Robust kinematic control of manipulator robots using dual quaternion representation

L.F.C. Figueredo, B.V. Adorno, J.Y. Ishihara, and G.A. Borges

***Abstract*— This paper addresses the H_∞ robust control problem for robot manipulators using unit dual quaternion representation, which allows an utter description of the end-effector transformation without decoupling rotational and translational dynamics. We propose three different H_∞ control criteria that ensure asymptotic convergence, whereas reducing the influence of disturbances upon the system stability. Also, with a new metric of dual quaternion error in $SE(3)$ we prove independence from robot coordinate changes. Simulation results highlight the importance and effectiveness of the proposed approach in terms of performance, robustness, and energy efficiency.**

I. INTRODUCTION

Kinematic control of manipulators have been extensively studied in the last fourty years and is one of the classic topics covered by most of robotics textbooks (for example, see [1], [2]). This type of control is suitable when the robot dynamics can be neglected, as in the case of stiff robots with harmonic drives operating at relatively low velocities.

In a pratical scenario, tasks are usually defined at the end-effector but the control signals are applied into actuators located at joint level. When the robot inverse kinematics is known, a desired configuration of the end-effector can be mapped into a desired configuration of the robot joints and trajectory planning can be used in conjunction to suitable controllers to ensure that the current vector of joint positions converge to a desired value. The inverse kinematics, however, is not easily obtained for the general case and, in addition, most often modern manipulator robots are already equipped with low level controllers at joint level. This way, a convenient approach, firstly introduced by Whitney for non-redundant manipulators [3], is to define control laws directly at the end-effector and then invert the differential foward kinematics model (FKM) in order to provide suitable references for the low level controllers. The differential FKM is usually written in the form of a Jacobian matrix which is fairly easy to invert—in the general case when the Jacobian matrix is not square, usually the Moore-Penrose pseudoinverse is used [2].

When defining the task at the end-effector level, it is important to use a suitable representation for the FKM. Minimal representations such as Euler angles plus Cartesian coordinates lead to singularities (i.e., ambiguities that make the inverse problem ill-posed), whereas non-singular representations usually cannot be used directly as control variables. Homogeneous transformation matrices (HTM) are a very popular choice for representing the FKM, but the control parameters must be extracted from the matrix, which requires additional calculations. Moreover, because of the approximative nature of the extracted rotational parameters, trajectory generation must be performed in order to guarantee small rotational errors and thus ensure stability [4].

Recently, an increasing interest has been given to unit dual quaternions, because they can completely represent the rigid motion in a more compact way than HTM (eight elements against twelve), it is straightforward to extract geometric parameters from a given unit dual quaternion (translation, axis of rotation, angle of rotation), and dual quaternions multiplications are less expensive than HTM multiplications [5, p. 42]. Furthermore, unit dual quaternions are more straightforward to use than unit quaternions plus Cartesian coordinates. This is thanks to the fact that a sequence of rigid motions can be represented by a sequence of dual quaternion multiplications (e.g., given two unit dual quaternions $\underline{\boldsymbol{x}}_1$ and $\underline{\boldsymbol{x}}_2$, a sequence of rigid motions is given by $\underline{\boldsymbol{x}}_f = \underline{\boldsymbol{x}}_1\underline{\boldsymbol{x}}_2$). On the other hand, if unit quaternions plus Cartesian coordinates are used, the calculations of rotation and position are made separately. The final rotation will be given by a sequence of unit quaternion multiplications and the position will be given by a much more complicated term; that is, given $\{\boldsymbol{r}_1, \boldsymbol{p}_1\}$ and $\{\boldsymbol{r}_2, \boldsymbol{p}_2\}$, where $\boldsymbol{r}$ represents rotation and $\boldsymbol{p}$ represents translation, the final rigid motion is given by $\{\boldsymbol{r}_1\boldsymbol{r}_2, \boldsymbol{p}_1 + \boldsymbol{r}_1\boldsymbol{p}_2\boldsymbol{r}_1^*\}$, where $\boldsymbol{r}_1^*$ is the conjugate of $\boldsymbol{r}_1$.

Nevertheless, few attention has been given to the control problem formulated directly in unit dual quaternion space. Among recent works, some different control strategies should be acknowledged thanks to their contribution to the analysis within dual quaternion space. One of them, introduced by [6]–[8], is based on a logarithmic mapping of the dual quaternion error, thus taking into account the unit dual quaternion Lie-group properties and its Lie-algebra. This approach has been extended to dynamic controllers with a metric invariant to changes in the body coordinate frame in [9], [10].

On the other hand, [11], [12] propose a control strategy based on the dual quaternion error mapping in an $\mathbb{R}^8$ manifold. This technique has the advantages of being considerably more intuitive and attractive from the control point of view, although it is not invariant to changes in the reference

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coordinate system.

All aforementioned works on control using dual quaternions provide feasible stability regions over a user defined parameter, but with no discussion on the performance, robustness, or energy efficiency related. In this context and motivated by the problem of optimizing the system performance, we state novel stabilization techniques that explicitly consider the system requirements into an H_∞ control design.

The H_∞ control theory, originated as a technique to reduce the feedback system sensitivity [13], allows the designer to incorporate robustness requirements, disturbance attenuation, and performance properties into one stabilization problem [14]. To the best of the authors knowledge, these concepts have never been exploited in previous works on control using dual quaternions. Therefore, the present paper brings an important contribution to control analysis for robot manipulators by developing H_∞ control schemes using the dual quaternion space to avoid decoupling the end-effector rotational and translational dynamics and representation singularities. Moreover, we propose a new error metric, which still maps the error in an $\mathbb{R}^8$ manifold, but is invariant to coordinate changes.

The paper is organized as follows: Section II presents the mathematical background related to dual quaternions and H_∞ control, whereas Section III presents the robot kinematic modeling. Section IV presents the main contributions of the paper; namely, a new error metric in dual quaternion that is invariant to changes in the reference coordinate system, and three different H_∞ control schemes. Section V reports simulations results used to validate the approach and lastly Section VI closes the paper.

