8 Vector Spaces

Definition and Examples

In the first part of the course we've looked at properties of the real n-space $\mathbb{R}^n$. We also introduced the idea of a field K in Section 3.1 which is any set with two binary operations $+$ and $\times$ satisfying the 9 field axioms. $\mathbb{R}$ is an example of a field but there are many more, for example $\mathbb{C}$, $\mathbb{Q}$ and $\mathbb{Z}_p$ (p a prime, with modulo p addition and multiplication).

In the second part of the course we will be looking at **vector spaces**. These will be a generalisation of $\mathbb{R}^n$ and we will see many other examples that have the same properties. We will start with an abstract definition listing the **vector space axioms**. All the properties we derive will then apply to any example that satisfies this definition.

Definition 8.1. *A* ***vector space*** *over a field K is a set V with* ***addition*** *and* ***scalar multiplication****, ie. $u+v \in V$ is defined for all $u, v \in V$ and $au \in V$ is defined for all $a \in K$ and $u \in V$, such that*

1.(i) $u + v = v + u$ *for all* $u, v \in V$

(ii) $(u+v)+w = u+(v+w)$ *for all* $u, v, w \in V$

(iii) there exists an element $\underline{0} \in V$ *such that* $u + \underline{0} = u$ *for all* $u \in V$

(iv) for each $u \in V$*, there exists a unique element* $-u \in V$ *such that* $u + (-u) = \underline{0}$

2.(i) $a(u+v) = au + av$ *for all* $a \in K$*, for all* $u, v \in V$

(ii) $(a+b)u = au + bu$ *for all* $a, b \in K$*, for all* $u \in V$

(iii) $a(bu) = (ab)u$ *for all* $a, b \in K$*, for all* $u \in V$

(iv) $1u = u$ *for all* $u \in V$*.*

The elements of V are called **vectors** and the elements of K are called **scalars**. We sometimes refer to V as a $K-$space.

Examples 8.2. 1. For all $n \geq 1$, $\mathbb{R}^n$ with the usual addition and scalar multiplication is a vector space over $\mathbb{R}$. More generally, let

$$K^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_i \in K \right\}$$

and define

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}, \qquad c\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix}$$

where $x_i, y_i, c \in K$. Then K^n is a vector space over K.

2. The set $M_{mn}(\mathbb{R})$ of all $m \times n$ matrices with entries in $\mathbb{R}$ with addition of matrices and scalar multiplication is a vector space over $\mathbb{R}$. More generally, let

$$M_{mn}(K) = \left\{ \begin{pmatrix} a_{11} & ... & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & ... & a_{mn} \end{pmatrix} \mid a_{ij} \in K \right\}$$

be the set of $m \times n$ matrices with entries in K and define

$$\begin{pmatrix} a_{11} & ... & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & ... & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & ... & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & ... & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & ... & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & ... & a_{mn} + b_{mn} \end{pmatrix},$$

$$c\begin{pmatrix} a_{11} & ... & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & ... & a_{mn} \end{pmatrix} = \begin{pmatrix} ca_{11} & ... & ca_{1n} \\ \vdots & & \vdots \\ ca_{m1} & ... & ca_{mn} \end{pmatrix}$$

where $(a_{ij}, b_{ij}, c \in K)$. Then $M_{mn}(K)$ is a vector space over K.

We write $M_n(K) = M_{nn}(K)$.

3. Let $\mathcal{P}_n$ denote the set of all polynomials of degree $\leq n$ with real coefficients:

$$\mathcal{P}_n = \{a_0 + a_1x + a_2x^2 + ... + a_nx^n |\, a_i \in \mathbb{R}, \forall i = 0, ..., n\}$$

and define

$$\sum_{i=0}^{n} a_ix^i + \sum_{i=0}^{n} b_ix^i = \sum_{i=0}^{n} (a_i + b_i)x^i$$

$$c(\sum_{i=0}^{n} a_ix^i) = \sum_{i=0}^{n} (ca_i)x^i.$$

Then $\mathcal{P}_n$ is a vector space over $\mathbb{R}$.

The zero vector is the zero polynomial with all coefficients equal to 0 and $-(\sum_{i=0}^{n} a_ix^i) = \sum_{i=0}^{n}(-a_i)x^i$.

