MATH45061: SOLUTION SHEET[1] VI

1.) The Eulerian rate of deformation tensor D is defined, in Cartesian components, to be

$$D_{IJ} = \frac{1}{2}\left(V_{I,J} + V_{J,I}\right).$$

In order to construct the (traceless) deviatoric tensor $\widetilde{\mathsf{D}}$, we first find the trace of D:

$$\text{trace}(\mathsf{D}) = D_{KK} = V_{K,K} = \boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V}.$$

If we subtract the trace(D) multiplied by the identity from D then we have

$$\widehat{\mathsf{D}} = \mathsf{D} - \text{trace}(\mathsf{D})\mathsf{I},$$

but

$$\text{trace}(\widehat{\mathsf{D}}) = \text{trace}(\mathsf{D}) - \text{trace}(\mathsf{D})\text{trace}(\mathsf{I}) \neq 0,$$

because trace(I) $= 3$, so we must also divide the second term by the trace of the identity to obtain our deviatoric tensor

$$\widetilde{\mathsf{D}} = \mathsf{D} - \frac{1}{3}\text{trace}(\mathsf{D})\mathsf{I},$$

$$\Rightarrow \quad \mathsf{D} = \widetilde{\mathsf{D}} + \frac{1}{3}\left(\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V}\right)\mathsf{I}.$$

2.) A compressible Newtonian fluid has the constitutive law

$$\mathsf{T} = -P\,\mathsf{I} + \lambda(\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V})\,\mathsf{I} + 2\mu\mathsf{D}, \tag{1}$$

and the heat flux is given by

$$\boldsymbol{Q} = -\kappa\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta. \tag{2}$$

These laws are consistent with the assumptions in the lecture notes that led to the constraints (you can go through the working again, if you like ... it's good for you!),

$$\widetilde{\mathsf{T}} : \mathsf{D} \geq 0 \quad \text{and} \quad -\frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta \geq 0,$$

where $\mathsf{T} = -P\,\mathsf{I} + \widetilde{\mathsf{T}}$. Using the constitutive laws (1) and (2) in the above gives

$$\lambda(\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V})\,\mathsf{I} : \mathsf{D} + 2\mu\mathsf{D} : \mathsf{D} \geq 0 \quad \text{and} \quad \frac{1}{\Theta}\kappa(\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta)^2 \geq 0.$$

The temperature $\Theta > 0$ and $(\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta)^2 \geq 0$, so the second inequality is only satisfied if $\kappa \geq 0$, as required. Using the fact that $\mathsf{I} : \mathsf{D} = \boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V}$, the first inequality becomes

$$\lambda(\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V})^2 + 2\mu\mathsf{D} : \mathsf{D} \geq 0.$$

The two terms in the above inequality do not vary independently so we cannot proceed further unless we separate the changes in volume from the other deformations

$$\mathsf{D} = \widetilde{\mathsf{D}} + \frac{1}{3}(\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V})\,\mathsf{I},$$

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and then the inequality becomes

$$\lambda(\boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V})^2 + 2\mu\left(\widetilde{\mathsf{D}} + \frac{1}{3}(\boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V})\,\mathsf{I}\right) : \left(\widetilde{\mathsf{D}} + \frac{1}{3}(\boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V})\,\mathsf{I}\right) \geq 0,$$

$$\Rightarrow \quad \lambda(\boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V})^2 + 2\mu\widetilde{\mathsf{D}} : \widetilde{\mathsf{D}} + 2\mu\frac{1}{9}(\boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V})^2\mathsf{I} : \mathsf{I} + 2\mu\frac{2}{3}(\boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V})\,\mathsf{I} : \widetilde{\mathsf{D}} \geq 0.$$

By construction $\widetilde{\mathsf{D}}$ is traceless so $\mathsf{I} : \widetilde{\mathsf{D}} = 0$ and $\mathsf{I} : \mathsf{I} = 3$, which means that

$$\left(\lambda + \frac{2}{3}\mu\right)(\boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V})^2 + 2\mu\widetilde{\mathsf{D}} : \widetilde{\mathsf{D}} \geq 0.$$

The two terms are now independent, which means that we can construct a process in which $\widetilde{\mathsf{D}} = 0$, but $(\boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V} \neq 0$, in which case

$$\left(\lambda + \frac{2}{3}\mu\right)(\boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V})^2 \geq 0;$$

and because the square term is positive (and not zero), we must have $(\lambda + \frac{2}{3}\mu) \geq 0$, as required. Similarly, we can construct an isochoric process for which $(\boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V} = 0$ and $\widetilde{\mathsf{D}} \neq 0$, and then

$$2\mu\widetilde{\mathsf{D}} : \widetilde{\mathsf{D}} \geq 0.$$

The term $\widetilde{\mathsf{D}} : \widetilde{\mathsf{D}}$ consists of the sum of squares so it must be positive and then we deduce that $\mu \geq 0$, as required.

