2-blocks with abelian defect groups

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Abstract

We give a classification, up to Morita equivalence, of 2-blocks of quasi-simple groups with abelian defect groups. As a consequence, we show that Donovan's conjecture holds for elementary abelian 2-groups, and that the entries of the Cartan matrices are bounded in terms of the defect for arbitrary abelian 2-groups. We also show that a block with defect groups of the form $C_{2^m} \times C_{2^m}$ for $m \geq 2$ has one of two Morita equivalence types and hence is Morita equivalent to the Brauer correspondent block of the normaliser of a defect group. This completes the analysis of the Morita equivalence types of 2-blocks with abelian defect groups of rank 2, from which we conclude that Donovan's conjecture holds for such 2-groups. A further application is the completion of the determination of the number of irreducible characters in a block with abelian defect groups of order 16. The proof uses the classification of finite simple groups.

1 Introduction

Let k be an algebraically closed field of prime characteristic ℓ, and let $\mathcal{O}$ be a discrete valuation ring with residue field k. Let G be a finite group, and let B be a block of the group algebra $\mathcal{O}G$ with defect group D. Assume that $\mathcal{O}$ contains a primitive $|D|$-th root of unity.

The motivation for this paper is Donovan's conjecture, which states that for a fixed ℓ-group D, there should only be a finite number of Morita equivalence classes of blocks with defect groups isomorphic to D. This conjecture is stated for blocks with respect to k, but it is also expected to hold for blocks with respect to $\mathcal{O}$ (however, there are key results used in reduction arguments for the conjecture that at present are only known for k).

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The main result is that every 2-block of a quasi-simple group with an abelian defect group is either one of a short list of exceptional cases or is Morita equivalent over $\mathcal{O}$ to either a block covered by a nilpotent block, or to a tensor product of a nilpotent block with a block with Klein 4-defect groups (see Theorem 6.1). Blocks covered by nilpotent blocks are treated in [34], where it is shown that they are Morita equivalent to their Brauer correspondent in the normaliser of a defect group. The Morita equivalences are achieved through the Bonnafé-Rouquier correspondence.

The above may be viewed as being in the spirit of Walter's classification of simple groups with abelian Sylow 2-subgroups, and it is hoped that it will eventually be used to tackle Donovan's conjecture for 2-blocks with arbitrary abelian defect groups. We begin here with some cases which we may tackle with tools already at our disposal.

We prove that Donovan's conjecture holds for 2-blocks with elementary abelian defect groups (Theorem 8.3). In this case we are restricted to blocks defined over k because we are reliant on the results of [24].

A conjecture of Brauer (from his Problem 22) represents a weak version of Donovan's conjecture. It states that for a given ℓ-group D, there is a bound on the entries of the Cartan matrix of a block with defect groups isomorphic to D. This conjecture has been reduced to quasi-simple groups by Düvel in [11]. We use our main result to show that the conjecture holds for all abelian 2-groups (Theorem 9.2).

We also consider the case that D is abelian of rank 2, so that D is isomorphic to a direct product $C_{2^m} \times C_{2^n}$ of two cyclic subgroups C_{2^m} and C_{2^n}. If $m \neq n$, then every block with defect group D is necessarily nilpotent. The case $m = n = 1$ is the Klein 4-case, which is already well known by [14]. We show the following:

Theorem 1.1 *Let G be a finite group and B a block of $\mathcal{O}G$ with defect group D. Suppose that $D \cong C_{2^m} \times C_{2^m}$ for some $m \geq 2$. Then B is Morita equivalent to either $\mathcal{O}D$ or $\mathcal{O}(D \rtimes C_3)$.*

By the remarks above, this completes the analysis of 2-blocks with abelian defect groups of rank 2, including the verification of Donovan's conjecture for these groups.

We are also able to show that Donovan's conjecture holds for groups of the form $C_{2^m} \times C_{2^m} \times C_2$ for $m \geq 3$ (Theorem 11.1).

Finally, we complete the determination of the number of irreducible characters and irreducible Brauer characters in a block with abelian defect groups of order 16 (Theorem 10.4). This was completed modulo one case in [26].

The structure of the paper is as follows: In Section 2 we collect together some results which we will use later. In Section 3 we give an analysis of finite groups of quotients of Levi subgroups with abelian Sylow 2-subgroups which will be important in later arguments.

As mentioned above, the main result uses the Bonnafé-Rouquier Morita equivalence. However, this equivalence only applies to finite groups of Lie type in the strict sense. In Section 4 we show that the Bonnafé-Rouquier Morita equivalence induces Morita equivalences on certain quotient groups, so that it applies to the associated simple group.

Section 5 contains the analysis of 2-blocks of finite groups of Lie type defined over fields of odd characteristic. The structure is presented in more detail than in the statement of the main result. In Section 6 we prove the main theorem. In Section 7 we consider blocks with homocyclic defect groups and prove Theorem 1.1. We prove Donovan's conjecture for elementary abelian 2-groups in Section 8.

In Section 9 we prove that the weak version of Donovan's conjecture holds for abelian 2-groups. In Section 10 we treat blocks with elementary abelian defect groups of order 16 and inertial index 15, so completing the work of [26]. Finally in Section 11 we prove that Donovan's conjecture holds for groups $C_{2^m} \times C_{2^m} \times C_2$ for $m \geq 3$.

2 Background material

For G a finite group, we will use the term "ℓ-block of G" to denote either a block of $\mathcal{O}G$ or kG; the base ring will be specified as needed.

We will make frequent use of the following, which apply for any prime ℓ:

Proposition 2.1 ([37]) *Let B be an ℓ-block of a finite group G and let $Z \leq Z(G)$ be an ℓ-subgroup. Let $\bar{B}$ be the unique block of G/Z corresponding to B. Then B is nilpotent if and only if $\bar{B}$ is nilpotent.*

Proposition 2.2 ([25]) *Let G be a finite group and $N \lhd G$. Let B be a block of $\mathcal{O}G$ with defect group D covering a nilpotent block b of $\mathcal{O}N$ with defect group $D \cap N$ and stabiliser H. Then there is a finite group L and $M \lhd L$ such that (i) $M \cong D \cap N$, (ii) $L/M \cong H/N$, (iii) there is a subgroup D_L of L with $D_L \cong D$ and $M \leq D_L$, and (iv) there is a central extension $\tilde{L}$ of L by an ℓ'-group, and a block $\tilde{B}$ of $\mathcal{O}\tilde{L}$ which is Morita equivalent to B and has defect group $\tilde{D} \cong D$.*

We will show that many blocks of finite groups of Lie type are covered by a nilpotent block. These blocks have very nice properties, as shown by Puig.

Definition 2.3 *We say that a block B of a finite group G is nilpotent-covered if there exists a group $\tilde{G}$ containing G as a normal subgroup, and a nilpotent block $\tilde{B}$ of $\tilde{G}$ covering B.*

Proposition 2.4 ([34]) *Every nilpotent-covered block B of $\mathcal{O}G$ with defect group D is Morita equivalent to its Brauer correspondent in $\mathcal{O}N_G(D)$.*

Proof. This is part of Corollary 4.3 of [34]. □

Lemma 2.5 *Let G be a finite group and $N \lhd G$ with $Z(N) \leq Z(G)$. Let $\bar{b}$ be a block of $N/Z(N)$, and let b be the unique block of N corresponding to $\bar{b}$ with $O_{\ell'}(Z(N))$ in its kernel. Then b is covered by a nilpotent block of G if and only if $\bar{b}$ is covered by a nilpotent block of $G/Z(N)$.*

Proof. This is an almost immediate corollary of Proposition 2.1. □

The main result of [23] applies particularly well to blocks with elementary abelian defect groups:

Proposition 2.6 *Let G be a finite group and let B be a block of kG with elementary abelian defect group D and suppose $N \trianglelefteq G$ with $G = ND$. If B covers a G-stable block b of kN, then there is an elementary abelian ℓ-group Q such that B is Morita equivalent to a block C of $k(N \times Q)$ with defect group $(D \cap N) \times Q \cong D$.*

Proof. We may write $D = (D \cap N) \times Q$ for some $Q \leq D$. By the main result of [23], $B \cong kQ \otimes_k b$ as k-algebras. Observe that $kQ \otimes_k b$ is a block of $N \times Q$ with defect group $D = (D \cap N) \times Q$. □

For dealing with finite simple groups, a powerful reduction is provided by a theorem of Bonnafé and Rouquier [4]. However, to effectively use the Bonnafé-Rouquier results, we will need also to have a version for simple Chevalley groups which are not groups of Lie type (the main cases in question are the simple groups of type E_7, where the finite group of Lie type has either a centre of order two or a normal subgroup of index two). This will be done in Section 4 using the following well known result. A proof is given for the convenience of the reader.

Lemma 2.7 *Let G and H be finite groups, b and c be block idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ respectively. Let M be an $(\mathcal{O}Gb, \mathcal{O}Hc)$-bimodule, and let Z be an ℓ-group embedded as a central subgroup of both G and H. Let $\bar{G} = G/Z$, $\bar{H} = H/Z$, and let $\bar{b}$ (respectively $\bar{c}$) be the image of b (respectively c) in $\mathcal{O}G/Z$ (respectively $\mathcal{O}H/Z$) under the canonical surjection $\mathcal{O}G \to \mathcal{O}G/Z$ (respectively $\mathcal{O}H \to \mathcal{O}H/Z$).*

Suppose that $zm = mz$ for all $z \in Z$ and all $m \in M$. Then $\mathcal{O} \otimes_{\mathcal{O}Z} M$ is an $(\mathcal{O}\bar{G}\bar{b}, \mathcal{O}\bar{H}\bar{c})$-bimodule via $\bar{g}(1 \otimes m)\bar{h} = 1 \otimes gmh$, $g \in G, h \in H, m \in M$. If $M \otimes_{\mathcal{O}Hc} -$ induces a Morita equivalence between $\mathcal{O}Hc$ and $\mathcal{O}Gb$, then $(\mathcal{O} \otimes_{\mathcal{O}Z} M) \otimes_{\mathcal{O}\bar{H}\bar{c}} -$ induces a Morita equivalence between $\mathcal{O}\bar{H}\bar{c}$ and $\mathcal{O}\bar{G}\bar{b}$.

Proof. The first assertion is straightforward. Since $zm^* = m^*z$ for all $z \in Z$, $m^* \in M^*$ we have similarly that $\mathcal{O} \otimes_{\mathcal{O}Z} M^*$ is an $(\mathcal{O}\bar{H}\bar{c}, \mathcal{O}\bar{G}\bar{b})$-bimodule via $\bar{h}(1 \otimes m^*)\bar{g} = 1 \otimes hm^*g$, $g \in G, h \in H, m^* \in M^*$. We also have that in $M \otimes_{\mathcal{O}Hc} M^*$, $z(m \otimes m^*) = (m \otimes m^*)z$ and in $M^* \otimes_{\mathcal{O}Gb} M$, $z(m^* \otimes m) = (m^* \otimes m)z$ for all $z \in Z$, $m \in M$, $m^* \in M^*$. Further, $(\mathcal{O} \otimes_{\mathcal{O}Z} M) \otimes_{\mathcal{O}\bar{H}\bar{c}} (\mathcal{O} \otimes_{\mathcal{O}Z} M^*) \cong \mathcal{O} \otimes_{\mathcal{O}Z} (M \otimes_{\mathcal{O}Hc} M^*)$ as $(\mathcal{O}\bar{G}\bar{b}, \mathcal{O}\bar{H}\bar{c})$-bimodule via $1 \otimes m \otimes 1 \otimes m^* \to 1 \otimes m \otimes m^*$, for $m \in M$, $m^* \in M^*$ and similarly, $(\mathcal{O} \otimes_{\mathcal{O}Z} M^*) \otimes_{\mathcal{O}\bar{G}\bar{b}} (\mathcal{O} \otimes_{\mathcal{O}Z} M) \cong \mathcal{O} \otimes_{\mathcal{O}Z} (M^* \otimes_{\mathcal{O}Gb} M)$ as an $(\mathcal{O}\bar{H}\bar{c}, \mathcal{O}\bar{G}\bar{b})$-bimodule. The result follows since $\mathcal{O} \otimes_{\mathcal{O}Z} \mathcal{O}Gb \cong \mathcal{O}\bar{G}\bar{b}$ as an $(\mathcal{O}\bar{G}\bar{b}, \mathcal{O}\bar{G}\bar{b})$-bimodule and $\mathcal{O} \otimes_{\mathcal{O}Z} \mathcal{O}Hc \cong \mathcal{O}\bar{H}\bar{c}$ as an $(\mathcal{O}\bar{H}\bar{c}, \mathcal{O}\bar{H}\bar{c})$-bimodule. □

This is easily extended to the following:

Lemma 2.8 *Let $G_1, \dots, G_t$ and $H_1, \dots, H_t$ be finite groups. Let b_i be a block idempotent of $\mathcal{O}G_i$ and c_i be a block idempotent of $\mathcal{O}H_i$ for $i = 1, \dots, t$. Write $G = G_1 \times \cdots \times G_t$ and $H = H_1 \times \cdots \times H_t$. Write $b = b_1 \cdots b_t$ and $c = c_1 \cdots c_t$, so that b is a block idempotent of $\mathcal{O}G$ and c is a block idempotent of $\mathcal{O}H$. Let M_i be an $(\mathcal{O}Gb_i, \mathcal{O}Hc_i)$-bimodule and let Z_i be an ℓ-group embedded as a central subgroup of both G_i and H_i. Write $\bar{G}_i = G_i/Z_i$ and $\bar{H}_i = H_i/Z_i$. Write $M = M_1 \otimes_{\mathcal{O}} \cdots \otimes_{\mathcal{O}} M_t$, an $(\mathcal{O}Gb, \mathcal{O}Hc)$-bimodule.*

Suppose that $z_i m_i = m_i z_i$ for all $z_i \in Z_i$ and all $m_i \in M_i$. Let $Z \leq Z_1 \times \cdots \times Z_t$, and write $\bar{G} = G/Z$ and $\bar{H} = H/Z$. Let $\bar{b}$ (resp. $\bar{c}$) be the block of $\bar{G}$ (resp. $\bar{H}$) corresponding to b (resp. c). Then $\mathcal{O} \otimes_{\mathcal{O}Z} M$ is an $(\mathcal{O}\bar{G}\bar{b}, \mathcal{O}\bar{H}\bar{c})$-bimodule via $\bar{g}(1 \otimes m)\bar{h} = 1 \otimes gmh$,

$g \in G, h \in H, m \in M$. If $M_i \otimes_{\mathcal{O}H_i c_i} -$ induces a Morita equivalence between $\mathcal{O}H_i c_i$ and $\mathcal{O}G_i b_i$ for each i, then $(\mathcal{O} \otimes_{\mathcal{O}Z} M) \otimes_{\mathcal{O}\bar{H}\bar{c}} -$ induces a Morita equivalence between $\mathcal{O}\bar{H}\bar{c}$ and $\mathcal{O}\bar{G}\bar{b}$.