The set $\mathcal{P}$ of all polynomials over $\mathbb{R}$ is also a vector space over $\mathbb{R}$.

4. Let $\mathcal{F}$ denote the set of all functions from $\mathbb{R} \to \mathbb{R}$ and for $f : \mathbb{R} \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$ define

$$(f+g)(x) = f(x) + g(x), \qquad (cf)(x) = c.f(x), \qquad \forall x \in \mathbb{R}.$$

Then $\mathcal{F}$ is a vector space over $\mathbb{R}$.

The zero is the constant function f_0 such that $f_0(x) = 0, \forall x \in \mathbb{R}$ and for any $f \in \mathcal{F}$, $-f$ is the function defined by

$$(-f)(x) = -f(x), \qquad \forall x \in \mathbb{R}.$$

5. The set $\mathbb{C}$ of complex numbers is a vector space over $\mathbb{R}$ with the usual addition of complex numbers and multiplication by real numbers.

6. An unusual example: Let U be a set. Consider the power set $\mathcal{P}(U) = \{A|\, A \subseteq U\}$. For $A, B \subseteq U$ define

$$A + B = (A \cup B) \backslash (A \cap B).$$

This definition satisfies conditions 1(i)-(iv) of Definition 8.1.

The zero in $\mathcal{P}(U)$ is $\emptyset$ and $-A = A$.

Consider the field $\mathbb{Z}_2 = \{0, 1\}$ and define

$$1.A = A, \qquad 0.A = \emptyset, \qquad \forall A \subseteq U.$$

We can show that 2(i)-(iv) of Definition 8.1 are satisfied.

Hence $\mathcal{P}(U)$ is a vector space over $\mathbb{Z}_2$.

Theorem 8.3. *Let V be a vector space over K. Then, for all $u \in V$ and all $a \in K$ we have:*

(i) $0u = \underline{0}$;

(ii) $a\underline{0} = \underline{0}$;

(iii) $(-1)u = -u$; *and*

(iv) *if* $au = \underline{0}$, *then* $a = 0$ *or* $u = \underline{0}$.

Proof. (i) $0u = (0+0)u = 0u + 0u$

$\Rightarrow \underline{0} = 0u - 0u = (0u + 0u) - 0u = 0u + (0u - 0u) = 0u.$

(ii) $a\underline{0} = a(\underline{0} + \underline{0}) = a\underline{0} + a\underline{0}$
$\Rightarrow \underline{0} = a\underline{0} - a\underline{0} = (a\underline{0} + a\underline{0}) - a\underline{0} = a\underline{0} + (a\underline{0} - a\underline{0}) = a\underline{0}$.

(iii) We need to show that $u + (-1)u = \underline{0}$. We have

$$u + (-1)u = 1u + (-1)u = (1-1)u = 0u = \underline{0}.$$

(iv) If $au = \underline{0}$ and $a \neq 0$, then there exists $a^{-1} \in K$. We have

$$a^{-1}(au) = a^{-1}\underline{0} = \underline{0}$$

but also $a^{-1}(au) = (a^{-1}a)u = 1u = u$ and therefore $u = \underline{0}$.

□

Subspaces

Definition 8.4. *A* ***non-empty*** *subset W of a $K-$space V is a* ***subspace*** *if*

(i) $u + v \in W, \qquad \forall u, v \in W$*; and*

(ii) $au \in W, \qquad \forall u \in W, \forall a \in K$.

Theorem 8.5. *A subspace W of a $K-$space V is itself a vector space over K with the same addition and scalar multiplication as in V.*

Proof. Since $W \neq \emptyset$, there exists $u \in W$ and then $\underline{0} = 0.u \in W$ by (ii) of Definition 8.4. For each $v \in W$ we have $-v = (-1)v \in W$. The remaining properties of a vector space hold in W because they hold in V and $W \subseteq V$. □

Examples 8.6. (i) In $\mathbb{R}^n$, let $W = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_i \in \mathbb{R}, x_1 = 0 \right\}$. Then W satisfies the two conditions of Definition 8.4 and is a subspace of $\mathbb{R}^n$.