II. Mathematical Background

Let $\hat{\imath}, \hat{\jmath}, \hat{k}$ be the three quaternionic units such that $\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = -1$ and $\hat{\imath}\hat{\jmath}\hat{k} = -1$. A unit quaternion $\boldsymbol{r} = \cos(\phi/2) + \sin(\phi/2)\boldsymbol{n}$ represents the rotation in $SO(3)$, where ϕ is the rotation angle around the rotation axis $\boldsymbol{n} = n_x\hat{\imath}+n_y\hat{\jmath}+n_z\hat{k}$ [15]. For unit quaternions, the inverse operation is given by the conjugate $\boldsymbol{r}^* = \cos(\phi/2) - \sin(\phi/2)\boldsymbol{n}$.

A rigid motion in $SE(3)$ can be represented by the unit dual quaternion $\underline{\boldsymbol{x}} = \boldsymbol{r} + \varepsilon(1/2)\boldsymbol{p}\boldsymbol{r}$, where $\boldsymbol{r}$ is a unit quaternion that represents the rotation, $\boldsymbol{p} = p_x\hat{\imath}+p_y\hat{\jmath}+p_z\hat{k}$ is a pure quaternion (i.e., a quaternion with real part equal to zero) that represents the translation, and ε is the nilpotent Clifford unit; that is, $\varepsilon \neq 0$ but $\varepsilon^2 = 0$ [16]. For unit dual quaternions, the inverse operation of $\underline{\boldsymbol{x}}$ is given by the conjugate $\underline{\boldsymbol{x}}^* = \boldsymbol{r}^*+\varepsilon(1/2)\boldsymbol{r}^*\boldsymbol{p}^*$, such that $\underline{\boldsymbol{x}}^*\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}\underline{\boldsymbol{x}}^* = 1$.

A sequence of rigid motions can be represented by a sequence of unit dual quaternion multiplications; for instance, given that $\underline{\boldsymbol{x}}_1^0$ represents frame $\mathcal{F}_1$ with respect to $\mathcal{F}_0$ and $\underline{\boldsymbol{x}}_2^1$ represents frame $\mathcal{F}_2$ with respect to $\mathcal{F}_1$, then the transformation from $\mathcal{F}_0$ to $\mathcal{F}_2$ is $\underline{\boldsymbol{x}}_2^0 = \underline{\boldsymbol{x}}_1^0\underline{\boldsymbol{x}}_2^1$. Alternatively, the transformation $\underline{\boldsymbol{x}}_2^0$ can be regarded as a coordinate change from the coordinate system $\mathcal{F}_0$ to the coordinate system $\mathcal{F}_2$.

It is well known that $SE(3)$ is a non-commutative group, so unit dual quaternion multiplication is not commutative either; that is, if $\underline{\boldsymbol{x}}$ and $\underline{\boldsymbol{y}}$ are unit dual quaternions, then $\underline{\boldsymbol{x}}\underline{\boldsymbol{y}} \neq \underline{\boldsymbol{y}}\underline{\boldsymbol{x}}$. Nonetheless, when dual quaternions are mapped into $\mathbb{R}^8$, Hamilton operators can be used to commute terms when performing dual quaternions multiplications. Let us consider the mapping

$$\text{vec} : \mathcal{H} \to \mathbb{R}^8, \tag{1}$$

where $\mathcal{H}$ is the set of dual quaternions. For the dual quaternion $\underline{\boldsymbol{z}} = \underline{\boldsymbol{x}}\underline{\boldsymbol{y}}$, the Hamilton operators are matrices that satisfy [17]

$$\begin{aligned}\text{vec}\,\underline{\boldsymbol{z}} &= \overset{+}{\boldsymbol{H}}(\underline{\boldsymbol{x}})\,\text{vec}\,\underline{\boldsymbol{y}}\\ &= \overset{-}{\boldsymbol{H}}(\underline{\boldsymbol{y}})\,\text{vec}\,\underline{\boldsymbol{x}}.\end{aligned}$$

In addition, using the definition of conjugate of dual quaternions it is easy to show that $\text{vec}\,\underline{\boldsymbol{x}}^* = \boldsymbol{C}_8\,\text{vec}\,\underline{\boldsymbol{x}}$, where $\boldsymbol{C}_8 = diag(1,-1,-1,-1,1,-1,-1,-1)$.

Let us now regard a continuous-time system

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B\boldsymbol{u}(t)) + B_w\boldsymbol{\omega}(t),\\ \boldsymbol{z}(t) &= C\boldsymbol{x}(t),\end{aligned}$$

where $\boldsymbol{x}(t)$ and $\boldsymbol{u}(t)$ denote the system's state and control input, whereas $\boldsymbol{\omega}(t)$ is an exogenous input representing the disturbance acting on the system, whose effect on the output, $\boldsymbol{z}(t)$, we want to minimize. Suppose that $\boldsymbol{u}(t){=}\boldsymbol{K}\boldsymbol{x}(t)$ is some stabilizing controller. If we set $\boldsymbol{x}(0){=}0$, the controlled closed-loop system defines a map from $\boldsymbol{\omega}(t){\in}L_2[0,\infty)$[1] to $\boldsymbol{z}(t){\in}L_2[0,\infty)$, [14]. The gain that defines the H_∞ norm of the closed-loop system is described as

$$sup\left\{\frac{\|\boldsymbol{z}(t)\|_2}{\|\boldsymbol{\omega}(t)\|_2},\ \boldsymbol{\omega}(t){\in}L_2\setminus\{0\}\right\}.$$

The H_∞ norm denotes the induced norm of the map $\boldsymbol{\omega}(t) \to \boldsymbol{z}(t)$, i.e., the supremum of the noise amplification upon the system output. If we introduce the index γ, such that

$$\|\boldsymbol{z}(t)\|_2 {\le} \gamma\,\|\boldsymbol{\omega}(t)\|_2\,,$$

then γ denotes an upper bound for the induced norm. The smaller the performance upper bound γ, yields a smaller influence of $\boldsymbol{\omega}(t)$ over $\boldsymbol{z}(t)$. Thus, the reduction of the index γ enlightens the aim of decoupling the disturbance influence from the output, $\boldsymbol{z}(t)$. The H_∞ control therefore strives to reduce the performance upper bound γ, through its relationship with $\boldsymbol{K}$, whereas maintaining the system internal stability.