3 On Levi subgroups with abelian Sylow 2-subgroups

Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 0$ and let $\mathbf{G}$ be a connected reductive group defined over $\mathbb{F}$ with a Steinberg endomorphism $F : \mathbf{G} \to \mathbf{G}$, and $G = \mathbf{G}^F$ the finite group of fixed points. Define q so that $\mathbb{F}_q$ is the field of definition of G.

Recall that $\mathbf{G} = Z^\circ(\mathbf{G})[\mathbf{G}, \mathbf{G}]$, where $Z^\circ(\mathbf{G})$ is the connected centre of $\mathbf{G}$, and that the derived subgroup $[\mathbf{G}, \mathbf{G}]$ of $\mathbf{G}$ is a semi-simple group, that is $[\mathbf{G}, \mathbf{G}]$ is a commuting product of simple groups, called the components of $\mathbf{G}$. We assume throughout this section that the fixed point subgroup of no F-orbit of components of $\mathbf{G}$ is isomorphic to a Suzuki or Ree group.

Lemma 3.1 *Suppose that p is odd and $[\mathbf{G}, \mathbf{G}]$ is simply connected (that is $[\mathbf{G}, \mathbf{G}]$ is a direct product of its components, each of which is simply connected).*

(i) If G has abelian Sylow 2-subgroups, then $\mathbf{G}$ is a torus.

(ii) Suppose that G has non-abelian Sylow 2-subgroups, but for some central 2-subgroup Z of G, G/Z has abelian Sylow 2-subgroups. Then all components of $\mathbf{G}$ are of type A_1. Further, if $\{\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_r\}$ is an F-orbit of components of $[\mathbf{G}, \mathbf{G}]$, then $q^r \equiv \pm 3 \pmod 8$, $Z \cap (\prod_{1 \le i \le r} \mathbf{X}_i)^F \neq 1$ and $Z^\circ(\mathbf{G})^F \cap (\prod_{1 \le i \le r} \mathbf{X}_i)^F = 1$.

Proof. The first statement is well-known. We prove (ii). Let $\{\mathbf{X}_1, \dots, \mathbf{X}_s\}$ be the set of components of $[\mathbf{G}, \mathbf{G}]$ and let $\{\mathbf{X}_1, \dots, \mathbf{X}_r\}$, $1 \le r \le s$ be an F-orbit of components such that $F(\mathbf{X}_i) = \mathbf{X}_{i+1}$ for $1 \le i \le r-1$ and $F(\mathbf{X}_r) = \mathbf{X}_1$. Then $(\prod_i \mathbf{X}_i)^F \cong \mathbf{X}_1^{F^r}$, and denoting by Z' the image of $Z \cap (\prod_i \mathbf{X}_i)^F$ under this isomorphism, $\mathbf{X}_1^{F^r}/Z'$ has abelian Sylow 2-subgroups. It follows that $\mathbf{X}_1$ is of type A_1, $\mathbf{X}_1^{F^r} \cong SL_2(q^r)$ with $q^r \equiv \pm 3 \pmod 8$ and that Z' is the unique central subgroup of order 2 of $\mathbf{X}_1^{F^r}$. In other words, $Z \cap (\prod_{1 \le i \le r} \mathbf{X}_i)^F$ is generated by $(z_1, F(z_1), \dots, F^{r-1}(z_1)) =: \zeta$ where z_1 is the unique involution in the centre of $\mathbf{X}_1$. It remains only to show that $\zeta \notin Z^\circ(\mathbf{G})$. Suppose the contrary. The multiplication map $\mu : Z^\circ(\mathbf{G}) \times [\mathbf{G}, \mathbf{G}] \to \mathbf{G}$ is surjective with kernel $\Delta(Z^\circ(\mathbf{G}) \cap [\mathbf{G}, \mathbf{G}])$, where $\Delta(Z^\circ(\mathbf{G}) \cap [\mathbf{G}, \mathbf{G}])$ is the diagonally embedded copy of $Z^\circ(\mathbf{G}) \cap [\mathbf{G}, \mathbf{G}]$ in $Z^\circ(\mathbf{G}) \times [\mathbf{G}, \mathbf{G}]$. Let A denote the inverse image under μ of G. So,

$$A = \{(u, g) : u \in Z^\circ(\mathbf{G}), g \in [\mathbf{G}, \mathbf{G}] \text{ such that } u^{-1}F(u) = gF(g^{-1})\}.$$

Let $\tau_1 : A \to \mathbf{X}_1$ be the (restriction to A of the) projection of $Z^\circ(\mathbf{G}) \times [\mathbf{G}, \mathbf{G}] = Z^\circ(\mathbf{G}) \times \mathbf{X}_1 \times \cdots \times \mathbf{X}_s$ onto $\mathbf{X}_1$. Since A contains $Z^\circ(\mathbf{G})^F \times [\mathbf{G}, \mathbf{G}]^F$, $\mathbf{X}_1^{F^r} \le \tau_1(A)$.

By the Lang-Steinberg theorem, there exist 2-elements $u \in Z^\circ(\mathbf{G})$ and $g \in [\mathbf{G}, \mathbf{G}]$ such that $u^{-1}F(u) = \zeta = gF(g^{-1})$. In particular, $(u, g) \in A$. Write $g = (x_1, \dots, x_s)$, $x_i \in \mathbf{X}_i$. The equation $\zeta = gF(g^{-1})$ implies that $x_1 = (z_1)^r F^r(x_1)$. Since $q^r \equiv \pm 3$

(mod 8), r is odd, hence $\tau_1(u,g) = x_1 \notin \mathbf{X}_1^{F^r}$. Consequently, a Sylow 2-subgroup of $\tau_1(A)$ properly contains a Sylow 2-subgroup of $\mathbf{X}_1^{F^r}$ and it follows that the Sylow 2-subgroups of $\tau_1(A)$ are non-abelian of order at least 16. Since $\mathbf{X}_1$ is of type A_1, any finite 2-subgroup of $\mathbf{X}_1$ is cyclic or quaternion. Hence the Sylow 2-subgroups of $\tau_1(A)$ are quaternion of order at least 16.

Let S be a Sylow 2-subgroup of A. By hypothesis, $[\mu(S), \mu(S)] \leq Z$ and the inverse image under μ of Z is a central subgroup of $Z^\circ(\mathbf{G}) \times [\mathbf{G}, \mathbf{G}]$ (note that $Z^\circ(\mathbf{G}) \times Z([\mathbf{G}, \mathbf{G}])$ is the full inverse image under μ of $Z(\mathbf{G})$ and $Z \leq Z(\mathbf{G})$). Hence, $[S, S] \leq Z(S)$ from which it follows that $[\tau_1(S), \tau_1(S)] \leq \tau_1(Z(S)) \leq Z(\tau_1(S))$, a contradiction. □

We now fix a maximal torus $\mathbf{T}$ of $\mathbf{G}$ and assume that $\mathbb{F}$ has odd characteristic. Let $X(\mathbf{T})$ be the group of rational characters of $\mathbf{T}$, $Y(\mathbf{T})$ the group of one-parameter subgroups of $\mathbf{G}$, and $<,>$: $X(\mathbf{T}) \times Y(\mathbf{T}) \to \mathbb{Z}$ the canonical exact pairing. Let $\Phi \subset X(\mathbf{T})$ be the set of roots of $\mathbf{G}$ with respect to $\mathbf{T}$, $\Phi^\vee$ the corresponding set of coroots and $I \subset \Phi$ a set of fundamental roots corresponding to a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$. For each $\alpha \in \Phi$, denote by $\mathbf{U}_\alpha$ the corresponding root subgroup of $\mathbf{G}$ and let $\phi_\alpha : SL_2(\mathbb{F}) \to \langle \mathbf{U}_\alpha, \mathbf{U}_{-\alpha} \rangle$ be a surjective homomorphism such that the image of the group of upper triangular matrices is $\mathbf{U}_\alpha$ and the image of the group of lower triangular matrices is $\mathbf{U}_{-\alpha}$ (see [10, Prop. 0.44]). If $\beta \in \Phi$ and $a \in \mathbb{F}^\times$, then $\beta(\phi_\alpha(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix})) = a^{<\beta, \alpha^\vee>}$.

For $J \subseteq I$, let Φ_J be the set of roots which are in the subspace of $\mathbb{R} \otimes X(\mathbf{T})$ generated by J. The group $\mathbf{M}_J := \langle \mathbf{U}_\alpha, \mathbf{U}_{-\alpha} : \alpha \in \Phi_J \rangle$ is the derived subgroup of the Levi subgroup $\mathbf{L}_J := \langle \mathbf{T}, \mathbf{M}_J \rangle$ of $\mathbf{G}$ and any Levi subgroup of $\mathbf{G}$ is conjugate to $\mathbf{L}_J$ for some $J \subseteq I$. All components of $\mathbf{M}_J$ are of type A_1 if and only if for each $\alpha, \beta \in J$ with $\alpha \neq \beta$, $< \alpha, \beta^\vee >= 0$ and in this case, $\mathbf{M}_J = \prod_{\alpha \in J} \langle \mathbf{U}_\alpha, \mathbf{U}_{-\alpha} \rangle$. Further, if $\mathbf{G}$ is simply connected, then the product $\prod_{\alpha \in J} \langle \mathbf{U}_\alpha, \mathbf{U}_{-\alpha} \rangle$ is direct and for all $\alpha \in J$, ϕ_α is an isomorphism. In particular, if $\mathbf{G}$ is simply connected and $char(\mathbb{F})$ is odd, then $z_\alpha := \phi_\alpha(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix})$ is the unique central element of order 2 in $\langle \mathbf{U}_\alpha, \mathbf{U}_{-\alpha} \rangle$.

Lemma 3.2 *Suppose that $char(\mathbb{F})$ is odd and that $\mathbf{G}$ is simple and simply connected. Let $\emptyset \neq J \subseteq I$ such that if $\alpha, \beta \in J$ with $\alpha \neq \beta$, then $< \alpha, \beta^\vee >= 0$. Let $z := \prod_{\alpha \in J} z_\alpha$ and suppose that $z \in Z(\mathbf{G})$.*

(i) Suppose that $\mathbf{G}$ is of type A_n, $n \geq 1$. Let $I = \{\alpha_1, \ldots, \alpha_n\}$, where α_i, α_{i+1}, $1 \leq i \leq n-1$ are consecutive nodes in the corresponding Dynkin diagram. Then n is odd and $J = \{\alpha_1, \alpha_3, \alpha_5, \ldots, \alpha_{n-2}, \alpha_n\}$.

(ii) Suppose that $\mathbf{G}$ is of type B_n, $n \geq 2$. Let $I = \{\alpha_1, \ldots, \alpha_n\}$, where α_i, α_{i+1}, $1 \leq i \leq n-1$ are consecutive nodes in the corresponding Dynkin diagram and there is a double arrow from α_{n-1} to α_n. Then $J = \{\alpha_n\}$.

(iii) Suppose that $\mathbf{G}$ is of type C_n, $n \geq 3$. Let $I = \{\alpha_1, \ldots, \alpha_n\}$, where α_i, α_{i+1}, $1 \leq i \leq n-1$ are consecutive nodes in the corresponding Dynkin diagram and there is a double arrow from α_n to α_{n-1}. If n is even, then $J = \{\alpha_1, \alpha_3, \ldots, \alpha_{n-3}, \alpha_{n-1}\}$. If n is odd, then $J = \{\alpha_1, \alpha_3, \ldots, \alpha_{n-2}, \alpha_n\}$.

(iv) *Suppose that* $\mathbf{G}$ *is of type* D_n, $n \geq 4$. *Let* $I = \{\alpha_1, \dots, \alpha_n\}$, *where* α_i, α_{i+1}, $1 \leq i \leq n-2$ *are consecutive nodes and* α_n *and* α_{n-2} *are connected. Then either* $J = \{\alpha_{n-1}, \alpha_n\}$ *or* n *is even and* J *is one of* $\{\alpha_1, \alpha_3, \dots, \alpha_{n-3}, \alpha_{n-1}\}$ *or* $\{\alpha_1, \alpha_3, \dots, \alpha_{n-3}, \alpha_n\}$.

(v) *Suppose* $\mathbf{G}$ *is of type* E_7. *Let* $I = \{\alpha_1, \dots, \alpha_7\}$, *such that* $I - \{\alpha_2\}$ *corresponds to the Dynkin diagram of type* A_6 *and* α_2 *is connected to* α_5. *Then* $J = \{\alpha_1, \alpha_2, \alpha_4\}$.

Proof. Let $\alpha, \beta \in I$. Then as explained above, $\beta(z_\alpha) = (-1)^{<\beta,\alpha^\vee>}$. By hypothesis, $z \in Z(\mathbf{G})$ whence $\prod_{\alpha \in J} \beta(z_\alpha) = \beta(z) = 1$ for all $\beta \in \Phi$ (see [10, Prop 0.35]). So, if $< \beta, \alpha^\vee >$ is odd (that is equals -1 or -3) and for all $\gamma \in J$ different from α, $< \beta, \gamma^\vee >$ is even (that is equals 0 or ± 2) , then $\alpha \notin J$. We will systematically use this observation. Also note that since the elements of J are pairwise orthogonal, J does not contain any pair of consecutive nodes.

Suppose first that $\mathbf{G}$ is of type A_n, $n \geq 1$. Then, $< \alpha_i, \alpha_j^\vee >$ is odd if and only if $j = i \pm 1$. So, α_2 is the unique element of I such that $< \alpha_1, \alpha_2^\vee >$ is odd. By the observation above $\alpha_2 \notin J$. We claim that $\alpha_1 \in J$. Indeed, suppose not and let i be the least integer such that $\alpha_i \in J$. Then $i \geq 3$ and $< \alpha_{i-1}, \alpha_j^\vee >$ is odd if and only if $j = i-2$ or i. Since $\alpha_{i-2} \notin J$, it follows that $i \notin J$, a contradiction. Thus, $\alpha_1 \in J$ and $\alpha_2 \notin J$. The result follows by repeating the argument.

Suppose that $\mathbf{G}$ is of type B_n, $n \geq 2$. Then, $< \alpha_i, \alpha_j^\vee >$ is odd if and only if $1 \leq i, j \leq n-1$ and $j = i \pm 1$ or $i = n$ and $j = n-1$. In particular, $< \alpha_n, \alpha_j^\vee >$ is odd if and only if $j = n-1$, hence $\alpha_{n-1} \notin J$. If $n = 2$, the result is proved. Suppose that $n \geq 3$. Then $< \alpha_{n-1}, \alpha_j^\vee >$ is odd if and only if $j = n-2$, hence $\alpha_{n-2} \notin J$. Suppose that $n \geq 4$ and let i be the greatest integer such that $i \leq n-3$ and $i \in J$. Then $< \alpha_{i+1}, \alpha_j^\vee >$ is odd if and only if $j = i$ or $j = i+2$. By maximality of i, $n-1 \geq i+2 \notin J$, hence $i \notin J$, a contradiction. Thus, $J = \{\alpha_n\}$.

