $f = u\phi \cdot g \cdot \alpha u^{-1}$ everywhere and u is Lipschitz.

We will show that u possesses some global degree of differentiability by showing that it is sufficiently differentiable along the stable and unstable manifolds of ϕ. That this is sufficient is contained in the following result.

Proposition 6.1 ([J2, Ll1])
Let ϕ be a C^r Anosov diffeomorphism on M ($1 \leq r \leq \infty$ or $r = \omega$). Let $f : M \to U \subset \mathbb{R}^n$ be uniformly C^r on stable and unstable manifolds. Then f is $C^{r-1+\varepsilon}$ for any $\varepsilon \in (0, 1)$.

This result was first proved for an Anosov diffeomorphism in the smooth case in [LMM]. Another proof of the smooth case, as well as a C^r version, can be found in [HK]. The techniques of [HK] were used in [Ll1] to deduce the analytic case. The proof of the C^r case given in [J2] has the advantage of being completely elementary: it only uses the facts that the stable and unstable manifolds are uniformly transverse, continuously varying C^r sub-manifolds.

In [NT2] the analogue of Theorem B is proved in the case where $\alpha = \mathrm{id}$ (indeed, they prove regularity results for cocycles taking values in a diffeomorphism group). This was done by viewing the associated skew product as a partially hyperbolic system. Recall that a diffeomorphism $\Phi : M \to M$ is *partially hyperbolic* if the tangent bundle TM has a continuous splitting $TM = E^s \oplus E^c \oplus E^u$ into $d\Phi$-invariant sub-bundles for which there exist constants $C > 0$, $0 < \lambda < 1 \leq \mu$ such that $\lambda\mu < 1$ and

$$\|d\Phi^n|_{E^s}\| \leq C_1\lambda^n,$$
$$\|d\Phi^{-n}|_{E^u}\| \leq C_2\lambda^n,$$
$$C_3\mu^{-n} \leq \|d\Phi^n|_{E^c}\| \leq C_4\mu^n,$$

for all $n > 0$ and some constants $C_1, \ldots, C_4 > 0$. Hence $d\Phi$ expands E^u more sharply than it expands E^c and contracts E^s more sharply than it contracts E^c.

The following proposition lists the properties of partially hyperbolic diffeomorphisms that we shall need. The case when $1 \leq r \leq \infty$ follows from Theorem 5.5 in [HPS] and the analytic case can be found in [NT2].

Proposition 6.2
Let Φ be a C^r partially hyperbolic diffeomorphism.

1. *The distributions E^s, E^u are integrable. We call the resulting manifolds the stable and unstable manifolds, respectively. They are C^r submanifolds and depend continuously on x in the C^r topology on embeddings.*

2. *The stable manifold $W^s(x)$ through x is characterised as follows. We have $y \in W^s(x)$ if and only if for some ν with $\lambda < \nu < \mu^{-1}$ we have*

$$\frac{d(\Phi^n x, \Phi^n y)}{\nu^n} \to 0$$

as $n \to \infty$. A similar criterion holds for unstable manifolds.

As in the Introduction, we define for $f : M \to G$ the diffeomorphism $\phi_f : M \times G \to M \times G : (x, y) \mapsto (\phi x, f(x)\alpha(y))$. By using Theorem 2.2 of [BP] and the unipotency assumption on α we have:

Proposition 6.3
Let $f : M \to G$ be C^r. Then ϕ_f is a C^r partially hyperbolic diffeomorphism and we can take $\mu = 1 + \delta$ in the definition of partial hyperbolicity, for any small $\delta > 0$. Moreover, if $W^s_f(x_0, y_0)$ denotes the stable manifold of ϕ_f through $(x_0, y_0) \in M \times G$ then $\pi W^s_f(x_0, y_0) = W^s(x_0)$, the stable manifold of ϕ through x. Hence we can write

$$W^s_f(x_0, y_0) = \{(x, \gamma^f_{(x_0,y_0)}(x)) \mid x \in W^s(x_0)\}$$

where $\gamma^f_{(x_0,y_0)} : M \to G$ is a C^r function whose first r derivatives depend continuously on (x_0, y_0).

Proof of Theorem B. Define $U : M \times G \to M \times G : (x, y) \mapsto (x, u(x)y)$. Then clearly U is Lipschitz and $U\phi_g = \phi_f U$.

