7 Diagonalization and Quadratic Forms

Diagonalization

Recall the definition of a diagonal matrix from Section 1.6.

Definition 7.1. A square matrix $A$ is diagonalizable if there exists an invertible matrix $P$ such that $P^{-1}AP$ is diagonal. We say that $P$ diagonalizes $A$.

Remark. Why is this interesting? For many applications we need to compute powers of matrices, for example

$$A^7 = AAAAAAA.$$

To do this by direct calculation is a lot of work, but if $A$ is diagonalizable, say $P^{-1}AP = D$ diagonal, then $A = PDP^{-1}$ so

$$A^7 = (PDP^{-1})^7 = PD^7P^{-1}.$$

and more generally, $A^k = PD^kP^{-1}$ for all $k$. We have seen in Exercise Sheet 5 that $D^k$ is easy to compute, so this gives a much easier way to work out $A^k$ for large $k$.

Examples 7.2. Consider

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}, P = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, P^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}.$$

Then we can check that

$$P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} =: D,$$

so $A$ is diagonalizable. Now for any $m$ we have

$$A^m = PD^mP^{-1} = P \begin{pmatrix} 2^m & 0 & 0 \\ 0 & 2^m & 0 \\ 0 & 0 & 1^m \end{pmatrix} P^{-1}$$

so to calculate $A^m$ we need only calculate the scalar power $2^m$ and then perform two matrix multiplications.
**Exercise.** With $A$ as above, work out $A^{16}$. Then try and do it directly without the “diagonalization”!

Given that diagonalizing a matrix is so useful, it is natural to ask which matrices can be diagonalized. To answer this question we will need a lemma giving yet another characterisation of invertible matrices.

**Lemma 7.3.** Let $P$ be an $n \times n$ square matrix. Then $P$ is invertible if and only if its columns (viewed as column $n$-vectors) form a set of $n$ linearly independent vectors.

**Proof.** See Section 14. □

**Theorem 7.4.** Let $A$ be an $n \times n$ matrix. Then $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors. A matrix $P$ diagonalizes $A$ if and only if $P$’s columns form a set of $n$ linearly independent eigenvectors for $A$. If it does, then the main diagonal entries of the diagonal matrix $P^{-1}AP$ are the eigenvalues of $A$ (in the order corresponding to the columns of $P$).

**Proof.** Suppose $P^{-1}AP = D$ is diagonal. Let $c_1, \ldots, c_n$ be the columns of $P$. By Lemma 7.3, the columns are linearly independent. Now $P^{-1}AP = D$ implies

$$AP = PD.$$ 

It follows from the definition of matrix multiplication that (i) the $i$th column of $PD$ is $D_{ii}c_i$ and (ii) the $i$th column of $AP = Ac_i$. Thus we have $Ac_i = D_{ii}c_i$, so each column $c_i$ is an eigenvector of $A$ corresponding to the eigenvalue $D_{ii}$.

Conversely, if $A$ has $n$ linearly independent eigenvectors $c_1, \ldots, c_n$ then let $P$ be the matrix with these as columns, and $D$ the diagonal matrix with the corresponding eigenvalues on the main diagonal. By Lemma 7.3, $P$ is invertible. Now reversing the argument above, the $i$th column of $AP$ is $Ac_i$ and the $i$th column of $PD$ is $D_{ii}c_i$ so $AP = PD$, so $P^{-1}AP = D$. □

**Example 7.5.** Let

$$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}, u = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ 

Then $A$ has eigenvalues 3 and $-1$, with corresponding eigenvectors $u$ and $v$ respectively (**exercise: check this**). It is easy to check that $\{u, v\}$ is linearly independent. If we let $P$ be the matrix whose columns are $u$ and $v$,

$$P = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \text{ then } P^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$
and one can check (exercise) that
\[ P^{-1}AP = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}. \]

**Definition 7.6.** An \( n \times n \) matrix \( A \) is called **orthogonally diagonalizable** if there is an **orthogonal** matrix \( P \) such that \( P^{-1}AP = P^TAP \) is diagonal.

**Theorem 7.7.** Let \( A \) be an \( n \times n \) matrix. Then the following are equivalent:

(i) \( A \) is orthogonally diagonalizable.

(ii) \( A \) has an orthonormal set of \( n \) eigenvectors;

(iii) \( A \) is symmetric.

**Proof.** (i) \( \iff \) (ii) This follows from Theorems 6.6, 6.8 and 7.4 (exercise: write down exactly how).

(i) \( \Rightarrow \) (iii) If (i) holds, say \( P^{-1}AP = D \) is diagonal with \( P \) orthogonal, then we have \( A = PDP^{-1} = PDP^T \). Clearly \( D \) is symmetric, so
\[ A^T = (PDP^T)^T = (P^T)^TD^TP^T = PDP^T = A \]
which means that \( A \) is symmetric.

(iii) \( \Rightarrow \) (ii) Omitted (see for example the textbook of Anton).