3.) The kinetic energy of the fluid is defined by

$$K(t) = \int_{\mathcal{D}} \frac{\rho}{2}\boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{V}\,\mathrm{d}\mathcal{V},$$

so

$$\frac{DK}{Dt} = \frac{D}{Dt}\int_{\mathcal{D}} \frac{\rho}{2}\boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{V}\,\mathrm{d}\mathcal{V},$$

and using the Reynolds transport theorem and conservation of mass (as in the lecture notes) we obtain

$$\frac{DK}{Dt} = \int_{\mathcal{D}} \frac{\rho}{2}\frac{D}{Dt}(\boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{V})\,\mathrm{d}\mathcal{V} = \int_{\mathcal{D}} \rho\boldsymbol{V}\boldsymbol{\cdot}\frac{D\boldsymbol{V}}{Dt}\,\mathrm{d}\mathcal{V}.$$

Using the balance of linear momentum to replace $\rho D\boldsymbol{V}/Dt$, we have

$$\frac{DK}{Dt} = \int_{\mathcal{D}} 2\mu\,(\boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\mathsf{D})\boldsymbol{\cdot}\boldsymbol{V} - \boldsymbol{V}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!\boldsymbol{R}}(P + \rho G)\,\mathrm{d}\mathcal{V},$$

after using the fact that $\boldsymbol{F} = -\boldsymbol{\nabla}_{\!\boldsymbol{R}}G$. If we write the integrand using index notation in Cartesian coordinates (for simplicity), we have

$$\frac{DK}{Dt} = \int_{\mathcal{D}} 2\mu D_{IK,K}V_I - V_K(P + \rho G)_{,K}\,\mathrm{d}\mathcal{V}$$

and using the product rule, we can write

$$\frac{DK}{Dt} = \int_{\mathcal{D}} (2\mu D_{IK} V_I - (P+\rho G)V_K)_{,K} - 2\mu D_{IK} V_{I,K} + (P+\rho G) V_{K,K} \, \mathrm{d}\mathcal{V}.$$

The last term vanishes because the fluid is incompressible $V_{K,K} = 0$ and we can use the divergence theorem on the first term to obtain

$$\frac{DK}{Dt} = \int_{\partial\mathcal{D}} (2\mu D_{IK} V_I - (P+\rho G)V_K)\, N_K \, \mathrm{d}\mathcal{S} - \int_{\mathcal{D}} 2\mu D_{IK} V_{I,K} \, \mathrm{d}\mathcal{V}.$$

The boundary is stationary which means that on the boundary $\boldsymbol{V} = \boldsymbol{0}$ and $\boldsymbol{V} \cdot \boldsymbol{N} = 0$, so the surface integral vanishes and we have

$$\frac{DK}{Dt} = -2\mu \int_{\mathcal{D}} D_{IK} V_{I,K} \, \mathrm{d}\mathcal{V} = -2\mu \int_{\mathcal{D}} \frac{1}{2} \left(D_{IK} V_{I,K} + D_{KI} V_{K,I} \right) \mathrm{d}\mathcal{V}.$$

Using the symmetry of D, we obtain the desired result

$$\frac{DK}{Dt} = -2\mu \int_{\mathcal{D}} D_{IK} \frac{1}{2} \left(V_{I,K} + V_{K,I} \right) \mathrm{d}\mathcal{V} = -2\mu \int_{\mathcal{D}} D_{IK} D_{IK} \, \mathrm{d}\mathcal{V},$$

$$\Rightarrow \quad \frac{DK}{Dt} + 2\mu \int_{\mathcal{D}} \mathsf{D} : \mathsf{D} \, \mathrm{d}\mathcal{V} = 0.$$

Incidentally, this result shows that the kinetic energy is conserved in an inviscid fluid, but that it decreases (is dissipated by viscosity) in a viscous fluid.