Suppose that $\mathbf{G}$ is of type C_n, $n \geq 3$. Then, $< \alpha_i, \alpha_j^\vee >$ is odd if and only if $1 \leq i, j \leq n-1$ and $j = i \pm 1$ or $i = n-1$ and $j = n$. Suppose first that $n-1 \in J$. Then $n-3 \in J$ as $< \alpha_{n-2}, \alpha_j^\vee >$ is odd if and only if $j = n-1$ or $n-3$. Continuing like this, we get that n is even and $J = \{\alpha_1, \alpha_3, \dots, \alpha_{n-1}\}$. Now suppose that $\alpha_{n-1} \notin J$. We claim that $\alpha_n \in J$. Indeed, suppose not and let i be the greatest integer such that $\alpha_i \in J$. Then $i \leq n-2$, and $< \alpha_{i+1}, \alpha_j^\vee >$ is odd if and only if either $j = i$ or $j = i+2$. Since $i+2 \notin J$, $i \notin J$, a contradiction. Thus, $\alpha_n \in J$ from which it follows that $n-2 \in J$. Continuing, one obtains that n is odd and $J = \{\alpha_1, \alpha_3, \dots, \alpha_n\}$.

Suppose that $\mathbf{G}$ is of type D_n. Then, $\alpha_{n-2} \notin J$ since $< \alpha_n, \alpha_j^\vee >$ is odd if and only if $j = n-2$. Suppose first that $\alpha_{n-3} \notin J$. Then it follows that $\alpha_i \notin J$ for any $i \leq n-3$ whence $J \subseteq \{\alpha_{n-1}, \alpha_n\}$ and consequently $J = \{\alpha_{n-1}, \alpha_n\}$. The case that $\alpha_{n-3} \in J$ leads to the conclusion that n is even and J is one of the two sets claimed.

Finally suppose that $\mathbf{G}$ is of type E_7. Then, $< \alpha_1, \alpha_j^\vee >$ is odd if and only if $j = 3$, hence $\alpha_3 \notin J$. Also, $< \alpha_4, \alpha_j^\vee >$ is odd if and only if $j = 3$ or $j = 5$, hence $\alpha_5 \notin J$. Since $< \alpha_6, \alpha_j^\vee >$ is odd if and only if $j = 7$ or $j = 5$, $\alpha_7 \notin J$. Since $< \alpha_7, \alpha_j^\vee >$ is odd if and only if $j = 6$, $\alpha_6 \notin J$. Similarly, it follows that $\alpha_2 \in J$ if and only if $\alpha_4 \in J$ if and only if $\alpha_1 \in J$. Hence $J = \{\alpha_1, \alpha_2, \alpha_4\}$ as claimed. □

Lemma 3.3 *Keep the notation and hypothesis of Lemma 3.2. Let* $\mathbf{L}_J = \langle \mathbf{T}, \mathbf{U}_\alpha, \mathbf{U}_{-\alpha} : \alpha \in J \rangle$ *be the Levi subgroup corresponding to* J.

(i) If $\mathbf{G}$ *is of type* B_n*,* $n \geq 2$*, or* C_n*,* $n \geq 3$ *and* n *even, then* $Z(\mathbf{L}_J)$ *is connected.*

(ii) If $\mathbf{G}$ *is of type* C_n*,* n *odd, then the components of* $\mathbf{L}$ *are not transitively permuted by* $N_W(W_J)$*.*

PROOF. (i) Let P be the subgroup of $X(\mathbf{T})$ generated by the α_i, $i \in J$. It suffices to show that $X(\mathbf{T})/P$ is torsion-free (see for instance [10, Lemma 13.14]). Keep the labelling of the fundamental roots introduced in Lemma 3.2. Let q_i, $1 \leq i \leq n$ be the set of fundamental weights corresponding to Φ, $\Phi^\vee$ and I. Thus q_i are vectors in $\mathbb{R} \otimes X(\mathbf{T})$ defined by $< q_i, \alpha_j^\vee >= \delta_{i,j}$, $1 \leq i, j \leq n$. Since $\mathbf{G}$ is simply connected, $X(\mathbf{T})$ equals the subgroup of $\mathbb{R} \otimes X(\mathbf{T})$ generated by the fundamental weights.

Suppose that $\mathbf{G}$ is of type B_n, $n \geq 2$. Then $\mathbb{R} \otimes X(\mathbf{T})$ may be identified with an n-dimensional Euclidean space with an orthonormal basis $e_1, \ldots, e_n$ such that under this identification,

$$\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n$$

and

$$q_1 = e_1, q_2 = e_1 + e_2, \ldots, q_{n-1} = e_1 + \cdots + e_{n-1}, q_n = \frac{1}{2}(e_1 + e_2 + \cdots + e_n).$$

So, $X(\mathbf{T})$ is generated by $e_1, e_2, \ldots e_n, q_n$ and $e_1, e_2, \ldots, e_{n-1}, q_n$ is a basis of $X(\mathbf{T})$. By Lemma 3.2, $P = \mathbb{Z}e_n$. So, $X(\mathbf{T})/P$ is free with basis $e_1, \ldots, e_{n-2}, q_n$.

Suppose that $\mathbf{G}$ is of type C_n, $n \geq 3$. Then $\mathbb{R} \otimes X(\mathbf{T})$ may be identified with an n-dimensional Euclidean space with an orthonormal basis $e_1, \ldots, e_n$ such that under this identification,

$$\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n$$

and

$$q_1 = e_1, q_2 = e_1 + e_2, \ldots, q_{n-1} = e_1 + \cdots + e_{n-1}, q_n = e_1 + e_2 + \cdots + e_n.$$

So, $e_1, e_2, \ldots, e_{n-1}, e_n$ is a basis of $X(\mathbf{T})$.

If n is even, then by Lemma 3.2,

$$P = \mathbb{Z}(e_1 - e_2) \oplus \mathbb{Z}(e_3 - e_4) \oplus \cdots \oplus \mathbb{Z}(e_{n-1} - e_n)$$

and $X(\mathbf{T})/P$ is free with generators $e_1 + P, e_2 + P, \ldots, e_{n-2} + P$.

(ii) Suppose that $\mathbf{G}$ is of type C_n, n odd. By Lemma 3.2, $J = \{\alpha_1, \alpha_3, \ldots, \alpha_{n-2}, \alpha_n\}$. Now, $\alpha_n = 2e_n$, $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$ and for any i, $1 \leq i \leq n$ and any $w \in W$, ${}^w(e_i) = \pm e_j$, hence α_n is not in the same W-orbit as α_i for any $i \leq n-1$. □

We assume from now on that $\mathbf{T}$ is F-stable. Recall that to any F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ is associated a pair (J, w), where $w \in W$, $J \subset I$ such that ${}^{wF}J = J$ and $\mathbf{L} = {}^g\mathbf{L}_J$ for some $g \in \mathbf{G}$ with $g^{-1}F(g) \in N_{\mathbf{G}}(\mathbf{T})$ whose image in $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ is w. Moreover, $\mathbf{L}^F \cong \mathbf{L}_J^{wF}$ (see [10, Prop. 4.3]).

Proposition 3.4 *Suppose that $\mathbb{F}$ has odd characteristic, $\mathbf{G}$ is simple and simply connected and that $\mathbf{T}$ is F-stable. Let $\mathbf{L}$ be an F-stable non-toral Levi subgroup of $\mathbf{G}$, Z a central 2-subgroup of G. Suppose that the Sylow 2-subgroups of $\mathbf{L}^F/Z$ are abelian. Then,*

(i) F has only one orbit on the set of components of $[\mathbf{L},\mathbf{L}]$.

(ii) $[\mathbf{L},\mathbf{L}]^F \cong SL_2(q^t)$, $q \equiv \pm 3 \pmod 8$, where t is odd and equals the number of components of $[\mathbf{L},\mathbf{L}]$. Further, $\mathbf{L}^F$ is a direct product of $[\mathbf{L},\mathbf{L}]^F$ and $Z^\circ(\mathbf{L})^F$.

Let P be a Sylow 2-subgroup of $\mathbf{L}^F$ and write $P = P_0 \times P_1$ where $P_0 \le Z^\circ(\mathbf{L})^F$ and $P_1 \le [\mathbf{L},\mathbf{L}]^F$. Then P_1 is quaternion of order 8, $Z \cap P_1$ is the central subgroup of $[\mathbf{L},\mathbf{L}]^F$ and $P_1/(Z \cap P_1)$ is a Klein 4-group. Moreover, one of the following holds

(a) $\mathbf{G}$ is of type A_n, $n+1 = 2t$, P_0 is trivial and Z has order 2. In particular, $P/Z \cong C_2 \times C_2$.

(b) $\mathbf{G}$ is of type D_n, $n = 2t$, G is of untwisted type $D_n(q)$ and either Z is of order 2 or a Klein 4-group. If $q \equiv 3 \pmod 4$, then P_0 has order 2 and P/Z is elementary abelian of order 8 or a Klein 4-group, depending on whether Z has order 2 or 4. If $q \equiv 1 \pmod 4$, then P_0 has order 4 and P/Z is isomorphic to $C_4 \times C_2 \times C_2$ or is elementary abelian of order 8, depending on whether Z has order 2 or 4.

(c) $\mathbf{G}$ is of type E_7 and $t = 3$.

Proof. Note that $|\mathbf{L}^F| = |Z^\circ(\mathbf{L})^F||[\mathbf{L},\mathbf{L}]^F|$. So, if $Z^\circ(\mathbf{L}) \cap [\mathbf{L},\mathbf{L}]^F = 1$, then $\mathbf{L}^F$ is a direct product of $Z^\circ(\mathbf{L})^F$ and $[\mathbf{L},\mathbf{L}]^F$. Thus, (ii) is a consequence of (i) and Lemma 3.1.

We prove (i). By Lemma 3.1 (applied to $\mathbf{L}$), Z and hence $O_2(G)$ is non-trivial. Hence, $\mathbf{G}$ is of type A_n, B_n, C_n, D_n or E_7. Similarly we may also assume that G is not of type ${}^3D_4(q)$. Let (J, w) be associated to $\mathbf{L}$ as explained above. The F-orbits of $\mathbf{L}$ on the components of $\mathbf{L}$ correspond to wF-orbits of J. Hence it suffices to prove that there is only one wF-orbit on J. Let $J_1 \subset J$ be a wF-orbit of J of size t. By Lemma 3.1, $[\mathbf{L}_{J_1},\mathbf{L}_{J_1}]^{wF} \cong [\mathbf{L},\mathbf{L}]^F \cong SL_2(q^t)$, where $q^t \equiv \pm 3 \pmod 8$. So, t is odd. Also by Lemma 3.1 and transport of structure, we have $[\mathbf{L}_{J_1},\mathbf{L}_{J_1}]^{wF} \cap Z \ne 1$ (here note that any $\mathbf{G}$-conjugate of Z equals Z). So, in particular $[\mathbf{L}_{J_1},\mathbf{L}_{J_1}] \cap Z(\mathbf{G}) \ne 1$. We now apply Lemma 3.2 and Lemma 3.3 to J_1 (note that by Lemma 3.1, all components of $\mathbf{L}_J$ have rank 1).

Suppose $\mathbf{G}$ is of type A_n. Then by Lemma 3.2, n is odd and $J_1 = \{\alpha_1, \alpha_3, \ldots, \alpha_n\}$. So, $n+1 = 2t$. It also follows that $J = J_1$, since J_1 is the only subset of J with the required properties. Thus, all statements for type A_n are proved except the assertion on P_1 being trivial. For this, we use the order formulas for $Z^\circ(\mathbf{L})^{wF}$ given in [7]. By Proposition 7 and 8 of [7], we see that $|Z^\circ(\mathbf{L})^F| = \frac{q^t-1}{q-1}$ if G is untwisted and $|Z^\circ(\mathbf{L})^F| = \frac{q^t+1}{q+1}$ if G is twisted. Since t is odd, in either case, we have that $|Z^\circ(\mathbf{L})^F|$ is odd. This proves the proposition for groups of type A_n.

Suppose $\mathbf{G}$ is of type B_n. Then by Lemma 3.2, $J = J_1 = \{\alpha_n\}$. By Lemma 3.3, $Z^\circ(\mathbf{L}_I)$ is connected, hence $Z \le Z^\circ(\mathbf{L}_I)$. But by Lemma 3.1 $Z \cap [\mathbf{L}_J,\mathbf{L}_J]^{wF} \ne 1$ and $Z^\circ(\mathbf{L}) \cap [\mathbf{L}_J,\mathbf{L}_J]^{wF} = 1$, a contradiction.

Suppose $\mathbf{G}$ is of type C_n. If n is even, then by Lemma 3.1 we are done by the same argument as for type B_n. Suppose that n is odd. Since $\mathbf{G}$ is of type C_n, we may assume that F acts trivially on I. Then, by Lemma 3.3 (ii), we get that there are at least two wF-orbits on J. But $Z \leq Z(\mathbf{G})$ is cyclic, and by Lemma 3.1, for each wF-orbit J' of J, $[\mathbf{L}_{J'}, \mathbf{L}_{J'}]^F$ intersects Z non-trivially, a contradiction.

Suppose $\mathbf{G}$ is of type D_n. Since $|J_1| = t$ is odd, $J_1 \neq \{\alpha_{n-1}, \alpha_n\}$. Hence, by Lemma 3.2, n is even and J_1 is one of $\{\alpha_1, \alpha_3, \ldots, \alpha_{n-3}, \alpha_{n-1}\}$ or $\{\alpha_1, \alpha_3, \ldots, \alpha_{n-3}, \alpha_n\}$. In any case $J = J_1$ and $n = 2t$. If G is of type ${}^2D_n$, we may assume that $F(\alpha_{n-1}) = \alpha_n$ and $F(\alpha_i) = \alpha_i$ for all i, $1 \leq i \leq n-2$. But then ${}^wF(J) \neq J$ for any $w \in W$, hence there is no F-stable Levi subgroup corresponding to J. By [7, Prop. 10], $Z^\circ(\mathbf{L})^F \cong Z^\circ(\mathbf{L}_I)^{wF}$ is cyclic of order $(q^t - 1)$ which proves all the assertions for the case $\mathbf{G}$ is of type D_n.

If $\mathbf{G}$ is of type E_7, then by Lemma 3.2, $J = J_1 = \{\alpha_1, \alpha_2, \alpha_4\}$. □

4 On Bonnafé-Rouquier equivalences and central extensions

We keep the notation of Section 3. We assume throughout this section also that the fixed point subgroup of no F-orbit of components of $\mathbf{G}$ is isomorphic to a Suzuki or Ree group. We assume in this section that ℓ is a prime different from p. We briefly recall the standard set-up for describing the representation theory of G.

Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$, and $\mathbf{G}^*$ be a group in duality with $\mathbf{G}$ with respect to $\mathbf{T}$ and with corresponding Steinberg homomorphism again denoted by F. Set $G^* = \mathbf{G}^{*F}$. By the fundamental results of Lusztig, the set of complex irreducible characters of G is a disjoint union of rational Lusztig series $\mathcal{E}(G, s)$, where s runs over semi-simple elements of G^* up to conjugation. Further, by results of Broué-Michel and Hiss, for any ℓ-block B of G, there is a unique G^*-conjugacy class of ℓ'-elements s such that B contains a complex irreducible character in $\mathcal{E}(G, s)$. We will call such an s a semi-simple label of B.

Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ such that $C_{\mathbf{G}^*}(s) \leq \mathbf{L}^*$, where $\mathbf{L}^*$ is a Levi subgroup of $\mathbf{G}^*$ in duality with $\mathbf{L}$. Then Lusztig induction provides a one-one correspondence between ℓ-blocks of $\mathbf{L}^F$ with semi-simple label s and ℓ-blocks of G with semi-simple label s. If a block B of G and a block C of $\mathbf{L}^F$ are related in this way, then B and C are said to be Bonnafé-Rouquier correspondents (see Definition 7.7 of [22]). By [4, Theorem B', §11], if B and C are Bonnafé-Rouquier correspondents, then B and C are Morita equivalent. We show next that this equivalence is preserved on passing to quotients by central ℓ-subgroups.

Proposition 4.1 *Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$, $L := \mathbf{L}^F$, B a block of $\mathcal{O}G$ and C a block of $\mathcal{O}L$ in Bonnafé-Rouquier correspondence with B. Let Z be a central ℓ-subgroup of G contained in L, and let $\bar{B}$ (respectively $\bar{C}$) be the block of $\mathcal{O}(G/Z)$ (respectively $\mathcal{O}(L/Z)$) corresponding to B (respectively C). Then $\bar{B}$ and $\bar{C}$ are Morita equivalent.*

Proof. Let $\mathbf{V}$ be the unipotent radical of some parabolic subgroup of $\mathbf{G}$ containing $\mathbf{L}$ as Levi complement. By [4, Theorem B', §11], C and B are isomorphic via a (B, C)-bimodule M which is isomorphic to a direct summand of the ℓ-adic cohomology module

$H^r_c(\mathbf{X}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathcal{O}$, where $\mathbf{X}$ is the (G, L)-variety consisting of cosets $g\mathbf{V}$ in $\mathbf{G}$ such that $g^{-1}F(g) \in \mathbf{V} \cdot {}^F\mathbf{V}$ and where the action of G is by left multiplication and that of L is by right multiplication. By Lemma 2.7 and the functoriality of ℓ-adic cohomology with respect to finite morphisms, it suffices to note that $zg\mathbf{V} = g\mathbf{V}z$ for all $g \in \mathbf{G}$ and all $z \in Z$ (here we use the fact that $Z(G) \leq Z(\mathbf{G})$). $\square$

The next result gives a sufficient condition for an ℓ-block of G to be nilpotent-covered.

Lemma 4.2 *Let B be a block of $\mathcal{O}G$ and let $s \in G^*$ be a semi-simple label of B. If $C^\circ_{\mathbf{G}^*}(s)$ is a torus, then there exists a finite group $\tilde{G}$ such that $G \trianglelefteq \tilde{G}$, $Z(G) \leq Z(\tilde{G})$ and a nilpotent block $\tilde{B}$ of $\mathcal{O}\tilde{G}$ covering B. If $C_{\mathbf{G}^*}(s)$ is a torus, then B is nilpotent.*

Proof. Let $\iota : \mathbf{G} \to \tilde{\mathbf{G}}$, where $\tilde{\mathbf{G}}$ is a connected reductive group with connected centre such that $[\tilde{\mathbf{G}}, \tilde{\mathbf{G}}] = [\mathbf{G}, \mathbf{G}]$, with corresponding Steinberg homomorphism again denoted F and let $\iota^* : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$ be the corresponding dual map (see Section 2, [3]). Set $\tilde{G} = \tilde{\mathbf{G}}^F$. Then $Z(G) = Z(\mathbf{G})^F \leq Z(\tilde{\mathbf{G}})^F$. Let $\tilde{B}$ be a block of $\mathcal{O}\tilde{G}$ covering B and let $\tilde{s} \in \tilde{\mathbf{G}}^{*F}$ be a semi-simple label of $\tilde{B}$. We may assume that $s = \iota^*(\tilde{s})$ and hence that $C_{\tilde{\mathbf{G}}^*}(\tilde{s}) = C^\circ_{\tilde{\mathbf{G}}^*}(\tilde{s})$ is the inverse image of $C^\circ_{\mathbf{G}^*}(s)$ in $\tilde{\mathbf{G}}^*$(see [3, Prop. 11.7]). So, $C_{\tilde{\mathbf{G}}^*}(\tilde{s})$ is a maximal torus of $\tilde{\mathbf{G}}^*$. By the Bonnafé-Rouquier theorem, it follows that $\tilde{B}$ is Morita equivalent to a block of $C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$. Since $C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ is an abelian group and Morita equivalence preserves nilpotency of blocks (see [32, 8.2]), $\tilde{B}$ is nilpotent. The second assertion follows by the same argument replacing $\tilde{\mathbf{G}}$ by $\mathbf{G}$. $\square$

5 On abelian 2-blocks of finite reductive groups in odd characteristic

We keep the notation and assumptions of the previous section. Our aim in this section is to determine the structure of all 2-blocks of G when p is odd and $\mathbf{G}$ is simply connected whose defect groups become abelian on passing to the quotient by a central 2-subgroup of G. This will be done in Proposition 5.3.

We first consider quasi-isolated blocks. Recall that the block B is called quasi-isolated if $C_{\mathbf{G}^*}(s)$ is not contained in any proper Levi subgroup of $\mathbf{G}^*$. In this case we call s a quasi-isolated element.

Lemma 5.1 *Suppose that $\mathbf{G} = PGL_{n+1}(\mathbb{F})$ is the adjoint group of type A_n and let s be a semi-simple quasi-isolated element of $\mathbf{G}$. Then, $o(s)$ is a divisor of $n+1$, $C^\circ_{\mathbf{G}}(s)$ is a Levi subgroup of $\mathbf{G}$ and $C^\circ_{\mathbf{G}}(s)/Z(C^\circ_{\mathbf{G}}(s))$ is a direct product of $o(s)$ adjoint groups of type $A_{\frac{n+1}{o(s)}-1}$, that is, a direct product of $o(s)$ copies of $PGL_{\frac{n+1}{o(s)}}(\mathbb{F})$.*

Proof. See table 2 of [2]. $\square$

Lemma 5.2 *Assume that p is odd and that $\mathbf{G}$ is simple and simply-connected. Let B be a quasi-isolated 2-block of G with semi-simple label $s \in \mathbf{G}^{*F}$.*

(a) *Suppose that B has abelian defect groups. Then one of the following holds.*

   (i) $\mathbf{G}$ *is of type* A_n*,* n *is even, and* $C^\circ_{\mathbf{G}^*}(s)$ *is a torus.*

   (ii) $\mathbf{G}$ *is of type* G_2*,* F_4*,* E_6 *or* E_8*,* $s = 1$ *and* B *is of defect* 0*.*

(b) *Suppose that B has non-abelian defect groups, but for some non-trivial central 2-subgroup Z of $\mathbf{G}^F$, the image $\bar{B}$ of B in G/Z has abelian defect groups. Then Z is cyclic of order 2 and one of the following holds.*

   (i) $\mathbf{G}$ *is of type* A_n*,* $n \equiv 1 \pmod 4$ *and the defect groups of* $\bar{B}$ *are* $C_2 \times C_2$*.*

   (ii) $\mathbf{G}$ *is of type* E_7*, and the defect groups of* $\bar{B}$ *are* $C_2 \times C_2$*.*

PROOF. Suppose first that $\mathbf{G}$ is of type B_n, $(n \geq 2)$, C_n, $(n \geq 3)$ or D_n, $(n \geq 4)$. The identity is the only quasi-isolated element of odd order in $\mathbf{G}^*$, (see for example table 2 of [2]), hence $s = 1$ and B is unipotent. Further, the only unipotent 2-block in classical groups is the principal block [6] (for the case $G = {}^3D_4(q)$, see the relevant section of [12]). In particular, the defect groups of B are the Sylow 2-subgroups of G. Since the Sylow 2-subgroups of G/Z are not abelian for any central subgroup of G, neither B nor $\bar{B}$ has abelian defect groups.

Suppose that $\mathbf{G}$ is of type G_2, F_4, E_6 or E_8. Then G has either a trivial centre or a centre of order 3, so the hypothesis of (b) does not hold. By the lists in [12] and [22], one sees that the only possibility for (a) to hold is $s = 1$ and B of defect zero.

Suppose that $\mathbf{G}$ is of type E_7 and $s = 1$. By [12], the defect groups of B are non-abelian. If Z is cyclic of order 2, then $\bar{B}$ has abelian defect groups only if $\bar{B}$ corresponds to lines 3 or 7 of the table on page 354 of [12] in which case the defect groups of B are dihedral of order 8 and those of $\bar{B}$ are isomorphic to $C_2 \times C_2$. If $\mathbf{G}$ is of type E_7 and $s \neq 1$, then by [22], neither B nor $\bar{B}$ has abelian defect groups.

It remains to consider the case that $\mathbf{G}$ is of type A_n. Then $C^\circ_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$. Let $\mathbf{H}$ be an F-stable Levi subgroup of $\mathbf{G}$ in duality with $C^\circ_{\mathbf{G}^*}(s)$. By [13, Proposition 1.5], the Sylow 2-subgroups of $\mathbf{H}^F$ are defect groups of B. Suppose first that the Sylow 2-subgroups of $\mathbf{H}^F$ are abelian. Then by Lemma 3.1(i) applied to $\mathbf{H}$, $\mathbf{H}$ and hence $C^\circ_{\mathbf{G}^*}(s)$ are tori. Since s has odd order, Lemma 5.1 implies that $n = o(s) - 1$ is even. Now suppose that the Sylow 2-subgroups of $\mathbf{H}^F$ are non-abelian, and let Z be a central 2-subgroup of G such that $\mathbf{H}^F/Z$ has abelian Sylow 2-subgroups (note that since $\mathbf{H}$ is a Levi subgroup of $\mathbf{G}$, $Z \leq \mathbf{H}$). By Lemma 3.1(ii) all components of $\mathbf{H}$ and hence of $C^\circ_{\mathbf{G}^*}(s)$ are of type A_1. Thus, by Lemma 5.1, $n \equiv 1 \pmod 4$. Since $\bar{B}$ has abelian defect groups and B has non-abelian defect groups, by Proposition 3.4 and its proof, the defect groups of $\bar{B}$ have order 4. On the other hand, since $\mathbf{H}^F/Z$ has a subgroup isomorphic to $C_2 \times C_2$, the defect groups of $\bar{B}$ are isomorphic to $C_2 \times C_2$ as required. □

The notation $A_n(q)$, $D_n(q)$ etc. in the following proposition stands for the simply connected version of the groups in question.

Proposition 5.3 *Suppose that $\mathbb{F}$ has odd characteristic, $\mathbf{G}$ is simple and simply connected. Let B be a 2-block of $\mathbf{G}$ with semi-simple label s and defect group P. Suppose that $Z \leq O_2(Z(G))$ is such that P/Z is abelian and that $C^\circ_{\mathbf{G}^*}(s)$ is not a torus. Then, one of the following holds.*

(i) $\mathbf{G}$ *is of type* G_2, F_4, E_6 *or* E_8, B *is unipotent and* $P = 1$.

(ii) $\mathbf{G}$ *is of type* E_7, B *is quasi-isolated,* $Z \neq 1$ *and* P/Z *is a Klein* 4*-group.*

(iii) $\mathbf{G}$ *is of type* A_n, $G \cong A_n(q)$ *or* ${}^2A_n(q)$, $n+1 = 2t$, t *odd,* $q \equiv \pm 3 \pmod 8$, P *is quaternion of order* 8 *and* P/Z *is a Klein* 4*-group.*

(iv) $\mathbf{G}$ *is of type* E_7 *or* E_8 *and there exists an* F*-stable Levi subgroup* $\mathbf{L}$ *of* $\mathbf{G}$*, and a* 2*-block* C *of* $\mathbf{L}^F$ *such that* B *and* C *are Bonnafé-Rouquier correspondents,* $[\mathbf{L}, \mathbf{L}]^F \cong E_6(q)$ *or* ${}^2E_6(q)$ *and* $Z^\circ(\mathbf{L})^F$ *contains a defect group of* C.

(v) $\mathbf{G}$ *is of type* D_n, $G \cong D_n(q)$, $n = 2t$, t *odd, and* $q \equiv \pm 3 \pmod 8$ *or* $\mathbf{G}$ *is of type* E_7*. In these cases, there exists an* F*-stable Levi subgroup* $\mathbf{L}$ *of* $\mathbf{G}$*, and a* 2*-block* C *of* $\mathbf{L}^F$ *such that* B *and* C *are Bonnafé-Rouquier correspondents and the following holds:* $\mathbf{L}^F = Z^\circ(\mathbf{L})^F \times [\mathbf{L}, \mathbf{L}]^F$ *and* $[\mathbf{L}, \mathbf{L}]^F \cong SL_2(q^t)$*. Let* C_0 *(respectively* C_1*) be the* 2*-block of* $Z^\circ(\mathbf{L})^F$ *(respectively* $[\mathbf{L}, \mathbf{L}]^F$*) covered by* C *and let* P_i, $i = 0, 1$ *be a defect group of* C_i*. Then* C_1 *is the principal block of* $[\mathbf{L}, \mathbf{L}]^F$, P_1 *is quaternion of order* 8, $Z = (Z \cap P_0) \times (Z \cap P_1)$ *and* $Z \cap P_1 \neq 1$*. In particular,* $P_1/(Z \cap P_1)$ *is a Klein* 4*-group. Further, if* $\mathbf{G}$ *is of type* D_n*, then* P_0 *is cyclic of order* 2 *or* 4.

Proof. Suppose first that $\mathbf{G}$ is of classical type A_n, B_n, C_n, D_n. Since s has odd order and 2 is the only bad prime for classical groups, $C^\circ_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$ which is also F-stable since $s \in \mathbf{G}^{*F}$. Denote by $\mathbf{L}$ an F-stable Levi subgroup of $\mathbf{G}$ in duality with $C^\circ_{\mathbf{G}^*}(s)$. By [12, Prop. 1.5] we may assume that P is a Sylow 2-subgroup of $\mathbf{L}$. So, by Proposition 3.4, $\mathbf{G}$ is not of type B_n or C_n. If $\mathbf{G}$ is of type A_n, then by Proposition 3.4, case (iii) of the proposition holds.

Suppose that $\mathbf{G}$ is of type D_n. Then $Z(\mathbf{G})$ is a 2-group, hence $C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s)$ is a 2-group. On the other hand, the exponent of $C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s)$ divides the order of s which is odd. Hence, $C_{\mathbf{G}^*}(s) = C^\circ_{\mathbf{G}^*}(s) = \mathbf{L}^*$ and there is a 2-block of $\mathbf{L}^F$, say C, which is a Bonnafé-Rouquier correspondent of B. Moreover, since s is central in $\mathbf{L}^*$, there exists a linear representation $\hat{s}$ of $\mathbf{L}^F$ with the same order as s and a unipotent block C' of $\mathbf{L}^F$ such that the characters of C are of the form $\hat{s} \otimes \chi$, where χ is a character of C'. Since the principal block in a group of type A_n is the only unipotent block, it follows from Proposition 3.4(b) that case (v) of the proposition holds.