**Quadratic Forms**

**Definition 7.8.** A quadratic form in \( n \) variables is a function \( f: \mathbb{R}^n \to \mathbb{R} \) of the form
\[ f(x) = f(x_1, \ldots, x_n) = \sum_{1 \leq i \leq j \leq n} c_{ij}x_i x_j \quad (*) \]
where \( x \in \mathbb{R}^n \) and \( c_{ij} \in \mathbb{R} (1 \leq i \leq j \leq n) \). Alternatively, a quadratic form is a homogeneous polynomial of degree 2 in \( n \) variables \( x_1, \ldots, x_n \).

**Examples 7.9.** The following are quadratic forms:

1. \( f(x_1) = x_1^2 \)
2. \( f(x_1, x_2) = 2x_1^2 + 3x_2^2 - x_1x_2 \)
3. \( f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3 \)
4. \( f(x_1, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 = \langle v | v \rangle \), where \( v = (x_1, \ldots, x_n) \). So the Euclidean inner product (see Chapter 6) gives rise to a quadratic form.

If we set \( a_{ii} = c_{ii} \) for \( i = 1, \ldots, n \) and \( a_{ij} = \frac{1}{2} c_{ij} \) for \( 1 \leq i < j \leq n \), then (*') becomes

\[
 f(x) = \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{1 \leq i < j \leq n} 2a_{ij} x_i x_j
\]

and we can write this as

\[
 f(x) = x^T A x
\]

where \( A \) is the symmetric \( n \times n \) matrix with \((i,j)\)-th entry equal to \( a_{ij} \). Then \( A \) is called the matrix of the quadratic form \( f \).

**Example 7.10.** Let \( f(x_1, x_2) = 2x_1^2 - 3x_2^2 - x_1 x_2 \). Then

\[
 f(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 2 & -1/2 \\ -1/2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\]

Now that a symmetric matrix is involved, we can take advantage of Theorem 7.7.

ie. there exists an orthogonal matrix \( Q \) such that

\[
 Q^T A Q = D = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}
\]

where \( D \) is a diagonal matrix and \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( A \).

Now let \( y = Q^{-1} x = Q^T x \). Then \( x = Qy \) and

\[
 f(x) = (Qy)^T A(Qy) = y^T Q^T A Q y = y^T D y
\]

and if \( y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \) then

\[
 y^T D y = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2,
\]

a quadratic form in variables \( y_1, \ldots, y_n \) with no cross terms. This process is called **diagonalization** of the quadratic form \( f \). We have just proved a famous theorem, namely
Theorem 7.11. (The Principal Axes Theorem) Every quadratic form \( f \) can be diagonalized. More specifically, if \( f(x) = x^T Ax \) is a quadratic form in \( x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \), then there exists an orthogonal matrix \( Q \) such that

\[
f(x) = x^T Ax = \lambda_1 y_1^2 + \ldots + \lambda_n y_n^2
\]

where \( \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = Q^T \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \) and \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of the matrix \( A \).

From the first part of the course we know that \( Q \) is the matrix whose columns are the unit eigenvalues of the matrix \( A \) of \( f \).

Example 7.12. Let \( f(x_1, x_2, x_3) = 2x_1x_2 + 2x_1x_3 + 2x_2x_3 \). The matrix of \( f \) is

\[
A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.
\]

The eigenvalues of \( A \) are 2, \(-1\), \(-1\) with corresponding unit eigenvectors

\[
\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \quad \begin{pmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}
\]

and so we have

\[
Q = \begin{pmatrix} 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \end{pmatrix}.
\]

If we set \( y = Q^T x \), we get

\[
y_1 = \frac{1}{\sqrt{3}}(x_1 + x_2 + x_3), \quad y_2 = \frac{1}{\sqrt{6}}(-2x_1 + x_2 + x_3), \quad y_3 = \frac{1}{\sqrt{2}}(x_2 - x_3).
\]

Then, expressed in terms of the variables \( y_1, y_2 \) and \( y_3 \), the quadratic form becomes \( 2y_1^2 - y_2^2 - y_3^2 \).

Definition 7.13. A quadratic form \( f : \mathbb{R}^n \to \mathbb{R} \) is **positive definite** if

\( f(x) > 0 \) for all \( x \neq 0 \).
An immediate consequence of the Principal Axes Theorem is the following:

**Theorem 7.14.** Let $f(x) = x^T Ax$ be a quadratic form with matrix $A$. Then $f$ is positive definite if and only if all the eigenvalues of $A$ are positive.

**Proof.** By the Principal Axes Theorem, there exists an orthogonal matrix $Q$ such that

$$f(x) = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$$

where $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = Q^T x$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$. If all the $\lambda_i$ are positive then $f(x) > 0$ except when $y = \mathbf{0}$. But this happens if and only if $x = \mathbf{0}$ because $Q^T$ is invertible. Therefore $f$ is positive definite.

On the other hand if one of the eigenvalues $\lambda_i \leq 0$, letting $y = e_i$ and $x = Q y$ we get $f(x) = \lambda_i \leq 0$ and so $f$ is not positive definite. \qed

We say that a symmetric matrix $A$ is **positive definite** if the associated quadratic form

$$f(x) = x^T Ax$$

is positive definite.

The Principal Axes Theorem has important applications in geometry.