III. Kinematics Modeling

Consider a serial manipulator with n joints, each joint attached to the beginning of a link. The unit dual quaternion $\underline{\boldsymbol{x}}_{i+1}^i$ represents the rigid transformation between the extremities of link i and is a function of the i-th joint angle θ_i. The forward kinematics relates the configuration of all joints to the configuration $\underline{\boldsymbol{x}}_m$ of the end-effector; that is, $\underline{\boldsymbol{x}}_m := \underline{\boldsymbol{x}}_1^0\underline{\boldsymbol{x}}_2^1\ldots\underline{\boldsymbol{x}}_n^{n-1}$.

Because $\underline{\boldsymbol{x}}_{i+1}^i$ is a function of θ_i, i.e., $\underline{\boldsymbol{x}}_{i+1}^i = f(\theta_i)$, the forward kinematics is a function of all joints—i.e.,

[1] L_2 is the Hilbert space of all square-integrable functions.

$\underline{\boldsymbol{x}}_{\mathrm{m}} = f(\theta_0, \ldots, \theta_{n+1})$—the differential forward kinematics is given by $\dot{\underline{\boldsymbol{x}}}_m = f'(\theta_0, \ldots, \theta_{n+1})$, where $f' \triangleq df/dt$. If the mapping (1) is used, then the differential kinematics is given by

$$\operatorname{vec} \dot{\underline{\boldsymbol{x}}}_m = \boldsymbol{J}\dot{\boldsymbol{\theta}}, \tag{2}$$

where $\boldsymbol{\theta} = [\theta_0 \ldots \theta_{n-1}]^T$ is the measured vector of joint variables and $\boldsymbol{J} = \partial f / \partial \boldsymbol{\theta}$ is the analytical Jacobian. Nonetheless, in practical applications, there exists several influences acting upon the system, such that the description of (2) is not perfect, and it is interesting to consider the effects of different disturbances over the system. In this case, we shall regard

$$\operatorname{vec} \dot{\underline{\boldsymbol{x}}}_m = \boldsymbol{J}\dot{\boldsymbol{\theta}} + \boldsymbol{B}\boldsymbol{\omega} \tag{3}$$

where $\boldsymbol{B}$ is a known matrix and $\boldsymbol{\omega}$ is the vector of exogenous disturbances, whose influence we want to minimize.

IV. H_∞ CONTROL STRATEGIES

In this section, we address three different control design strategies for the kinematic model (3). As previously stressed, the measured and desired attitude and position configurations can be compactly and utterly represented in dual quaternion space, and its dynamics related to the angular and linear joint vector velocity $\dot{\boldsymbol{\theta}}(t)$. In this framework, the aim of the control scheme is to synthesize a joint based feedback controller to make the current configuration of the robot end-effector converge to a desired reference without decoupling the rotational and translational dynamics.

A. Error definition

Given the desired unit dual quaternion configuration $\underline{\boldsymbol{x}}_d$ and the current configuration $\underline{\boldsymbol{x}}_m$, we define the spatial difference in $SE(3)$ as

$$\underline{\boldsymbol{x}}_e = \underline{\boldsymbol{x}}_m^* \underline{\boldsymbol{x}}_d. \tag{4}$$

When $\underline{\boldsymbol{x}}_m$ is equal to $\underline{\boldsymbol{x}}_d$, the spatial difference $\underline{\boldsymbol{x}}_e$ equals 1. In this context, we introduce a novel error metrics

$$\underline{\boldsymbol{e}} = 1 - \underline{\boldsymbol{x}}_e, \tag{5}$$

such that, if $\underline{\boldsymbol{x}}_m$ converges to $\underline{\boldsymbol{x}}_d$, the dual quaternion error $\underline{\boldsymbol{e}} \to 0$. Rewriting (4)-(5), we have

$$\underline{\boldsymbol{e}} = (\underline{\boldsymbol{x}}_d^* - \underline{\boldsymbol{x}}_m^*)\, \underline{\boldsymbol{x}}_d,$$

which can be mapped into the $\mathbb{R}^8$ manifold, using (1), as

$$\operatorname{vec} \underline{\boldsymbol{e}} = \bar{\boldsymbol{H}}\, (\underline{\boldsymbol{x}}_d) \boldsymbol{C}_8 \operatorname{vec} (\underline{\boldsymbol{x}}_d - \underline{\boldsymbol{x}}_m)\,. \tag{6}$$

For $\underline{\boldsymbol{x}}_d$ constant, the first derivative of (6) yields

$$\operatorname{vec} \dot{\underline{\boldsymbol{e}}} = -\bar{\boldsymbol{H}}\, (\underline{\boldsymbol{x}}_d) \boldsymbol{C}_8 \operatorname{vec} \dot{\underline{\boldsymbol{x}}}_m.$$

Since the dual quaternion dynamics $\dot{\underline{\boldsymbol{x}}}_m$ is given by the robot forward kinematic model (2), then

$$\begin{aligned} \operatorname{vec} \dot{\underline{\boldsymbol{e}}} &= -\bar{\boldsymbol{H}}\, (\underline{\boldsymbol{x}}_d) \boldsymbol{C}_8 \boldsymbol{J}\dot{\boldsymbol{\theta}} \\ &= -\boldsymbol{N}\dot{\boldsymbol{\theta}}, \end{aligned} \tag{7}$$

where $\boldsymbol{N} = \bar{\boldsymbol{H}}\, (\underline{\boldsymbol{x}}_d) \boldsymbol{C}_8 \boldsymbol{J}$.