Thus, $\mathbf{G}$ is of the exceptional type. Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ and $\mathbf{L}^*$ a Levi subgroup of $\mathbf{G}^*$ in duality with $\mathbf{L}$ such that s is a quasi-isolated element of $\mathbf{L}^*$ and $C_{\mathbf{G}^*}(s) = C_{\mathbf{L}^*}(s)$. Let C be a 2-block of $\mathbf{L}^F$ in Bonnafé-Rouquier correspondence with B.

Suppose that all components of $\mathbf{L}$ are of type A. Then, by the same argument as in the beginning of the proof, $C^\circ_{\mathbf{G}^*}(s) = C^\circ_{\mathbf{L}^*}(s)$ is a Levi subgroup of $\mathbf{L}^*$, hence of $\mathbf{G}^*$ which is also F-stable. Let $\mathbf{H} \leq \mathbf{L} \leq \mathbf{G}$ be an F-stable Levi subgroup in duality with $C^\circ_{\mathbf{L}^*}(s)$. Again, by [12, Prop. 1.5], a Sylow 2-subgroup, say P' of $\mathbf{H}^F$ is a defect group of C. By [22, Theorem 1.3] $P'/Z \cong P/Z$ is abelian. By Proposition 3.1, $Z \neq 1$, hence $\mathbf{G}$ is of type E_7. In this case, $Z(\mathbf{G})$ is of order 2. So, by the same argument as for the case D_n above, $C_{\mathbf{G}^*}(s) = C^\circ_{\mathbf{L}^*}(s) = \mathbf{L}^*$. Now again by the same argument as given above for type D_n and using Proposition 3.4(c), we get that case (v) of the proposition holds.

We assume from now on that $\mathbf{G}$ is of exceptional type and $\mathbf{L}$ has a component which is not of type A. Let Δ be the set of components of $[\mathbf{L},\mathbf{L}]$ and let $Orb_F(\Delta)$ be the set of F-orbits of Δ. Then,

$$[\mathbf{L},\mathbf{L}] = \prod_{\delta\in Orb_F(\Delta)} \prod_{\mathbf{X}\in\delta} \mathbf{X}$$

so that

$$[\mathbf{L},\mathbf{L}]^F = \prod_{\delta\in Orb_F(\Delta)} (\prod_{\mathbf{X}\in\delta} \mathbf{X})^F.$$

Let E be a block of $[\mathbf{L},\mathbf{L}]^F$ covered by C and for each $\delta \in Orb_F(\Delta)$, let E_δ be the corresponding component block of $(\prod_{\mathbf{X}\in\delta}\mathbf{X})^F$ and D_δ a defect group of E_δ.

Let $\iota^* : \mathbf{L}^* \to [\mathbf{L},\mathbf{L}]^*$ be a map dual to the inclusion $[\mathbf{L},\mathbf{L}] \to \mathbf{L}$ chosen to be compatible with F. Then

$$[\mathbf{L},\mathbf{L}]^* = \prod_{\delta\in Orb_F(\Delta)} \prod_{\mathbf{X}\in\delta} \mathbf{X}^*$$

where for each $\delta \in Orb_F(\Delta)$, and $\mathbf{X} \in \delta$, $\mathbf{X}^*$ is dual to $\mathbf{X}$. Let $s \in \mathbf{L}^{*F}$ be a semi-simple label of C and let $\bar{s} = \prod_{\delta\in Orb_F(\Delta)} \prod_{\mathbf{X}\in\delta} \bar{s}_{\mathbf{X}}$ be the image of s under ι^*. Then for each $\delta \in Orb_F(\Delta)$, $\prod_{\mathbf{X}\in\delta} s_{\mathbf{X}}$ is a semi-simple label of E_δ. Since s is quasi-isolated in $\mathbf{G}^*$, $\bar{s}$ is quasi-isolated in $[\mathbf{L},\mathbf{L}]^F$ (see [2, Prop. 2.3]), and consequently, $\prod_{\mathbf{X}\in\Delta} s_{\mathbf{X}}$ is quasi-isolated in $\prod_{\mathbf{X}\in\delta}\mathbf{X}^*$. Further, if $\mathbf{X} \in \delta$, then through the isomorphism of $(\prod_{\mathbf{Y}\in\delta}\mathbf{Y})^F$ with $\mathbf{X}^{F^{|\delta|}}$ induced by projection onto $\mathbf{X}$, E_δ is identified with a block of $\mathbf{X}^{F^{|\delta|}}$ with quasi-isolated semi-simple label $s_{\mathbf{X}}$.

Let $\delta \in Orb_F(\Delta)$ and $\mathbf{X} \in \delta$ be such that $\mathbf{X}$ is not of type A_n. By Lemma 5.2, $\mathbf{X}$ is not of type B_n, C_n or D_n. If $\mathbf{X}$ is of type G_2 or F_4, then $\mathbf{L} = \mathbf{G}$, $Z = 1$ and by Lemma 5.2, B is unipotent, and $P = 1$, so (i) of the proposition holds.

Suppose $\mathbf{X}$ is of type E_6. Then $\mathbf{G}$ is of type E_6, E_7 or E_8 and by rank considerations, $\delta = \{\mathbf{X}\}$. Further, $Z \cap X = 1$. So D_δ is abelian and by Lemma 5.2 , E_δ is a unipotent block of defect 0. If $\mathbf{G}$ is of type E_6, then $\mathbf{G} = \mathbf{L} = \mathbf{X}$, whence B is a unipotent block with trivial defect groups, that is (i) holds. Suppose $\mathbf{G}$ is of type E_7. By rank considerations, $\mathbf{L} = Z^\circ(\mathbf{L})\mathbf{X}$. Let $M = Z^\circ(\mathbf{L})^F\mathbf{X}^F$. Then M is a normal subgroup of $\mathbf{L}^F$ and $\mathbf{L}^F/M$ is abelian of order $|Z^\circ(\mathbf{L})^F \cap \mathbf{X}^F|$. In particular $[\mathbf{L}^F : M]$ is 1 or 3, which means that $D \leq M$. Since $D_\delta = 1$, and since D is a defect group of a block of M covered by C and $Z^\circ(\mathbf{L})^F$ and $\mathbf{X}^F$ commute with each other, we may assume that $D \leq Z^\circ(\mathbf{L})^F$. So (iv) of the proposition holds.

Now suppose that $\mathbf{G}$ is of type E_8. Either $\mathbf{L} = Z^\circ(\mathbf{L})\mathbf{X}$ or $[\mathbf{L},\mathbf{L}] = \mathbf{Y} \times \mathbf{X}$, where $\mathbf{Y}$ is of type A_1. In the former case, we argue as above to conclude that case (iv) holds. Since $Z = 1$, by Lemma 5.2, $\mathbf{L}$ has no component of type A_1. This rules out the latter case.

Suppose that $\mathbf{X}$ is of type E_7. Then $\mathbf{G}$ is of type E_7 or of type E_8 and δ consists of a single component. Suppose $\mathbf{G}$ is of type E_7. Then $\mathbf{L} = \mathbf{G}$ and hence by Lemma 5.2, B is quasi-isolated and $P/Z \cong C_2 \times C_2$, that is case (ii) holds. The case that $\mathbf{G} = E_8$ is ruled out by Lemma 5.2 since if $\mathbf{G}$ is of type E_8, then $Z = 1$.

Finally, suppose that $\mathbf{X}$ is of type E_8. Then, $\mathbf{G} = \mathbf{L}$ is of type E_8 and by Lemma 5.2, B is unipotent and P is trivial, so case (i) holds. □

Remark. Some special cases of the above theorem are given in [20, Theorem 1.3]. We point out that (iii) of [20, Theorem 1.3] is not correct for the classical groups of type $D_n(q)$ and $r = 3$: the above analysis shows that if $n = 2t$, $q \equiv -3 \pmod 8$, then there do exist non-nilpotent 2-blocks with elementary abelian defect groups of order 8 in the simple group of type $D_n(q)$. This does not pose a problem for the conclusion of Theorem 1.1 of [20] as by part (ii) of the above proposition such a block has 3 simple modules, and the conclusion of Theorem 1.1 of [20] for groups of type $D_n(q)$ follows by Theorem 5.1 and Proposition 3.2 of [20].

We record the following for later use.

Proposition 5.4 *Suppose that $\mathbb{F}$ has odd characteristic, $\mathbf{G}$ is simple and simply connected. Let B be a 2-block of G with semi-simple label s and defect group P. Suppose that $Z \leq O_2(Z(G))$ is such that P/Z is abelian and that $C^\circ_{\mathbf{G}^*}(s)$ is a torus, but $C_{\mathbf{G}^*}(s)$ is not connected. Then, $\mathbf{G}$ is of type A_n or E_6. Further, if $\mathbf{G}$ is of type E_6, then there exists an F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ and a 2-block C of $\mathbf{L}^F$ in Bonnafé-Rouquier correspondence with B such that $[\mathbf{L}, \mathbf{L}]^F$ is a direct product of at most two groups of type A_2. In particular, if $\mathbf{G}$ is of type E_6 (so that $Z = 1$) and P is elementary abelian of order 16 then a defect group P' of C has the form $P' = P_1 \times P_2$, $P_1 \cong P_2 \cong C_2 \times C_2$ and both P_1 and P_2 are invariant under $N_{\mathbf{L}^F}(P')$.*

Proof. We freely use the notation of the proof of Proposition 5.3. Let $\mathbf{L}$ be a Levi subgroup of $\mathbf{G}$ in duality with an F-stable Levi subgroup of $\mathbf{G}^*$ containing $C^*_{\mathbf{G}}(s)$ and such that s is quasi-isolated in $\mathbf{L}$. The exponent of $C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s) = C_{\mathbf{L}^*}(s)/C^\circ_{\mathbf{L}^*}(s)$ is a divisor of the order of s (see [10, 13.15]) and also $C_{\mathbf{L}^*}(s)/C^\circ_{\mathbf{L}^*}(s)$ is isomorphic to a subgroup of $Z(\mathbf{L})/Z^\circ(\mathbf{L}) \leq Z(\mathbf{G})/Z^\circ(\mathbf{G})$. Since s has odd order and $Z(\mathbf{G})$ is a 2-group unless $\mathbf{G}$ is of type A_n or E_6, the first assertion follows.

Now, suppose that $\mathbf{G}$ is of type E_6, so $C_{\mathbf{L}^*}(s)/C^\circ_{\mathbf{L}^*}(s) = Z(\mathbf{L})/Z^\circ(\mathbf{L})$ has order 3 and 3 divides the order of s. By [2, Table 3], $\mathbf{G}^* \neq \mathbf{L}^*$. So, $\mathbf{L}$ is a proper Levi subgroup of $\mathbf{G}$ and in particular has semi-simple rank at most 5. Let X be a component of $[\mathbf{L}, \mathbf{L}]$. Then, $s_{\mathbf{X}}$ is a quasi-isolated element of $\mathbf{X}^*$, so as explained in the proof of Proposition 5.3, $\mathbf{X}$ is not of classical type D_n. Thus, $\mathbf{X}$ is of type A_n. By Lemma 5.1, $n+1 = o(s_{\mathbf{X}})$. Since $1 \leq n \leq 5$ and $o(s_{\mathbf{X}})$ is odd and a multiple of 3, it follows that $n = 2$.

Hence all components of $[\mathbf{L}, \mathbf{L}]$ are of type A_2 and there are at most two components. Since the 2-rank of special linear or unitary groups in odd dimension is 2, the final assertion is immediate from the structure of $\mathbf{L}^F$. □

6 Structure Theorem

Our aim in this section is to prove the following.

Theorem 6.1 *Let $\ell = 2$ and let G be a quasi-simple group. If B is a block of $\mathcal{O}G$ with abelian defect group P, then one (or more) of the following holds:*

(i) $G/Z(G)$ is one of $A_1(2^a)$, ${}^2G_2(q)$ (where $q \geq 27$ is a power of 3 with odd exponent), or J_1, B is the principal block and P is elementary abelian.

(ii) G is Co_3, B is a non-principal block, $P \cong C_2 \times C_2 \times C_2$ (there is one such block).

(iii) There exists a finite group $\tilde{G}$ such that $G \trianglelefteq \tilde{G}$, $Z(G) \le Z(\tilde{G})$ and such that B is covered by a nilpotent block of $\tilde{G}$.

(iv) B is Morita equivalent to a block C of $\mathcal{O}L$ where $L = L_0 \times L_1$ is a subgroup of G such that the following holds: The defect groups of C are isomorphic to P, L_0 is abelian and the block of $\mathcal{O}L_1$ covered by C has Klein 4-defect groups. In particular, B is Morita equivalent to a tensor product of a nilpotent block and a block with Klein 4-defect groups.

Proof. If P is central in G, then B is nilpotent and we are in case (iii). Hence we may assume that P is non-central. We consider first the case $|P| \le 8$ or P is cyclic. If P is cyclic or if $P \cong C_4 \times C_2$, then B is again nilpotent, hence we are in the situation of (iii). If P is a Klein 4-group, then we are in case (iv) (with $L_1 = L = G$). Thus, we may assume that P is a non-cyclic group of order at least 8 and that if $|P| = 8$, then P is elementary abelian.

Write $\bar{G} := G/Z(G)$. Suppose $\bar{G}$ is a sporadic group. If B is the principal block, then $\bar{G} = G = J_1$ and P is elementary abelian of order 8, hence we are in case (i). The non-principal 2-blocks of quasi-simple groups with sporadic quotient are listed in [29], and the only possibility with $|P| \ge 8$ is $\bar{G} = G = Co_3$ and P elementary abelian of order 8. Thus, B is as in (ii).

Suppose $\bar{G} := G/Z(G)$ is an alternating group A_n, $n \ge 5$. The trivial group and $C_2 \times C_2$ are the only abelian 2-groups occurring as defect groups of 2-blocks of $\bar{G}$, hence $G \ne \bar{G}$. If G is a double cover of $\bar{G}$, then the lift of a Klein 4-block of $\bar{G}$ has non-abelian defect groups of order 8 and the lift of a defect zero block has central defect groups. Suppose that G is an exceptional cover of A_n. If $\bar{G} = A_6, A_7$ and G is a 3- or 6-fold covering of $\bar{G}$, then the Sylow 2-subgroups of $G/Z(G)$ are non-abelian of order 8 and consequently G has no blocks with P as defect group for which we have not accounted.

Suppose $\bar{G}$ is a finite group of Lie type in characteristic 2 not isomorphic to any of ${}^2F_4(2)'$, $B_2(2)'$ or $G_2(2)'$ or $PSp_4(2)'$. Then by [9, Proposition 8.7], the non-trivial defect groups of 2-blocks are Sylow 2-subgroups of G. The only possibility is $\bar{G} = G = SL_2(2^a)$ and B the principal block and we are in case (i).

If $\bar{G}$ is the Tits group ${}^2F_4(2)'$, then $\bar{G} = G$ and by [15], G has precisely three blocks, two of defect 0 and the principal block. If $\bar{G}$ is isomorphic to $PSp_4(2)'$, then $\bar{G} \cong A_6$, a case we have already considered. If $\bar{G} \cong G_2(2)'$, then $\bar{G} \cong {}^2A_2(3)$, hence $\bar{G} = G \cong {}^2A_2(3)$, a case which will be handled below. If $\bar{G}$ is isomorphic to $PSL_2(4)$, then $\bar{G} \cong A_5$ a case that has been handled above.