4.) a.) For cylindrical polar coordinates the position is given by

$$\boldsymbol{R} = (R\cos\Theta, R\sin\Theta, Z)^T,$$

which means that if we choose $\xi^1 = R$, $\xi^2 = \Theta$, $\xi^3 = Z$,

$$\begin{aligned}
\boldsymbol{G}_1 &= (\cos\Theta, \sin\Theta, 0)^T = \boldsymbol{e}_R, \\
\boldsymbol{G}_2 &= (-R\sin\Theta, R\cos\Theta, 0)^T = R\boldsymbol{e}_\Theta, \\
\boldsymbol{G}_3 &= (0,0,1)^T = \boldsymbol{e}_Z.
\end{aligned}$$

It follows that $\boldsymbol{G}^1 = \boldsymbol{e}_R$, $\boldsymbol{G}^2 = \frac{1}{R}\boldsymbol{e}_\Theta$, $\boldsymbol{G}^3 = \boldsymbol{e}_Z$. If we write the velocity in the form $\boldsymbol{V} = V^i \boldsymbol{G}_i$, then where $V^i = \boldsymbol{V} \cdot \boldsymbol{G}^i$, then

$$V^1 = F(R)\, \boldsymbol{e}_\Theta \cdot \boldsymbol{e}_R = 0, \quad V^2 = F(R)\, \boldsymbol{e}_\Theta \cdot \frac{1}{R} \boldsymbol{e}_\Theta = \frac{F(R)}{R}, \quad V^3 = F(R)\, \boldsymbol{e}_\Theta \cdot \boldsymbol{e}_Z = 0.$$

b.) The rate of deformation tensor is given by

$$D^{ij} = \frac{1}{2} \left(G^{jk}\, V^i||_k + G^{ik}\, V^j||_k \right),$$

where $V^i||_j = V^i_{,j} + \Gamma^i_{jk} V^k$. The only non-zero Christoffel symbols are $\Gamma^2_{12} = 1/R$, $\Gamma^1_{22} = -R$ (see example sheet 1), so for the given velocity field we have

$$V^1||_2 = V^1_{,2} + \Gamma^1_{22} V^2 = -F(R),$$

$$V^2||_1 = V^2_{,1} + \Gamma^2_{12} V^2 = \frac{F'(R)}{R} - \frac{F(R)}{R^2} + \frac{1}{R}\frac{F(R)}{R} = \frac{F'(R)}{R},$$

and all other components $V^i||_j$ are zero. The only non-zero entries of the contravariant metric tensor are $G^{11} = G^{33} = 1$, $G^{22} = 1/R^2$, so

$$D^{12} = \frac{1}{2}\left(G^{22}\, V^1||_2 + G^{11}\, V^2||_1\right) = \frac{1}{2}\left(-\frac{F(R)}{R^2} + \frac{F'(R)}{R}\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{F(r)}{R}\right),$$

and all other values of D^{ij} are zero. The constitutive relation is given in a general coordinate-free notation; and in Cartesian components, it would be written

$$T_{IJ} = -P\delta_{IJ} + 2\mu D_{IJ}. \tag{3}$$

Converting equation (3) into components in the general basis via the tensor transformation laws gives

$$T^{ij} = \frac{\partial \xi^i}{\partial X_I} T_{IJ} \frac{\partial \xi^j}{\partial X_J} = -P\frac{\partial \xi^i}{\partial X_I}\delta_{IJ}\frac{\partial \xi^j}{\partial X_J} + 2\mu\frac{\partial \xi^i}{\partial X_I}D_{IJ}\frac{\partial \xi^j}{\partial X_J},$$

$$\Rightarrow T^{ij} = -P\frac{\partial \xi^i}{\partial X_I}\frac{\partial \xi^j}{\partial X_I} + 2\mu D^{ij} = -PG^{ij} + 2\mu D^{ij}.$$

Thus, the stress tensor is

$$T^{ij} = \begin{pmatrix} -P & \mu\,(F(R)/R)' & 0 \\ \mu\,(F(R)/R)' & -P/R^2 & 0 \\ 0 & 0 & -P \end{pmatrix},$$

as required.