It is interesting to highlight that the convergence properties from $\underline{\boldsymbol{x}}_m$ to $\underline{\boldsymbol{x}}_d$ are utterly related to the definition of the novel error metrics, $\underline{\boldsymbol{e}}$, which is invariant with respect to coordinate changes. For instance, let us assume that both base frame and desired set point have been transformed by a coordinate change represented by the unit dual quaternion $\underline{\boldsymbol{y}}$; that is,

$$\begin{aligned} \underline{\boldsymbol{x}}'_m &= \underline{\boldsymbol{y}} \underline{\boldsymbol{x}}_m \\ \underline{\boldsymbol{x}}'_d &= \underline{\boldsymbol{y}} \underline{\boldsymbol{x}}_d. \end{aligned}$$

Since $\underline{\boldsymbol{e}} = 1 - \underline{\boldsymbol{x}}_e$, the error in the new coordinate system is given by

$$\begin{aligned} \underline{\boldsymbol{e}}' &= 1 - \underline{\boldsymbol{x}}'_e = 1 - \underline{\boldsymbol{x}}'^*_m \underline{\boldsymbol{x}}'_d \\ &= 1 - \underline{\boldsymbol{x}}^*_m \underline{\boldsymbol{y}}^* \underline{\boldsymbol{y}} \underline{\boldsymbol{x}}_d \\ &= \underline{\boldsymbol{e}}, \end{aligned}$$

which is independent of $\underline{\boldsymbol{y}}$. In this context, and using the novel error metrics (6), to asymptotically stabilize the system (7) is to assume $\underline{\boldsymbol{x}}(t) \to \underline{\boldsymbol{x}}_{\boldsymbol{d}}(t)$ as $t \to \infty$, independently from the choice of the robot base coordinate systems and from coordinate changes.

In a more practical scenario, the existence of disturbances leads the more accurate differential kinematics description (3). The error dynamics in this case, can be described by

$$\operatorname{vec} \dot{\underline{\boldsymbol{e}}}(t) = -\boldsymbol{N}\dot{\boldsymbol{\theta}} - \boldsymbol{B}_w \boldsymbol{\omega}(t), \tag{8}$$

where $\boldsymbol{B}_w = \bar{\boldsymbol{H}}\, (\underline{\boldsymbol{x}}_d)\, \boldsymbol{C}_8 \boldsymbol{B}$.

B. H_∞ control design

For the H_∞ control, we seek a joint based control law that makes the dual quaternion configuration $\underline{\boldsymbol{x}}_m(t)$ asymptotically converges to $\underline{\boldsymbol{x}}_d(t)$, whereas ensuring disturbances attenuation properties, i.e., attenuating the influence of any exogenous signal $\boldsymbol{\omega}(t) \in L_2[0, \infty)$. In this context and based on [14], the following definition describes the robust performance in the H_∞ sense.

Definition 1: For a prescribed scalar $\gamma > 0$, the robust control performance is achieved with an H_∞ norm bound γ, if the following hold

(1) The error dynamics (8) is asymptotically stable for $\boldsymbol{\omega}(t) \equiv 0$;

(2) Under the assumption of zero initial conditions, the disturbance influence on the error, $\operatorname{vec} \underline{\boldsymbol{e}}(t)$, is attenuated below a desired level γ, $||\operatorname{vec} \underline{\boldsymbol{e}}(t)||_2 \leq \gamma ||\boldsymbol{\omega}(t)||_2$ for all nonzero $\boldsymbol{\omega}(t) \in L_2[0, \infty)$.

Considering the system description and Definition 1, we state a solution for the H_∞ control problem in the following criterion.

Theorem 1 (LMI-based H_∞ controller): For a prescribed $\gamma > 0$, there exist a joint based controller such that (8) achieves robust stability with H_∞ performance γ, in the sense of Definition 1, if there exist matrices $\boldsymbol{P} = \boldsymbol{P}^T > 0$ and $\boldsymbol{G} = \boldsymbol{G}^T > 0$, such that

$$\begin{bmatrix} (-2\boldsymbol{G} + \boldsymbol{I}) & -\boldsymbol{P}\boldsymbol{B}_w \\ -\boldsymbol{B}_w^T\boldsymbol{P} & -\gamma^2\boldsymbol{I} \end{bmatrix} < 0. \tag{9}$$

Moreover, if the above conditions are satisfied, a stabilizing joint vector velocity control is given by $\dot{\boldsymbol{\theta}}(t) = \boldsymbol{N}^+ \left(\boldsymbol{P}^{-1}\boldsymbol{G}\right) \operatorname{vec} \underline{\boldsymbol{e}}(t)$, where $\boldsymbol{N}^+$ is the pseudo-inverse of $\boldsymbol{N}$.

Proof: Let us choose as Lyapunov function candidate the positive definite quadratic function:

$$\boldsymbol{V}(t) = \operatorname{vec} \underline{\boldsymbol{e}}(t)^T \boldsymbol{P} \operatorname{vec} \underline{\boldsymbol{e}}(t). \tag{10}$$

Taking the time-derivative of (10) with respect to t along the trajectory (8) yields

$$\begin{aligned} \dot{\boldsymbol{V}}(t) &= \operatorname{vec} \dot{\underline{\boldsymbol{e}}}(t)^T \boldsymbol{P} \operatorname{vec} \underline{\boldsymbol{e}}(t) + \operatorname{vec} \underline{\boldsymbol{e}}(t)^T \boldsymbol{P} \operatorname{vec} \dot{\underline{\boldsymbol{e}}}(t) \\ &= 2 \operatorname{vec} \underline{\boldsymbol{e}}(t)^T \boldsymbol{P} \operatorname{vec} \dot{\underline{\boldsymbol{e}}}(t) \\ &= -2 \operatorname{vec} \underline{\boldsymbol{e}}(t)^T \boldsymbol{P} \left(\boldsymbol{N}\dot{\boldsymbol{\theta}}(t) + \boldsymbol{B}_w\boldsymbol{\omega}(t)\right) \end{aligned}$$