(ii) In $M_n(K)$ let
$$W = \{A \in M_n(K) | \, A = A^T\}$$
be the subset of all symmetric matrices. Then W is a subspace because the sum of two symmetric matrices is symmetric and any scalar multiple of a symmetric matrix is also symmetric.

(iii) For all n, $\mathcal{P}_n$ is a subspace of $\mathcal{P}$ because the sum of two polynomials with degree $\leq n$ is a polynomial of degree $\leq n$ and any scalar multiple of a polynomial of degree $\leq n$ has degree $\leq n$.

Also $\mathcal{P}_n$ is a subspace of $\mathcal{P}_m$ for all $n \leq m$.

(iv) In the space of real-valued functions $\mathcal{F}$ let

$$W = \{f \in \mathcal{F} |\, f(1) = 0\}.$$

Then W is a subspace of $\mathcal{F}$ because if $f, g \in W$ then

$$(f + g)(1) = f(1) + g(1) = 0 + 0 = 0$$

and for all $a \in \mathbb{R}$,

$$(af)(1) = af(1) = a0 = 0.$$

(v) In any vector space V, the subset $\{\underline{0}\}$ is a subspace, called the **zero subspace**.

Theorem 8.7. *Let W_1 and W_2 be subspaces of the vector space V. Then $W_1 \cap W_2$ is also a subspace of V.*

Proof. First note that $\underline{0} \in W_1 \cap W_2$ and so $W_1 \cap W_2 \neq \emptyset$. Let $u, v \in W_1 \cap W_2$. Then $u + v \in W_1$ and $u + v \in W_2$ because W_1, W_2 are subspaces. Hence $u + v \in W_1 \cap W_2$.

Similarly if $a \in K$ and $u \in W_1 \cap W_2$, then $au \in W_1$ and $au \in W_2$ and so $au \in W_1 \cap W_2$. Therefore $W_1 \cap W_2$ satisfies the two conditions of Definition 8.4 and is a subspace of V. □

Definition 8.8. *Let W_1 and W_2 be subspaces of a vector space V. Then the set*

$$W_1 + W_2 = \{u + v |\, u \in W_1, v \in W_2\}$$

*is called the **sum** of W_1 and W_2 in V.*

Theorem 8.9. *The sum of two subspaces of a vector space V is a subspace of V.*

Proof. We have $\underline{0} = \underline{0} + \underline{0} \in W_1 + W_2$.

Let $u + v, u' + v' \in W_1 + W_2$, where $u, u' \in W_1$ and $v, v' \in W_2$. Then

$$(u + v) + (u' + v') = (u + u') + (v + v') \in W_1 + W_2$$

and for any $c \in K$,

$$c(u + v) = cu + cv \in W_1 + W_2$$

because W_1 and W_2 are subspaces of V. □

Example 8.10. In $M_n(\mathbb{R})$ consider the subsets

$$W_1 = \{A \in M_n(\mathbb{R}) |\, A = A^T\}, \qquad W_2 = \{B \in M_n(\mathbb{R}) |\, B = -B^T\}.$$

We have already seen that W_1 is a subspace and it's not hard to show that W_2 is a subspace. We have

$$W_1 \cap W_2 = \{A \in M_n(\mathbb{R}) |\, A = A^T \text{ and } A = -A^T\}.$$

So if $A \in W_1 \cap W_2$, then $A^T = -A^T$ and we get $A^T = 0$ and $A = 0$. Therefore $W_1 \cap W_2 = \{\underline{0}\}$.

Let $A \in M_n(\mathbb{R})$. Then we can write

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

and by properties of the transpose we have $\frac{1}{2}(A+A^T) \in W_1$ and $\frac{1}{2}(A-A^T) \in W_2$. Therefore $A \in W_1 + W_2$ and $W_1 + W_2 = M_n(\mathbb{R})$.

We can extend Definition 8.8 to the sum of more than two subspaces:

Definition 8.11. *Let $W_1, W_2, ..., W_t$ be subspaces of the vector space V. Then*

$$W_1 + W_2 + ... + W_t = \{w_1 + w_2 + ... + w_t |\, w_i \in W_i, i = 1, ..., t\}$$

is the sum of the subspaces $W_1, ..., W_t$.

An easy induction on t and Theorem 8.9 show that this sum is a subspace of V.