Suppose that $\bar{G}$ is a finite group of Lie type in odd characteristic. By [15] no faithful 2-block of an exceptional covering of $\bar{G}$ has non-central defect groups which are abelian. Hence we may assume that G is a non-exceptional covering of $\bar{G}$. So we have $G = \mathbf{G}^F/Z$, where $\mathbf{G}$ is a simple and simply-connected group defined over an algebraic closure of the field of p elements for p an odd prime, $F : \mathbf{G} \to \mathbf{G}$ is a Steinberg endomorphism, and $Z \le Z(\mathbf{G}^F)$. By replacing G by a suitable central extension and B by a suitable (and Morita equivalent) lift with isomorphic defect groups if necessary, we may assume that Z is a 2-group. Let $\hat{B}$ be a 2-block of $\mathbf{G}^F$ lifting B. Then $\hat{B}$ has a defect group $\hat{P}$ such that $Z \le \hat{P}$ and $\hat{P}/Z = P$.

If $\bar{G} = {}^2G_2(q)$, q a power of 3, then, $\bar{G} = G$ and the Sylow 2-subgroups of G are elementary abelian of order 8. Further, the principal block is the unique 2-block of G with maximal defect groups (see for instance [20, Proposition 15.2]), and so we are in case (i).

Hence we may assume that $\mathbf{G}^F$ is not ${}^2G_2(q)$. Suppose first that a semi-simple label of $\hat{B}$ has a connected centraliser which is a torus. Then by Lemma 4.2, there exists a finite group H and a nilpotent block $\hat{E}$ of H covering $\hat{B}$ such that $Z \leq Z(\mathbf{G}^F) \leq Z(H)$. Let $\tilde{B}$ be the block of $\tilde{G} := H/Z$ corresponding to $\hat{E}$. Then $\tilde{B}$ covers B and $\tilde{B}$ is nilpotent. Since $\mathbf{G}^F$ is quasi-simple, $Z(G) = Z(\mathbf{G}^F)/Z \leq Z(H)/Z \leq Z(\tilde{G})$. Hence we are in case (iii).

So, we may assume that the connected centraliser of a semi-simple label of $\hat{B}$ is not a torus. We apply Lemma 5.3 to $\hat{B}$. If $\hat{B}$ is as in case (ii) or (iii) of Lemma 5.3, then case (iv) of the theorem holds. If $\hat{B}$ is as in case (i) or (iv) of Lemma 5.3, then $\hat{B}$ is nilpotent, hence so is B by [32, 8.2] and we are in case (iii) of the theorem. Suppose $\hat{B}$ is as in case (v) of Lemma 5.3. Let $\mathbf{L}$ and $\hat{C}$ be as in Lemma 5.3 (replacing C by $\hat{C}$) and let C be the block of $L := \mathbf{L}^F/Z$ corresponding to $\hat{C}$. By Lemma 5.3, L is a direct product of $L_0 := Z^\circ(\mathbf{L})/(Z \cap P_0)$ and $L_1 := [\mathbf{L}, \mathbf{L}]^F/(Z \cap P_1)$ and the 2-block of L_1 covered by C has Klein 4-defect groups. By Lemma 2.7, B and C are Morita equivalent. The defect groups of B and C are isomorphic by [22, Theorem 1.3]. So, we are in case (iv) of the theorem. □

Remark. Whenever case (iv) of the above theorem holds, then setting $W = O_2(Z(G))$, we have $W \leq \mathbf{L}^F/Z$ and by the proof of Lemma 2.7, the Morita equivalence between B and C may be realised by a bimodule on which $\Delta(W)$ acts trivially.

The principal 2-blocks of the ${}^2G_2(q)$ all have elementary abelian defect groups of order 8 and by [27] they all have the same decomposition matrices. It seems to be folklore that they are all Morita equivalent, but we are unable to find a reference for this. We can however easily show that there are only finitely many Morita equivalence classes amongst them:

Lemma 6.2 *Consider the groups ${}^2G_2(3^{2m+1})$ for $m \geq 1$. There are only finitely many Morita equivalence classes of blocks amongst the principal 2-blocks of these groups.*

Proof. By [27] the principal 2-blocks all have the same decomposition matrices. Write B_m for the principal 2-block of ${}^2G_2(3^{2m+1})$ and let fB_mf be an associated basic algebra. By the proof of [19, 1.4], there is an $\mathbb{F}_2$-algebra A_m such that $fB_mf \cong k \otimes_{\mathbb{F}_2} A_m$. Now A_m has dimension equal to the sum of the entries of the Cartan matrix, i.e., 76. Hence $|A_m| = 2^{76}$, and so there are only finitely many possibilities for A_m, and hence for the Morita equivalence class of B_m. □

7 Proof of Theorem 1.1

Let us keep the notation of Theorem 1.1. In particular, we suppose that D is an abelian 2-group of rank 2, so that D is isomorphic to a direct product $C_{2^m} \times C_{2^n}$ of two cyclic subgroups C_{2^m} and C_{2^n}. Write b for the unique block of $\mathcal{O}N_G(D)$ with Brauer correspondent B. The following facts are known:

- If $m \neq n$, then $\mathrm{Aut}(D)$ is a 2-group. Thus B is a nilpotent block. By the main result of [30], B is then Morita equivalent to the group algebra $\mathcal{O}D$.

- If $m = n = 1$, then D is a Klein 4-group. This case was completed by Erdmann in [14] for blocks defined over k, and extended to blocks defined over $\mathcal{O}$ by Linckelmann in [28], where it is proved that B is Morita equivalent to the group algebra $\mathcal{O}D$, to the group algebra $\mathcal{O}A_4$ or to the principal block of the group algebra $\mathcal{O}A_5$. (Here, A_n denotes the alternating group of degree n.) Therefore, in the following we will assume that $m = n > 1$.

- If $D \in \mathrm{Syl}_2(G)$ (and $m = n > 1$), then G is solvable by a theorem of Brauer (Theorem 1 of [5]).

- In general, B is perfectly isometric to b. This is stated without explicit proof in Remark 1.6 of [35], and a proof is given in Satz 3.3 of [36]. We note that this result does not use the classification of finite simple groups. Consequently, if $m = n > 1$, then B is either nilpotent (in which case $l(B) = 1$ and $k(B) = |D|$) or $l(B) = 3$ and $k(B) = (|D| + 8)/3$.

The following are well-known, but we state them here for convenience:

Lemma 7.1 *Let $D = C_{2^m} \times C_{2^m}$. Then $\mathrm{Aut}(D)$ is a $\{2,3\}$-group, where $|\mathrm{Aut}(D)|_3 = 3$. If $\theta \in \mathrm{Aut}(D)$ has order three, then θ transitively permutes the three involutions in D.*

Let G be a finite group such that $D \lhd G$. Let B be a block of $\mathcal{O}G$ with defect group D and let b_D be a block of $\mathcal{O}C_G(D)$ with $(b_D)^G = B$. Write $N_G(D, b_D)$ for the stabiliser of b_D in $N_G(D)$ and $E_B = N_G(D, b_D)/C_G(D)$. Then B is Morita equivalent to the group algebra $\mathcal{O}(D \rtimes E_B)$.

If $O_2(Z(G)) \neq 1$, then B is nilpotent.

We now give the proof of Theorem 1.1.

Proof. Let B be a counterexample with $(|G : Z(G)|, |G|)$ minimised in the lexicographic ordering. By the first Fong reduction and minimality, B is quasi-primitive, that is, for any normal subgroup N of G, there is a unique block of N covered by B. By the second Fong reduction and minimality, $O_{2'}(G)$ is cyclic and central in G.

Now suppose that $N := O_2(G) \neq 1$, so that $N \subseteq D$. Then B covers a unique block B_C of $C := C_G(N)$, and B_C has defect group D. Since $1 \neq N \leq D \cap Z(C)$, the block B_C is nilpotent. Thus by Proposition 2.2 and minimality, C is nilpotent. Since G/C is isomorphic to a subgroup of $\mathrm{Aut}(N)$, which is a $\{2,3\}$-group, G is solvable. But we have $O_{2'}(G) \leq Z(G)$, so $C_G(N) \leq NZ(G)$, and $D = N$ since D is abelian. Hence by Lemma 7.1 we have a contradiction to minimality.

Hence $O_2(G) = 1$, so that $F(G) = O_{2'}(G) = Z(G) =: Z$. Let $N \lhd G$ such that N/Z is a minimal normal subgroup of G/Z, and let B_N be the unique block of N covered by B. If B_N is nilpotent, then we can use Proposition 2.2 again to obtain a contradiction to minimality. Thus we may assume that B_N is not nilpotent, and in particular the defect group $D \cap N$ of B_N is nontrivial. Moreover, we have $D \cap N \cong C_{2^t} \times C_{2^t}$ for some $t \leq m$. This implies that N/Z is the only minimal normal subgroup of G/Z, and so $N = F^*(G)$ (the generalised Fitting subgroup).

Assume next that G has a normal subgroup K of index 2. Let B_K be the unique block of K covered by B. Then B_K is G-stable, and B is the only block of G covering B_K. Moreover, $D \cap K$ is a defect group of B_K and $DK/K \in \mathrm{Syl}_2(G/K)$, so $G = DK$. Hence $2 = |G/K| = |DK/K| = |D/D \cap K|$. This implies that $D \cap K$ is a direct product of two non-isomorphic cyclic groups. Hence $\mathrm{Aut}(D \cap K)$ is a 2-group, and so B_K is nilpotent. By Proposition 2.2 and minimality, K is nilpotent. Then G is solvable, a contradiction since $O_2(G) = 1$ and $O_{2'}(G) \leq Z(G)$.

Hence $G = O^2(G)$. Let $L_1, \ldots, L_t$ denote the components of G. We have seen that these are permuted transitively by G, and $L := L_1 * \cdots * L_t \lhd G$. Let B_L be the unique block of L covered by B, and let B_i be the unique block of L_i covered by B_L ($i = 1, \ldots, t$). Then B_L has defect group $D \cap L$, and B_i has defect group $D \cap L_i$ ($i = 1, \ldots, t$). Thus $D \cap L = (D \cap L_1) \times \cdots \times (D \cap L_t)$, where $D \cap L_1, \ldots, D \cap L_t$ are conjugate in G (since $B_1, \ldots, B_t$ are). This implies that $t \leq 2$. Since $G = O^2(G)$, we must have $t = 1$ (by consideration of the kernel of the permutation action). This shows that G has a unique component L, so that the layer $E(G) = L$ is quasi-simple and $F^*(G) = L * Z$.

Suppose that $G \neq L$. By the Schreier Conjecture G/L is solvable. Since $G = O^2(G)$, it follows that there is a normal subgroup N of G such that $|G : N| = w$ for some odd prime w. Let B_N be the unique block of N covered by B. Now B_N is G-stable and has defect group D. Suppose that B is the unique block of G covering B_N. Now B_N has either 1 or 3 irreducible Brauer characters, according to whether B_N has inertial index 1 or 3 respectively. If the inertial index is 1, then B_N is nilpotent, and Proposition 2.2 and minimality may be applied to obtain a contradiction. Hence $l(B_N) = 3$. Similarly $l(B) = 3$, since if $l(B) = 1$, then B is nilpotent. If $w > 3$, then each irreducible Brauer character in B_N must be G-stable, and applying Clifford theory (noting that each simple module extends to G since G/N is cyclic and of odd order) we obtain $l(B) = 3w > 3$, a contradiction. If $w = 3$, then the irreducible Brauer characters in B_N are either all G-stable or are permuted transitively. If they are all G-stable, then as above we have $l(B) = 3w = 9$, a contradiction. Hence the three irreducible Brauer characters in B_N are permuted transitively and by Clifford theory induce to an irreducible Brauer character, which must lie in B, and further we must have $l(B) = 1$, a contradiction. Hence B is not the unique block of G covering B_N. In this case, since w is an odd prime, every irreducible Brauer character in B_N is G-stable, and it follows by Clifford theory (again using the fact that G/N is a cyclic $2'$-group) that B_N is covered by w blocks of G and that each of the three irreducible Brauer characters in B is an extension of a distinct irreducible Brauer character in B_N. Hence we have a bijection given by restriction between the irreducible Brauer characters of B and those of B_N, and by [18, 4.1] B and B_N are Morita equivalent. This gives a contradiction to minimality.

A final application of the second Fong correspondence and minimality show that $Z(G) \leq [G, G]$. Hence we have shown that $G = L$, i.e., G is quasi-simple, and that $Z(G)$ is cyclic of odd order. Proposition 6.1 and Proposition 2.4 give an immediate contradiction. □

Corollary 7.2 *Let $\ell = 2$, G a finite group and let B be a block of $\mathcal{O}G$ with abelian defect group D of rank 2. If D is homocyclic of order at least 16, then there are precisely* *two Morita equivalence classes of blocks with defect group D. If D is a Klein 4-group, then there are precisely three Morita equivalence classes, and if D is not homocyclic, then B must be nilpotent.*

Corollary 7.3 *Let D be an abelian 2-group of rank 2. Then Donovan's conjecture holds for D.*

8 Donovan's conjecture for 2-blocks with elementary abelian defect groups

In this section, by an ℓ-block of a finite group G, we will mean a block of kG. The following is immediate from [11, 1.11] and Proposition 2.2:

Proposition 8.1 *Let P be an abelian ℓ-group for a prime ℓ. In order to verify Donovan's conjecture for P, it suffices to verify that there are only a finite number of Morita equivalence classes of quasi-primitive blocks B with defect group $D \cong P$ of finite groups G satisfying the following conditions:*

(i) $F(G) = Z(G)$,

(ii) $O_{\ell'}(G) \leq [G,G]$,

(iii) $G = \langle D^g : g \in G \rangle$,

(iv) every component of G is normal in G,

(v) if $N \leq G$ is a component, then $N \cap D \neq Z(N) \cap D$,

(vi) if $N \lhd G$ and B covers a nilpotent block of N, then $N \leq Z(G)$.

Applying Proposition 2.6 we can reduce further to:

Corollary 8.2 *Let P be an elementary abelian ℓ-group for a prime ℓ. In order to verify Donovan's conjecture for P, it suffices to verify that there are only a finite number of Morita equivalence classes of blocks B with defect group $D \cong P$ of finite groups G satisfying the following conditions:*

(i) G is a central product $G_1 * \cdots * G_t$ of quasi-simple groups;

(ii) the block b_i of G_i covered by B is not nilpotent.

Proof. It suffices to consider groups G of the form given in Proposition 8.1. If G has this form, then by the Schreier conjecture there is $N \lhd G$ with N a central product of quasi-simple groups and G/N solvable. By condition (iii) in Proposition 8.1 we have $O^{\ell'}(G) = G$. By Proposition 2.6 we may assume that $O^{\ell}(G) = G$. Hence, since G/N is solvable, we may assume that $G = N$ and the result follows from the conditions listed in Proposition 8.1. □

Theorem 8.3 *Donovan's conjecture holds for elementary abelian 2-groups.*

Proof. Let P be an elementary abelian 2-group.