Choosing $\dot{\boldsymbol{\theta}}(t) := \boldsymbol{N}^+\boldsymbol{K}\boldsymbol{P} \operatorname{vec} \underline{\boldsymbol{e}}(t)$, gives

$$\begin{aligned} \dot{\boldsymbol{V}}(t) &= -2 \operatorname{vec} \underline{\boldsymbol{e}}(t)^T \boldsymbol{P} \left(\boldsymbol{N}\boldsymbol{N}^+\boldsymbol{K}\boldsymbol{P} \operatorname{vec} \underline{\boldsymbol{e}}(t) + \boldsymbol{B}_w\boldsymbol{\omega}(t)\right) \\ &= -2 \operatorname{vec} \underline{\boldsymbol{e}}(t)^T \boldsymbol{P} \left(\boldsymbol{K}\boldsymbol{P} \operatorname{vec} \underline{\boldsymbol{e}}(t) + \boldsymbol{B}_w\boldsymbol{\omega}(t)\right). \end{aligned}$$

For any $\boldsymbol{K} > 0$, it is easy to see that $\dot{\boldsymbol{V}}(t) < 0$ holds for $\boldsymbol{\omega}(t) \equiv 0$, which in turn implies $\boldsymbol{V}(t) \to 0$ as $t \to \infty$. Hence, $\operatorname{vec} \underline{\boldsymbol{e}}(t)$ converges to zero, and the first condition in Definition 1 is satisfied. Now, let us define $\dot{\boldsymbol{V}}_{H_\infty}(t) := \dot{\boldsymbol{V}}(t) + \operatorname{vec} \underline{\boldsymbol{e}}(t)^T \operatorname{vec} \underline{\boldsymbol{e}}(t) - \gamma^2\boldsymbol{\omega}^T(t)\boldsymbol{\omega}(t)$, such that

$$\begin{aligned} \dot{\boldsymbol{V}}_{H_\infty}(t) = &- \operatorname{vec} \underline{\boldsymbol{e}}(t)^T \left(2\boldsymbol{P}\boldsymbol{K}\boldsymbol{P} - \boldsymbol{I}\right) \operatorname{vec} \underline{\boldsymbol{e}}(t) \\ &- 2 \operatorname{vec} \underline{\boldsymbol{e}}(t)^T \boldsymbol{P}\boldsymbol{B}_w\boldsymbol{\omega}(t) - \gamma^2\boldsymbol{\omega}^T(t)\boldsymbol{\omega}(t), \end{aligned}$$

which can also be written as

$$\dot{\boldsymbol{V}}_{H_\infty}(t) = \begin{bmatrix} \operatorname{vec} \underline{\boldsymbol{e}}(t) \\ \boldsymbol{\omega}(t) \end{bmatrix}^T \begin{bmatrix} (-2\boldsymbol{G} + \boldsymbol{I}) & -\boldsymbol{P}\boldsymbol{B}_w \\ -\boldsymbol{B}_w^T\boldsymbol{P} & -\gamma^2\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \operatorname{vec} \underline{\boldsymbol{e}}(t) \\ \boldsymbol{\omega}(t) \end{bmatrix},$$

for $\boldsymbol{G} = \boldsymbol{P}\boldsymbol{K}\boldsymbol{P}$. Note that the condition (9) holds if and only if the term $\dot{\boldsymbol{V}}_{H_\infty}(t)$ is negative definite. Thus, we must have

$$\dot{\boldsymbol{V}}(t) + \operatorname{vec} \underline{\boldsymbol{e}}^T(t) \operatorname{vec} \underline{\boldsymbol{e}}(t) - \gamma^2\boldsymbol{\omega}^T(t)\boldsymbol{\omega}(t) < 0,$$

and integrating the inequality from 0 to t, yields

$$\int_0^t \left[\dot{\boldsymbol{V}}(t) + \operatorname{vec} \underline{\boldsymbol{e}}^T(t) \operatorname{vec} \underline{\boldsymbol{e}}(t) - \gamma^2\boldsymbol{\omega}^T(t)\boldsymbol{\omega}(t)\right] dt =$$

$$\boldsymbol{V}(t) - \boldsymbol{V}(0) + \int_0^t \left[\operatorname{vec} \underline{\boldsymbol{e}}^T(t) \operatorname{vec} \underline{\boldsymbol{e}}(t) - \gamma^2\boldsymbol{\omega}^T(t)\boldsymbol{\omega}(t)\right] dt < 0.$$

Now, under zero initial conditions and given the Lyapunov function positiveness properties, we have

$$\int_0^t \operatorname{vec} \underline{\boldsymbol{e}}^T(t) \operatorname{vec} \underline{\boldsymbol{e}}(t) dt - \int_0^t \gamma^2\boldsymbol{\omega}^T(t)\boldsymbol{\omega}(t) dt < 0,$$

for all $t>0$. Note that $\boldsymbol{\omega}(t) \in L_2[0, \infty)$, thus $\operatorname{vec} \underline{\boldsymbol{e}}(t)$ is also L_2 and the inequality converges, which in turn is equivalent to $|| \operatorname{vec} \underline{\boldsymbol{e}}(t)||_2 < \gamma||\boldsymbol{\omega}(t)||_2$. Hence, the conditions in Definition 1 are satisfied, and the proof is completed. ■

Theorem 1 provides a feasible solution for the H_∞ problem that ensures the asymptotic stability of the dual quaternion error dynamics, while attenuating all exogenous disturbances. The resulting control scheme is obtained through the solution of an LMI[2]. Alternatively, we can state a new control procedure based on the solution of an algebraic Riccati equation,

$$\boldsymbol{A}_{ARE}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_{ARE} - \boldsymbol{P}\boldsymbol{M}\boldsymbol{P} + \boldsymbol{C}_{ARE} = 0. \tag{11}$$