We claim that $UW^s_g(x_0, y_0) \subset W^s_f(U(x_0, y_0))$. To see this, let $(x, y) \in W^s_g(x_0, y_0)$. Then if $(1 + \delta)\lambda < \nu < 1$ we have

$$\begin{aligned}
\frac{d(\phi^n_f U(x, y), \phi^n_f(x_0, y_0))}{\nu^n} &= \frac{d(U\phi^n_g(x, y), U\phi^n_g(x_0, y_0))}{\nu^n} \\
&\leq |U|_{\mathrm{Lip}} \frac{d(\phi^n_g(x, y), \phi^n_g(x_0, y_0))}{\nu^n} \\
&\to 0
\end{aligned}$$

as $n \to \infty$. By Proposition 6.2.2, $U(x, y) \in W^s_f(U(x_0, y_0))$.

Take $(x, y) \in W^s_g(x_0, y_0)$. Then

$$(x, y) = (x, \gamma^g_{(x_0,y_0)}(x)). \tag{15}$$

Moreover, as $U(x,y) \in W^s_f(U(x_0,y_0))$, we can write

$$(x, u(x)y) = (x, \gamma^f_{U(x_0,y_0)}(x)). \tag{16}$$

Combining (15) and (16) we see that

$$u(x) = \gamma^f_{U(x_0,y_0)}(x)\gamma^g_{(x_0,y_0)}(x)^{-1}$$

and the right-hand side is uniformly C^r along stable manifolds. As is remarked above, this suffices to prove the theorem. ❑

Remarks.

1. The assumption that the Anosov diffeomorphism is transitive is only used to deduce that measurable solutions are Lipschitz.

2. Let α be an automorphism of G such that the spectrum of $d\alpha$ is contained in the annulus $\{z \in \mathbb{C} \mid \lambda < |z| < \lambda^{-1}\}$. Let f, g be C^r functions and suppose $f = u\phi \cdot g \cdot \alpha u^{-1}$ everywhere for some $u \in C^\theta(G)$, $\theta > \log\mu/\log\lambda^{-1}$. Then the above proof shows that u is $C^{r-1+\varepsilon}$ for any $\varepsilon \in (0,1)$.

§7 A counter-example

Consider the coboundary equation $f = u\phi \cdot u^{-1}$ a.e. where u is measurable. Recently, Nicol, Pollicott and Walkden [NPW] proved that u must be Hölder when G is a connected Lie group (provided f satisfies a partial hyperbolicity assumption on the semi-simple part of G). For the equation $f = u\phi \cdot g \cdot u^{-1}$ a.e. taking values in a nilpotent Lie group, u must again be Hölder [NPW]. We give an example to show that this is false for soluble Lie groups.

Choose a hyperbolic or quasi-hyperbolic toral automorphism $\alpha \in \mathrm{Aut}(\mathbb{R}^k/\mathbb{Z}^k)$. By the examples at the end of §5, choose a hyperbolic diffeomorphism ϕ and analytic functions $f, g : \Lambda \to \mathbb{R}^k/\mathbb{Z}^k$ for which there exists a measurable solution $u : \Lambda \to \mathbb{R}^k/\mathbb{Z}^k$ to $f = u\phi \cdot g \cdot \alpha u^{-1}$ but u has no continuous version. We may assume that $g = 0$.

Let G denote the semi-direct product $\mathbb{Z} \ltimes \mathbb{R}^k/\mathbb{Z}^k$ with group structure given by $(n,g) + (m,h) = (n+m, g + \alpha^n h)$. Then G is soluble.

Define $\bar{f} = (1,f), \bar{g} = (1,0), \bar{u} = (1,u) : \Lambda \to G$. Then $\bar{f}, \bar{g}$ are analytic and $\bar{u}$ is measurable. Moreover, one can easily check that $\bar{f} = \bar{u}\phi \cdot \bar{g} \cdot \bar{u}^{-1}$ a.e. and by the assumption on u, $\bar{u}$ has no continuous version.

Remarks.

1. Suppose G_1 and G_2 are Lie groups (not necessarily compact) such that G_1 acts smoothly on G_2 by automorphisms. Suppose that the Livšic regularity theorem (Theorem 1.1) holds for the coboundary equation taking values in G_1 and G_2. Then one can show that the regularity theorem for the coboundary equation continues to hold for the semi-direct product $G_1 \ltimes G_2$.

2. In [NT2], an example is given (based on one due to de la Llave) of an $Gl(d,\mathbb{R})$-valued cocycle equation $f = u\phi \cdot g \cdot u^{-1}$ where f, g are smooth, u is continuous but u is not C^r.

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