We may assume initially that we have a block B of a group G as in Corollary 8.2, so that G is a central product $G_1 * \cdots * G_t$ of quasi-simple groups. By taking an appropriate central extension of G by a group of odd order, we may assume that $G = E/Z$, where $E \cong G_1 \times \cdots \times G_t$ and $Z \leq Z(G)$ is a 2-group. Write $Z_i = O_2(Z(G_i))$.

Let B_i be the (unique) block of G_i covered by B. Note that B_i has elementary abelian defect groups, and no B_i is nilpotent. Let D be a defect group for B (with $D \cong P$). Then $D_i := D \cap G_i$ is a defect group for B_i. Since B_i is not nilpotent, $|D_i/Z_i| > 2$,

Let B_E be the unique block of E corresponding to B. Then $B_E \cong B_1 \otimes \cdots \otimes B_t$ and B_E has defect group $D_E \cong D_1 \times \cdots \times D_t$ (in particular D_E is elementary abelian), with $D_E/Z \cong D$.

Then, B_i and G_i belong to one (or more) of the classes (i)–(iv) in Theorem 6.1. We will define a new group H containing a copy of Z and a block C of H such that C is Morita equivalent to B_E via a bimodule satisfying the conditions in Lemma 2.7.

Suppose first that G_i satisfies (i), (ii) or (iii) of Theorem 6.1. Then write $H_i = G_i$ and $C_i = B_i$.

Suppose B_i and G_i are as in (iv) of Theorem 6.1. Then there is a Morita equivalence with a block C_i of a finite group H_i containing Z_i, such that $H_i \cong A_i \times L_i$ where A_i is abelian and C_i covers a block of L_i with Klein 4-defect groups. Further this Morita equivalence is realised by a bimodule M_i such that $z_i m_i = m_i z_i$ for all $z_i \in Z_i$ and all $m_i \in M_i$.

in case (iv) of Theorem 6.1.

We now have a Morita equivalence satisfying the conditions of Lemma 2.7 from B_E to a block C of the direct product $H = H_1 \times \cdots \times H_t$, where C covers the block C_i of H_i, $Z \leq Z(H)$, and C_i is as in (i)–(iv). By Lemma 2.7 this gives a Morita equivalence between B and the unique block $C_{H/Z}$ of H/Z corresponding to C. Thus it suffices to assume that $H = E$ and $B_E = C$.

Relabelling as necessary to account for the direct product in case (iv), and noting that blocks of abelian groups are necessarily nilpotent, we may write H as a direct product of groups H_i, with block C_i of H_i covered by C satisfying one or more of the following:

(a) C_i is a nilpotent-covered block;

(b) C_i has defect groups $C_2 \times C_2$ and $O_2(Z(H_i)) = 1$;

(c) H_i and C_i are as in (i) or (ii) of Theorem 6.1.

Note that we may assume $O_2(Z(H_i)) = 1$ in case (b), since otherwise C_i is nilpotent and so belongs to case (a). By checking in [8] we see that in case (c) $Z(H_i)$ has odd order.

It follows that Z is contained in the direct product of factors of type (a), i.e., we may express G as a direct product $(U/Z) \times V \times W$, where U is a direct product of groups satisfying (a), V is a direct product of groups satisfying (b), and W is a direct product of groups satisfying (c). Further $B \cong C_{U/Z} \otimes C_V \otimes C_W$, where C_U, C_V, C_W

are the blocks of U, V, W resp. covered by C, and $C_{U/Z}$ is the unique block of U/Z corresponding to C_U.

Now a tensor product of nilpotent-covered blocks is nilpotent-covered, and by Lemma 2.5 a quotient of such a block by a central 2-subgroup is also nilpotent-covered. Hence $C_{U/Z}$ is nilpotent-covered. So by Proposition 2.4, $C_{U/Z}$ is Morita equivalent to a block with normal elementary abelian defect group, of which there are only finitely many possibilities for Morita equivalence classes.

C_V is a tensor product of a bounded number of blocks with Klein 4-defect groups, and so there are only a finite number of possibilities for the Morita equivalence class of C_V.

By Lemma 6.2 there are only a finite number of Morita equivalence classes of blocks of groups satisfying (i) and (ii) of Theorem 6.1 with elementary abelian defect groups of order at most $|P|$, and of course the number of factors in W is bounded in terms of $|P|$, hence there are only finitely many possibilities for the Morita equivalence class of C_W.

Since $B \cong C_{U/Z} \otimes C_V \otimes C_W$, the result follows. $\square$

9 On the weak Donovan conjecture for abelian 2-blocks

A weak version of Donovan's conjecture is the following.

Conjecture 9.1 *Let D be a finite ℓ-group. There is a bound on the Cartan invariants of blocks of finite groups with defect group D which depends only on D.*

Theorem 9.2 *The weak Donovan conjecture holds for 2-blocks with abelian defect groups.*

PROOF. By [11, Theorem 3.2], it suffices to prove that the weak Donovan conjecture holds for all abelian defect 2-blocks of quasi-simple groups (note that the reductions employed in [11] all work in the realm of abelian defect groups). By [27, Theorem 3.9] the Cartan invariants of the principal 2-blocks of the Ree groups are at most 8. The proof follows by Theorem 6.1. $\square$

10 On numerical invariants for 2-blocks with elementary abelian defect groups of order 16

It is shown in [26] that if a block B has elementary abelian defect groups of order 16, then $k(B)$ is either 8 or 16, and that in all but one case, $k(B)$, $k_0(B)$ and $l(B)$ are determined given the action of the inertial quotient on D and certain cocycles. It remains to show that if the inertial quotient has order 15, then $k(B) = 16$ (and consequently $l(B) = 15$, $k_0(B) = 16$).

Lemma 10.1 *Let B be a block of a finite group G with elementary abelian defect group D of order 16 and inertial index 15. Let $N \lhd G$ have odd prime index, and let b be a G-stable block of N covered by B. Let B_D be a block of $C_G(D)$ with $(B_D)^G = B$ and b_D be a block of $C_N(D)$ with $(b_D)^G = B$.*

(i) If $C_G(D) \leq N$, then $N_N(D, b_D)/C_N(D) \leq N_G(D, B_D)/C_G(D)$ and $|N_G(D, B_D) : N_N(D, b_D)|$ divides $|G : N|$.

(ii) If $C_G(D) \not\leq N$, then $N_N(D, b_D)/C_N(D) \cong N_G(D, B_D)/C_G(D)$.

Either B is the unique block of G covering b, or there are $|G : N|$ blocks covering b.

Proof. Part (i) is immediate.

(ii) Recall that the blocks of $C_N(D)$ with defect group D correspond to inflations of irreducible characters in blocks of defect zero of $C_N(D)/D$ (and similarly for $C_G(D)$). Let θ be the canonical character of b_D. Suppose that $C_G(D) \not\leq N$. So $|C_G(D) : C_N(D)| \neq 1$.

Consider first the case $C_G(D) \leq N_G(D, b_D)$. Then θ extends to $|G : N|$ irreducible characters of $C_G(D)$, each inflated from a block of defect zero of $C_G(D)/D$. Hence b_D is covered by $|G : N|$ blocks of $C_G(D)$. Let θ_1 be the irreducible character of B_D extending θ. We have $N_G(D, B_D) \leq N_G(D, b_D)$, so $|N_N(D, b_D) : C_N(D)| \geq |N_G(D, B_D) : C_G(D)|$. Since C_{15} is a maximal subgroup of odd order of $GL_4(2)$, it follows that $|N_N(D, b_D) : C_N(D)| = |N_G(D, B_D) : C_G(D)|$. The same is true for each block of $C_G(D)$ covering b_D. It follows that $N_G(D)$ possesses $|G : N|$ blocks covering $(b_D)^{N_N(D)}$, and so G possesses $|G : N|$ blocks covering b by [17].

Now consider the case $C_G(D)$ is not in $N_G(D, b_D)$. Then B_D covers $|G : N|$ many $C_G(D)$-conjugates of b_D. We have $|N_G(D, B_D) : N_G(D, b_D)| = |G : N|$, so $|N_N(D, b_D) : C_N(D)| = |N_G(D, B_D) : C_G(D)|$. The same is true for each block of $C_N(D)$ covered by B_D. It follows that $N_N(D)$ possesses $|G : N|$ blocks covered by $(B_D)^{N_G(D)}$, and so N possesses $|G : N|$ blocks covered by b by [17], contradicting the G-stability of b. $\square$

When it comes to reducing the desired result to the consideration of quasi-simple groups, we will be unable to rule out the case that we have a quasi-simple normal subgroup of index 3. We must consider this situation in more detail. This is the object of the next results.

For a group G and a subgroup X of G, denote by $\mathrm{Aut}(G)_X$ the subgroup of $\mathrm{Aut}(G)$ which leaves X invariant. Denote by $\overline{\mathrm{Aut}(G)_X}$ the image of $\mathrm{Aut}(G)_X$ in $\mathrm{Aut}(X)$ through the restriction map. Denote by $\mathrm{Aut}_G(X)$ the subgroup of $\mathrm{Aut}(X)$ of automorphisms induced by conjugation by elements of G. So $\mathrm{Aut}_G(X)$ is naturally isomorphic to $N_G(X)/C_G(X)$.

Proposition 10.2 *Let q be an odd prime power and n a natural number. Let $G = SL_n(q)/Z_0$ (respectively $SU_n(q)/Z_0$), where Z_0 is a central subgroup of $SL_n(q)$ (respectively $SU_n(q)$) and suppose that $Z(G) = O_{2'}(G)$. Let $P \leq G$ be a defect group of a 2-block of G and suppose that P is elementary abelian of order $2^t \geq 8$. Set $u = t + 2$ if n is even and $u = t + 1$ if n is odd. Then $\overline{\mathrm{Aut}(G)_P} = \mathrm{Aut}_G(P)$ and $\mathrm{Aut}_G(P)$ is a subquotient of S_u, where S_u denotes the symmetric group on u letters.*

Proof. Let us first consider the case that $G = SL_n(q)/Z_0$. By the statements and proofs of [20, Lemmas 12.4, 13.4], t is even, $q \equiv -3 \pmod 8$. Moreover, replacing P by a G-conjugate if necessary, P has the following structure: Consider $GL_n(q)$ in its natural matrix representation. There is a decomposition

$$n = n_1 + \cdots + n_u,$$

into odd natural numbers n_i such that denoting by a_i the diagonal element of $GL_n(q)$ with entry -1 in positions $n_1 + \cdots + n_{i-1} + 1, \ldots, n_1 + \cdots + n_i$ and entry 1 elsewhere, and by T_0 the subgroup of $GL_n(q)$ generated by the elements $a_i a_j$, $1 \leq i, j \leq u$, $P = (T_0 Z_0)/Z_0$ (here n_0 is to be taken to be 0).

Let $\sigma \in \mathrm{Aut}(G)_P$. Since $PSL_n(q)$ is simple (as $n \geq 3$), σ lifts to an automorphism of $SL_n(q)$. We denote the lift of σ also by σ. Since T_0 is the unique Sylow 2-subgroup of the inverse image of P in $SL_n(q)$, $\sigma \in \mathrm{Aut}(SL_n(q))_{T_0}$. Thus, in order to prove the first assertion, it suffices to prove that $\mathrm{Aut}(SL_n(q))_{T_0} = \mathrm{Aut}_{SL_n(q)}(T_0)$.

Let H be the subgroup of diagonal matrices of $GL_n(q)$. Let $q = p^r$, p a prime and let $\varphi : SL_n(q) \to SL_n(q)$ be the automorphism which raises every matrix entry to the p-th power. Let $\tau : SL_n(q) \to SL_n(q)$ be the transpose inverse map. By the structure of the automorphism groups of $SL_n(q)$ (see for instance [16, Theorems 2.5.12, 2.5.14]), $\mathrm{Out}(SL_n(q))$ is generated by the images of τ, φ and $\mathrm{Aut}_H(SL_n(q))$ in $\mathrm{Out}(SL_n(q))$ On the other hand, τ, φ and all elements of $\mathrm{Aut}_H(SL_n(q))$ act as the identity on T_0. Thus, $\mathrm{Aut}(SL_n(q))_{T_0} = \mathrm{Aut}_{SL_n(q)}(T_0)$ and the first assertion is proved.

It has been shown above that there is a surjective homomorphism from $\mathrm{Aut}(SL_n(q))_{T_0}$ to $\mathrm{Aut}(G)_P$ and that $\mathrm{Aut}(SL_n(q))_{T_0} = \mathrm{Aut}_{SL_n(q)}(T_0)$. Thus, in order to prove the second assertion, it suffices to prove that $\mathrm{Aut}(SL_n(q))_{T_0}$ is isomorphic to a subgroup of S_u.

Let V be an $\mathbb{F}_q$-vector space underlying the natural matrix representation of $GL_n(q)$ and for each i, $1 \leq i \leq u$, let V_i be the -1 eigenspace of a_i. So $V = \oplus_{1 \leq i \leq u} V_i$ and $dim(V_i) = n_i$, $1 \leq i \leq u$. Since T_0 is generated by pairs of involutions $a_i a_j$, the -1 eigenspaces of elements of T_0 are precisely of the form $\oplus_{i \in I} V_i$, where I ranges over subsets of even cardinality of $\{1, \ldots, u\}$.

Let $g \in N_{SL_n(q)}(T_0)$, and let $i, j, k \in \{1, \ldots, u\}$ be pairwise distinct (this is possible since $|P| \geq 8$ implies that $u \geq 4$). Since $V_i \oplus V_j$ is the -1-eigenspace of $a_i a_j$, ${}^g(V_i \oplus V_j)$ is the -1 eigenspace of ${}^g(a_i a_j) \in T_0$. Hence, ${}^g(V_i \oplus V_j)$ (and similarly ${}^g(V_j \oplus V_k)$) is a direct sum of an even number of the V_i's. Since ${}^g V_i = {}^g(V_i \oplus V_j) \cap {}^g(V_i \oplus V_k)$, and since for any two subsets J, K of $\{1, \ldots, u\}$, $(\oplus_{t \in J} V_t) \cap (\oplus_{t \in K} V_t) = \oplus_{t \in J \cap K} V_t$ we have that ${}^g V_i$ is also a direct sum of some of the V_i's, say ${}^g V_i = \oplus_{t \in I'} V_t$. So, $V_i = \oplus_{t \in I'} {}^{g^{-1}} V_t$. But by the same argument as before, applied to g^{-1}, it follows that I' consists of a single element. Hence, for any $g \in N_{SL_n(q)}(T_0)$ and any i, $1 \leq i \leq u$, ${}^g V_i = V_j$ for some j, $1 \leq j \leq u$. Further, again by considering triples of three indices i, j, k one sees that $g \in C_{SL_n(q)}(T_0)$ if and only if ${}^g V_i = V_i$ for all $i \in I$. Thus, $\mathrm{Aut}_{SL_n(q)}(T_0) \cong N_{SL_n(q)}(T_0)/C_{SL_n(q)}(T_0)$ is isomorphic to a subgroup of S_u as required.