Theorem 2 (ARE-based H_∞ controller): For a given scalar $\alpha>1$, and a prescribed $\gamma > 0$, a joint vector velocity H_∞ control that stabilizes the error dynamics, whereas ensuring disturbance attenuation properties in the sense of Definition 1, is given by $\dot{\boldsymbol{\theta}}(t) = \boldsymbol{N}^+\overline{\boldsymbol{K}} \operatorname{vec} \underline{\boldsymbol{e}}(t)$, where $\overline{\boldsymbol{K}} = \frac{1}{2}\left(\frac{1}{\gamma^2}\boldsymbol{B}_w\boldsymbol{B}_w^T + \boldsymbol{M}\right)\boldsymbol{P}$, for any positive definite matrix $\boldsymbol{M}$, and with the matrix $\boldsymbol{P}$ given as the solution of the ARE (11), for $\boldsymbol{A}_{ARE} = \boldsymbol{0}$, and $\boldsymbol{C}_{ARE} = \alpha\boldsymbol{I}$.

Proof: From the Schur complement of the LMI condition in Theorem 1, we have that (9) holds if and only if

$$\begin{aligned} &\boldsymbol{I} - 2\boldsymbol{P}\boldsymbol{K}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{B}_w\frac{1}{\gamma^2}\boldsymbol{B}_w^T\boldsymbol{P} \\ &\qquad = \boldsymbol{P}\left(\frac{1}{\gamma^2}\boldsymbol{B}_w\boldsymbol{B}_w^T - 2\boldsymbol{K}\right)\boldsymbol{P} + \boldsymbol{I} < 0, \end{aligned}$$

where $\boldsymbol{P}\boldsymbol{K}\boldsymbol{P} = \boldsymbol{G}$. Then, choosing

$$\boldsymbol{K} := \tfrac{1}{2}\left(\tfrac{1}{\gamma^2}\boldsymbol{B}_w\boldsymbol{B}_w^T + \boldsymbol{M}\right), \tag{12}$$

we obtain the inequality

$$-\boldsymbol{P}\boldsymbol{M}\boldsymbol{P} + \boldsymbol{I} < \boldsymbol{0}, \tag{13}$$

which is satisfied by the Riccati equation in (11), if $\boldsymbol{C}_{ARE}$ is chosen such that $\boldsymbol{C}_{ARE} > \boldsymbol{I}$. ■

To further simplify the choice of a feasible gain $\boldsymbol{M} > \boldsymbol{0}$, we may consider $\boldsymbol{M} = \sigma\boldsymbol{I}$, where σ is a positive scalar. In this particular case, the following corollary presents an optimal choice of σ in the sense of minimizing the norm of the control gain $\overline{\boldsymbol{K}}$ in Theorem 2.

Corollary 1 (Closed-form solution): For a prescribed $\gamma > 0$, a joint vector velocity H_∞ control that stabilizes the error dynamics, whereas ensuring disturbance attenuation properties in the sense of Definition 1, is given by $\dot{\boldsymbol{\theta}}(t) = \boldsymbol{N}^+\overline{\boldsymbol{K}} \operatorname{vec} \underline{\boldsymbol{e}}(t)$, where

$$\overline{\boldsymbol{K}} = \frac{1}{\gamma}\left(\boldsymbol{B}_w\boldsymbol{B}_w^T + \left[\frac{\sqrt{2}}{4}\boldsymbol{B}_w^T\boldsymbol{B}_w\right]\boldsymbol{I}\right)\frac{\alpha}{\sqrt{\boldsymbol{B}_w^T\boldsymbol{B}_w\sqrt{2}}}, \tag{14}$$

for any scalar $\alpha > 1$. Moreover, the norm of $\overline{\boldsymbol{K}}$ is given by $||\overline{\boldsymbol{K}}||_2 = \frac{1}{\gamma}\alpha\sqrt{\frac{1}{2}(1+\sqrt{8})\boldsymbol{B}_w^T\boldsymbol{B}_w}$.

[2]Interior point based algorithms, as the LMI Control Toolbox from Matlab, can solve this convex problem in polynomial time.

Proof: From (13) and taking the positive matrix $\boldsymbol{M}$ to be $\boldsymbol{M}{=}\sigma\boldsymbol{I}$, where σ is a positive scalar, we have $-\sigma\boldsymbol{P}\boldsymbol{P}+\boldsymbol{I}{<}\boldsymbol{0}$, which is satisfied if

$$-\sigma\boldsymbol{P}\boldsymbol{P}+\alpha^2\boldsymbol{I}{=}\boldsymbol{0}$$

holds for $\alpha>1$. A trivial solution for the above equation is $\boldsymbol{P}{=}\frac{\alpha}{\sqrt{\sigma}}\boldsymbol{I}$. Then, substituting this feasible $\boldsymbol{P}$ in $\overline{\boldsymbol{K}}$ yields

$$\overline{\boldsymbol{K}}=\frac{1}{2}\frac{\alpha}{\sqrt{\sigma}}\left(\frac{1}{\gamma^2}\boldsymbol{B}_w\boldsymbol{B}_w^T{+}\sigma\boldsymbol{I}\right). \tag{15}$$

The Frobenius norm of $\overline{\boldsymbol{K}}$ is given by

$$\begin{aligned}||\overline{\boldsymbol{K}}||_2&{=}\frac{1}{2}\frac{\alpha}{\sqrt{\sigma}}||\frac{1}{\gamma^2}\boldsymbol{B}_w\boldsymbol{B}_w^T{+}\sigma\boldsymbol{I}||_2\\&{=}\frac{1}{2}\frac{\alpha}{\sqrt{\sigma}}\sqrt{tr\left(\frac{\boldsymbol{B}_w\boldsymbol{B}_w^T\boldsymbol{B}_w\boldsymbol{B}_w^T}{\gamma^4}{+}\frac{2\sigma}{\gamma^2}\boldsymbol{B}_w\boldsymbol{B}_w^T{+}\sigma^2\boldsymbol{I}\right)}.\end{aligned}$$

Using trace properties, we are looking for the σ that minimizes $||\overline{\boldsymbol{K}}||_2$, i.e., $\hat{\sigma}=arg\min\limits_{\sigma}||\overline{\boldsymbol{K}}||_2$,

$$\begin{aligned}\hat{\sigma}&{=}arg\min_{\sigma}\left\{\frac{\alpha}{2}\sqrt{\frac{1}{\sigma}\left(\frac{\left[\boldsymbol{B}_w^T\boldsymbol{B}_w\right]^2}{\gamma^4}+\frac{2\left[\boldsymbol{B}_w^T\boldsymbol{B}_w\right]}{\gamma^2}\sigma+8\sigma^2\right)}\right.\\&{=}arg\min_{\sigma}\left\{\frac{1}{\sigma}\left(\frac{\left[\boldsymbol{B}_w^T\boldsymbol{B}_w\right]^2}{\gamma^4}+\frac{2\left[\boldsymbol{B}_w^T\boldsymbol{B}_w\right]}{\gamma^2}\sigma+8\sigma^2\right)\right\}.\end{aligned}$$

After some manipulation, we find $\hat{\sigma}=\frac{\sqrt{2}}{4\gamma^2}\boldsymbol{B}_w^T\boldsymbol{B}_w$, and the minimum norm given by $||\overline{\boldsymbol{K}}||_2=\frac{1}{\gamma}\alpha\sqrt{\frac{1}{2}(1{+}\sqrt{8})\boldsymbol{B}_w^T\boldsymbol{B}_w}$. Thus, replacing σ in (15) yields the control gain given in (14). ■

Corollary 1 provides a straightforward solution for the H_∞ control problem. The resulting joint based control strategy is easier to implement than the one of Theorem 1, as the control gain $\overline{\boldsymbol{K}}$ in (14) is a closed-form expression. Also, the solution is based on the optimal value for $\boldsymbol{M}{=}\sigma\boldsymbol{I}$, in the sense of reducing the gain norm, which in turn reduces the control effort required to maintain the H_∞ performance.

Remark 1: Both position and orientation configurations are regarded in an unified framework, which allows more efficient control techniques compared to conventional decoupled-based control. This is thanks to the fact that in our unified framework the error is invariant with respect to the choice of coordinate system (recall Section IV-A), whereas in the decoupled-based approach the error in position will always be dependent on the choice of coordinate system; that is, given the error

$$\boldsymbol{e}_p=\boldsymbol{p}_d-\boldsymbol{p}_m \tag{16}$$

between the desired and measured positions, a coordinate change given by a rotation $\boldsymbol{r}$ will result in $\boldsymbol{e}_p=\boldsymbol{r}\left(\boldsymbol{p}_d-\boldsymbol{p}_m\right)\boldsymbol{r}^*$, which is clearly different from the original position error (16). This way, using unit dual quaternions with an invariant error definition, all the proposed control strategies ensure the asymptotically convergence of the error to zero, while satisfying the attenuation properties described in Definition 1, and with a behavior that is independent of the choice of coordinate system. The results from Theorem 1, which can be shown to be equivalent to Theorem 2, are more general and less conservative than Corollary 1. However, the proposed conditions yield any feasible result, without regard to the control effort. In this context, Corollary 1, although being slightly more conservative, provides an easy to implement closed-form gain expression that also represents the minimum norm solution for (12) with $M{=}\sigma I$. To the best of the authors knowledge, this is the first work to exploit any of the aforementioned characteristics in the control design within the dual quaternion space.

V. Experiments

In this section, we present some experimental results in order to demonstrate the effectiveness of the proposed H_∞ control criteria presented in Section IV. The H_∞ control criteria is applied to a six-link manipulator. To this aim, we have derived for a Comau SMART SiX robot, the forward kinematics model in the dual quaternion space and the corresponding Jacobian matrix.

In the experiment, we seek to make the initial dual quaternion configuration $\underline{\boldsymbol{x}}_{\text{init}}$ converge to a desired configuration $\underline{\boldsymbol{x}}_d$, whereas reducing the influence of an exogenous disturbance over the system in the H_∞ sense. As initial configuration, we regard all joint positions equal to $\theta_{\text{init}}=\begin{bmatrix}-54 & 36 & -90 & 0 & 90 & 0\end{bmatrix}^T$, which corresponds to the following vector in dual quaternion space:

$$\begin{aligned}\operatorname{vec}\underline{\boldsymbol{x}}_{\text{init}}=[&-0.275\quad-0.432\quad 0.847\quad-0.140\\&-0.184\quad-0.387\quad-0.197\quad 0.362]^T.\end{aligned}$$

The desired end-effector position and orientation configuration is given by

$$\begin{aligned}\operatorname{vec}\underline{\boldsymbol{x}}_d=[&0.0\quad 0.707\quad 0.707\quad 0.0\\&0.282\quad-0.380\quad 0.380\quad 0.282]^T.\end{aligned}$$

The exogenous disturbance acting on the system is supposed to be $\omega(t)=1.1\cos(5t)$, with $B=[1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]^T$. The control requirement is to maintain the noise to error amplification less than the upper bound $\gamma=0.002$. All controllers, from Theorem 1 and 2, and Corollary 1, yield feasible results for this control problem. In this example, we shall consider the controller resulting from Corollary 1, which minimizes the Frobenius norm of (12) with $M{=}\sigma I$.