This proves the proposition in case G is a quotient of $SL_n(q)$. The case of $SU_n(q)$ is similar and we omit the details. $\square$

Proposition 10.3 *Let H be a finite group with $Z := Z(H) = O_{2'}(H)$ cyclic of odd order and G a quasi-simple group such that $Z \leq G \trianglelefteq H$ and $H/Z \leq \mathrm{Aut}(G/Z)$, $[H : G] = 3$. Let A be a 2-block of H and B an H-stable 2-block of G covered by A. Suppose that the defect groups of A and B are elementary abelian of order 16. Then the inertial index of A is not 15.*

Proof. Note that since $H/Z \leq \mathrm{Aut}(G/Z)$, $Z = Z(G)$. Let $D \cong C_2 \times C_2 \times C_2 \times C_2 \leq G$ be a defect group of A and of B. Note that by Lemma 10.1, the inertial quotient of B contains a subgroup isomorphic to C_5 and hence $\mathrm{Aut}_G(D)$ contains a subgroup of order 5.

Again, we go through the various possibilities for G and $\bar{G} = G/Z$. If $\bar{G}$ is an alternating or sporadic group, then G does not have a block with defect group D. If $\bar{G}$ is a finite group of Lie type in characteristic 2, not isomorphic to any of ${}^2F_4(2)'$, $B_2(2)'$ or $PSp_4(2)$ then $\bar{D} := DZ(G)/Z(G) \cong D$ is a Sylow 2-subgroup of $\bar{G}$, hence $\bar{G} = PSL_2(2^4)$. But $\bar{G} \neq SL_2(2^4)$ as $\mathrm{Out}(SL_2(2^4))$ is a 2-group. The cases that $\bar{G}$ is isomorphic to one of ${}^2F_4(2)'$, $B_2(2)'$ or $PSp_4(2)$ can be handled as in the proof of Theorem 6.1, as can the case that G is an exceptional extension of $\bar{G}$.

Suppose that $\bar{G}$ is a finite group of Lie type in odd characteristic and that G is a non-exceptional extension of $\bar{G}$. By Proposition 3.4, if $\bar{G}$ is a symplectic or orthogonal group, then B is nilpotent, a contradiction. If $\bar{G}$ is a projective special linear or unitary group, then since neither S_5 nor S_6 contain an element whose order is divisible by 15, we get a contradiction by Lemma 10.2.

Thus $\bar{G}$ is of exceptional type. Since B is not nilpotent and $\bar{G}$ is not of type A_n, by Proposition 5.4, B is Morita equivalent to a block C of a finite group L with defect group $D' \cong D$ such that D' is a product of two factors of rank 2 each of which is invariant in $N_L(D')$. It has been shown above that $\mathrm{Aut}_G(D)$ contains a subgroup of order 5. In particular, $\mathrm{Aut}_G(D)$ does not leave invariant any proper non-trivial direct factor of D and by [26], B is not Morita equivalent to a block whose inertial quotient does leave a non-trivial factor of D invariant. □

Theorem 10.4 *Let B be a block of a finite group G with elementary abelian defect group D of order 16 and inertial quotient C_{15}. Then $k(B) = k_0(B) = 16$ and $l(B) = 15$.*

Proof. It is clear from Brauer's second main theorem that $k(B) - l(B) = 1$.

Let B be a counterexample to $k(B) = 16$ with $(|G : Z(G)|, |G|)$ minimised in the lexicographic ordering. Hence by [26] $k(B) = 8$.

By the first Fong reduction and minimality, B is quasi-primitive. By the second Fong reduction and minimality, $O_{2'}(G)$ is cyclic and central in G.

Suppose that $N \lhd G$ with $N \cap D \neq 1$. Since $N_G(D)$ acts transitively on the non-trivial elements of D, it follows that $D \leq N$. In particular, if $N = O_2(G)$, then $D = O_2(G)$. But then $k(B) = 16$, contradicting our choice of B. Hence $O_2(G) = 1$.

By Proposition 2.2, if $N \lhd G$ with $N \cap D = 1$, then by minimality $N \leq Z(G)$.

If $N = O^2(G) \neq G$, then since B is quasi-primitive there is a unique block b of N covered by B (and B is the unique block of G covering b). But then $D \cap N$ is a defect group of b and $DN/N \in \mathrm{Syl}_2(G/N)$, so by the above $D \cap N = 1$. But then $N \leq Z(G)$, a contradiction. Hence $O^2(G) = G$.

Let N be a normal subgroup of G minimal subject to strictly containing $Z(G)$. Then $D \leq N$, and $N = F^*(G)$. Let $L_1, \ldots, L_t$ denote the components of G. We have seen that these are permuted transitively by G, and $L := L_1 * \cdots * L_t \lhd G$. Then $D \leq L$. Let B_L be the unique block of L covered by B, and let B_i be the unique block of L_i covered by B_L ($i = 1, \ldots, t$). Then B_L has defect group $D \cap L$, and B_i has defect group $D \cap L_i$ ($i = 1, \ldots, t$). Thus $D \cap L = (D \cap L_1) \times \cdots \times (D \cap L_t)$, where $D \cap L_1, \ldots, D \cap L_t$ are conjugate in G (since $B_1, \ldots, B_t$ are). This implies that $t \in \{1, 2, 4\}$. However the existence of an element of order 15 transitively permuting the non-trivial elements of D then forces $t = 1$. Hence L is quasi-simple and $F^*(G) = Z(G)L$.

By the Schreier conjecture, $G/F^*(G)$ is solvable. Suppose $N \leq G$ with $|G : N| = w$, where w is prime. Then w is odd. Let b be the unique block of N covered by B. Note that since C_{15} is a maximal odd order subgroup of $GL_4(2)$, the inertial quotient of b must be a subgroup of C_{15}, and further if $w > 5$, then it must be C_{15}, in which case $k(b) = 16$ by minimality. By [26], in any case $k(b) = 8$ or 16.

Suppose first that B is the unique block of G covering b. Consider the action of G on the irreducible characters of b. If $w \geq 11$, there is a fixed $\theta \in \mathrm{Irr}(b)$, and by Clifford theory $k(B) \geq w$, a contradiction. Suppose $w = 7$. Since $k(b) = 16$, there are at least two fixed $\theta \in \mathrm{Irr}(b)$, and so $k(B) > 14$, a contradiction.

Suppose $w = 5$ and that $k(b) = 16$. We must turn to Brauer characters to obtain a contradiction. We have $l(b) = 15$. Then either $l(B) > 25$ or $l(B) = 3$, according to whether there are fixed points or not. In either case we have a contradiction.

Suppose that $w = 5$ and that $k(b) = 8$. Then $l(b) = 3$, from which it follows that $l(B) = 15$, a contradiction.

The case $w = 3$ is ruled out by Proposition 10.3.

Suppose that B is not the unique block of G covering b. Then by Lemma 10.1, B is one of w blocks covering b, and b has inertial index 15. Hence $k(b) = 16$ by minimality. Write t for the number of G-orbits of $\mathrm{Irr}(b)$. Then by Clifford theory the number of irreducible characters in blocks of G covering b is $(16 - tw)w + t$. On the other hand, there are w blocks covering b, each with 8 irreducible characters. Hence $t = \frac{8w}{w^2-1}$. But $\frac{8w}{w^2-1}$ is only an integer when $w = 3$. Again, the case $w = 3$ is ruled out by Proposition 10.3.

We have shown that G is quasi-simple with centre of odd order. Then we are in one of the cases (i), (iii) or (iv) of Theorem 6.1. In case (i), $G \cong PSL_2(16)$, where it is indeed the case that the principal block has inertial index 15. Checking using [15], we see that $k(B) = 16$ in this case. In case (iii), by Proposition 2.4, B is Morita equivalent to a block of $N_G(D)$, and we see that $k(B) = 16$. In case (iv) we again have $k(B) = 16$, since Morita equivalence preserves the defect of a block and a block with Klein 4-defect group has four irreducible characters, and we are done. $\square$

11 Donovan's conjecture for groups of the form $C_{2^m} \times C_{2^m} \times C_2$ for $m \geq 3$

In this section, by an ℓ-block of a finite group G, we will mean a block of kG.

Theorem 11.1 *Let $D = C_{2^m} \times C_{2^m} \times C_2$, where $m \geq 3$. Then Donovan's conjecture holds for D.*

Proof. It suffices to consider blocks B of groups G satisfying the conditions in Proposition 8.1. Let D be a defect group for B, and write $D = P \times Q$, where $P \cong C_{2^m} \times C_{2^m}$ and $Q \cong C_2$.

We show that we may further assume that $O^2(G) = G$. Suppose that $N \lhd G$ with $|G : N| = 2$. Since B is quasi-primitive it follows that $G = ND$. If $N \cap D \cong P$, then $G = N \rtimes Q$ and by [23], B is Morita equivalent to a block of $N \times Q$ with defect group D. It then follows by Theorem 1.1 that there are only two Morita equivalence classes of such blocks. Hence $N \cap D \cong C_{2^m} \times C_{2^{m-1}} \times C_2$. Since $m \geq 3$, it follows that $\mathrm{Aut}(N \cap D)$ is a 2-group, so that B covers a nilpotent block of N, and we are done in this case by Proposition 2.2. Hence we may suppose that $O^2(G) = G$. It follows by Schreier's conjecture that we may assume that $G/Z(G)$ is a direct product of simple groups. Further, we may take $O_2(Z(G)) = 1$ or $D = P \times O_2(Z(G))$, as otherwise B would correspond to a nilpotent block of $G/O_2(Z(G))$, and would itself be nilpotent by Proposition 2.1. Further, it is clear that we may also take G to have a single component, i.e., that G is quasi-simple. The result follows by Theorem 6.1. □

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References

[1] J. An, 2-*weights for classical groups*, J. Reine Angew. Math. **439** (1993), 159–204.

[2] C. Bonnafé, *Quasi-isolated elements in reductive groups*, Comm. Algebra **33** (2005), 2315–2337.

[3] C. Bonnafé, *Sur les caractères des groupes réductifs finis à centre non connexe: applications aux groupes spéciaux linéaires et unitaires*, Astérisque **306** (2006).

[4] C. Bonnafé and R. Rouquier, *Catégories dérivées et variétés de Deligne-Lusztig*, Publ. Math. Inst. Hautes Études Sci. **97** (2003), 1–59.

[5] R. Brauer, *Some applications of the theory of blocks of characters of finite groups. II*, J. Algebra **1** (1964), 307–334.

[6] M. Cabanes and M. Enguehard, *Unipotent blocks of finite reductive groups of a given type*, Math. Z. **213** (1993), 479–490.

[7] R. W. Carter, *Centralisers of semisimple elements in the finite classical groups* Proc. London Math. Soc. **42** (1981), 1–41.

[8] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*, Oxford University Press, Eynsham, 1985.

[9] D. A. Craven, C. W. Eaton, R. Kessar and M. Linckelmann, *The structure of blocks with a Klein four defect group*, Math. Z. **268** (2011), 441–476.

[10] F. Digne and J. Michel, *Representations of Finite Groups of Lie Type*. LMS Student Texts, 21. Cambridge University Press, Cambridge, 1991.

[11] O. Düvel, *On Donovan's conjecture*, J. Algebra **272** (2004), 1–26.

[12] M. Enguehard, *Sur les l-blocs unipotents des groupes réductifs finis quand l est mauvais*, J. Algebra **230** (2000), 334–377.

[13] M. Enguehard, *Vers une décomposition de Jordan des blocs des groupes réductifs finis*, J. Algebra **319** (2008), 1035–1115.

[14] K. Erdmann, *Blocks whose defect groups are Klein four groups: a correction*, J. Algebra **76** (1982), 505–518.

[15] The GAP group, GAP - Groups, algorithms, and programming, version 4.4, http://www.gap-system.org, 2005.

[16] D. Gorenstein, R. Lyons, and R. Solomon, The classification of finite simple groups. Number 3, Mathematical Surveys and Monographs **40**, American Mathematical Society, Providence, RI, 1998.

[17] M. E. Harris and R. Knörr, *Brauer correspondence for covering blocks of finite groups*, Comm. Algebra **13** (1985), 1213-1218.

[18] A. Hida and S. Koshitani, *Morita equivalent blocks in non-normal subgroups and p-radical blocks in finite groups*, J. London Math. Soc. (2) **59** (1999), 541–556.

[19] R. Kessar, *A remark on Donovan's conjecture*, Arch. Math. **82** (2004), 391–394.

[20] R. Kessar, S. Koshitani and M. Linckelmann, *Conjectures of Alperin and Broué for 2-blocks with elementary abelian defect groups of order* 8, J. Reine Angew. Math. **671** (2012), 85–130.

[21] R. Kessar and M. Linckelmann, *On stable equivalences and blocks with one simple module*, J. Algebra **323** (2010), 1607–1621.

[22] R. Kessar and G. Malle, *On quasi-isolated blocks and Brauer's height zero conjecture*, Ann. of Math. **178** (2013), 321–384 .

[23] S. Koshitani and B. Külshammer, *A splitting theorem for blocks*, Osaka J. Math. **33** (1996), 343–346.

[24] B. Külshammer, *Donovan's conjecture, crossed products and algebraic group actions*, Israel J. Math. **92** (1995), 295–306.

[25] B. Külshammer and L. Puig, *Extensions of nilpotent blocks*, Invent. Math. **102** (1990), 17–71.

[26] B. Külshammer and B. Sambale, *The* 2*-blocks of defect* 4, Represent. Theory **17** (2013), 226–236.

[27] P. Landrock and G. Michler, *Principal 2-blocks of the simple groups of Ree type*, Trans. Amer. Math. Soc. **260** (1980), no. 1, 83–111.

[28] M. Linckelmann, *The source algebras of blocks with a Klein four defect group*, J. Algebra **167** (1994), 821–854.

[29] F. Noeske, *ADGC for Sporadic Groups*, `http://www.math.rwth-aachen.de/~Felix.Noeske/tabular.pdf`

[30] L. Puig, *Nilpotent blocks and their source algebras*, Invent. Math. **93** (1988), 77–116.

[31] L. Puig, *Une correspondance de modules pour les blocs à groupes de défaut abéliéns*, Geom. Dedicata **37** (1991), 9–43.

[32] L. Puig, On the Local Structure of Morita and Rickard Equivalences between Brauer Blocks, Birkhäuser Verlag, Basel, 1999.

[33] L. Puig, *Source algebras of p-central group extensions*, J. Algebra **235** (2001), 359–398.

[34] L. Puig, *Nilpotent extensions of blocks*, Math. Z. **269** (2011), 115-136

[35] L. Puig and Y. Usami, *Perfect isometries for blocks with abelian defect groups and cyclic inertial quotients of order* 4, J. Algebra **172** (1995), 205–213.

[36] B. Sambale, 2*-Blöcke mit metazyklischen und minimal nichtabelschen Defektgruppen*, PhD thesis, Friedrich-Schiller-Universität Jena, 2010.

[37] A. Watanabe, *On nilpotent blocks of finite groups*, J. Algebra **163** (1994), 128–134.