The first result regards the time response of each component of the dual quaternion vector, shown in Fig. 1. Besides the large disturbance, each component asymptotically converges to its desired reference. The end-effector position are

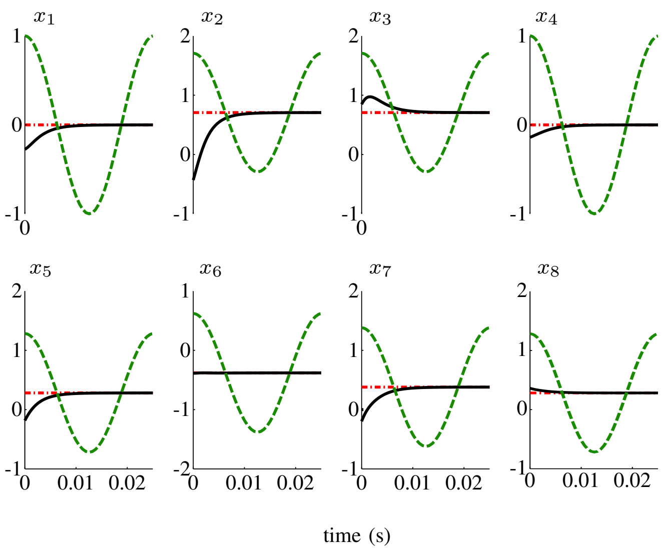

Fig. 1. Exogeneous disturbance (*green dashed curve*), desired dual quaternion coefficient (*red dash-dotted curve*), and measured dual quaternion coefficient (*black solid curve*).

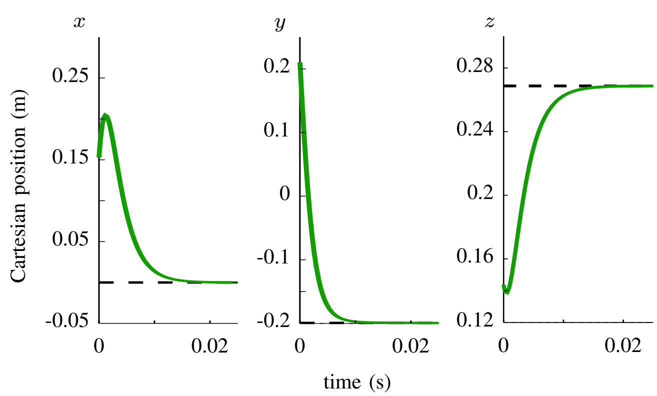

Fig. 2. Cartesian position: reference (*black dashed curve*) and measured value (*solid green*).

shown in Fig. 2, which also demonstrates the convergence efficiency of the control technique. Fig. 3 regards the evolution of the convergence error, which is described by the evolution of its norm, i.e., $||\operatorname{vec}\underline{\boldsymbol{e}}(t)||_2$. It is clear that the H_∞ control technique succeeded in reducing the disturbance influence upon the closed-loop system, whereas maintaining the system stability. Indeed, under zero initial conditions and numerically computing the error and noise norms, the resulting noise to error amplification is $\frac{\int_{t_0}^{t_1}||\operatorname{vec}\underline{\boldsymbol{e}}(t)||_2}{\int_{t_0}^{t_1}||\underline{\boldsymbol{\omega}}(t)||_2} = 0.0013$, which is less than the calculated upper bound $\gamma = 0.002$.

Moreover, in order to allow comparison with previous results, we have empirically chosen a controller, using the results from [11], with the same control performance obtained with Corollary 1, i.e., with a noise to error attenuation of 0.0013. The control effort of this brute-force manually chosen controller is shown in Fig. 4 (light green). The resulting control is up to 5 times more expensive than the

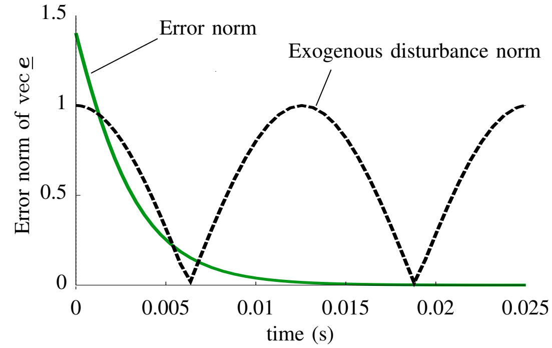

Fig. 3. Error norm with noise.

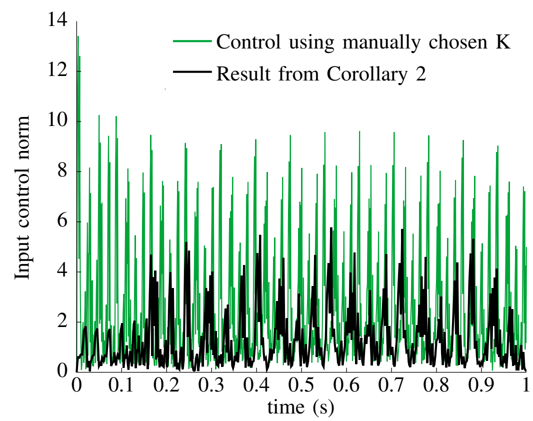

Fig. 4. Comparison from control effort .

controller obtained with Corollary 1, which minimizes a given energy criterion. The result enlightens the importance of the proposed robust control techniques in comparison to previous results.

VI. Conclusions

In this paper, we have presented new control strategies for robot manipulators directly in dual quaternion space. From a compact and complete representation of the end-effector position and orientation using unit dual quaternion, we proposed novel stabilization techniques for the asymptotic convergence of the dual position to a desired reference without decoupling rotational and translational dynamics. The control criteria also ensure disturbance attenuation properties in the H_∞ sense, and are invariant with regard to coordinate changes. The H_∞ control procedures may either rely on the solution of an LMI, or on the solution of an algebraic Riccati equation. Alternatively, we also proposed an easy to implement closed-form solution for the H_∞ problem, which additionally minimizes a given control effort criterion. The advantages and benefits from the proposed criteria are further highlighted with experimental results that illustrates the efficiency of the proposed methods.

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