c.) The balance of linear momentum equation is

$$\rho\frac{DV^i}{Dt} = \rho\left[\frac{\partial V^i}{\partial t} + V^j V^i||_j\right] = T^{ij}||_j + \rho F^i,$$

where $T^{ij}||_j = T^{ij}_{,j} + \Gamma^i_{jk}T^{jk} + \Gamma^j_{jk}T^{ik}$.
Thus for $i = 1, 2, 3$, we have (only showing the non-zero Christoffel symbol terms)

$$\rho V^2 V^1||_2 = T^{11}_{,1} + T^{12}_{,2} + T^{13}_{,3} + \Gamma^1_{22}T^{22} + \Gamma^2_{21}T^{11},$$

$$0 = T^{21}_{,1} + T^{22}_{,2} + T^{23}_{,3} + \Gamma^2_{12}T^{12} + \Gamma^2_{21}T^{21} + \Gamma^2_{21}T^{21},$$

$$0 = T^{31}_{,1} + T^{32}_{,2} + T^{33}_{,3} - \rho g,$$

which, on using the explicit form for the stress tensor, become

$$-\rho\frac{F^2}{R} = -\frac{\partial P}{\partial R} - R\frac{P}{R^2} + \frac{P}{R} = -\frac{\partial P}{\partial R}, \tag{4}$$

$$0 = [\mu(F/R)']' - \frac{1}{R^2}\frac{\partial P}{\partial \Theta} + 3\frac{\mu}{R}(F/R)', \tag{5}$$

$$0 = -\frac{\partial P}{\partial Z} - \rho g. \tag{6}$$

If we assume that the solution is axisymmetric then P is not a function of Θ and letting $y = T^{12}$, equation (5) becomes

$$y' + 3y/R = 0 \quad \Rightarrow \int \frac{y'}{y}\,\mathrm{d}y = -\int \frac{3}{R}\,\mathrm{d}R \quad \Rightarrow \ln y = -3\ln R + C,$$

$$\Rightarrow y = T^{12} = \frac{K}{R^3},$$

where $\ln K = C$. It follows from direct integration of equations (4, 5) that

$$P = -P_0 + \int^R \rho \frac{F^2}{s}\,\mathrm{d}s + \rho g z,$$

which gives the required answer.

The corresponding physical component of the stress is given by

$$\sigma_{(12)} = \sqrt{G_{22}/G^1}T^{12} = \sqrt{R^2}T^{12} = \frac{K}{R^2}.$$

d.) The outer unit normal to the cylinder is given by $\boldsymbol{e}_R$, so that the contravariant components of the traction exerted **on** the cylinder of fluid are

$$T^j = T^{ij}N_i = T^{1j},$$

because $N_1 = 1$ and $N_2 = N_3 = 0$. Thus, the only non-zero components are T^1 and T^2 and the traction exerted **on** the fluid is then

$$\boldsymbol{T} = T^{11}\boldsymbol{G}_1 + T^{12}\boldsymbol{G}_2.$$

It follows that the traction exerted **on** the fluid is

$$\boldsymbol{T} = -P\boldsymbol{e}_R + \frac{K}{R^2}\boldsymbol{e}_\Theta.$$

The total torque is given by

$$\int \boldsymbol{R}\times\boldsymbol{T}\,\mathrm{d}V = \int R\boldsymbol{e}_R \times \left(-P\boldsymbol{e}_R + \frac{K}{R^2}\boldsymbol{e}_\Theta\right)\,\mathrm{d}A,$$

$$= \int \frac{K}{R}\boldsymbol{e}_Z R\,\mathrm{d}\Theta\,\mathrm{d}Z = 2\pi K H \boldsymbol{e}_Z,$$

where H is the height of the cylinder. Thus, the torque per unit length is $T\boldsymbol{e}_Z = 2\pi K\boldsymbol{e}_Z$, which implies that $K = T/2\pi$.

e.) The first invariant is the trace of the rate of deformation tensor ${}_{\mathsf{D}}I_1 = D^i_i = 0$, the second strain invariant is

$${}_{\mathsf{D}}I_2 = \frac{1}{2}\left(({}_{\mathsf{D}}I_1)^2 - D^{ij}G_{jl}G_{im}D^{lm}\right) = -\frac{1}{4}R^2\left[\frac{\partial(F/R)}{\partial R}\right]^2.$$

Thus,

$$\mu \propto R^{n-1}\left[\frac{\partial(F/R)}{\partial R}\right]^{n-1} = \alpha R^{n-1}\left[\frac{\partial(F/R)}{\partial R}\right]^{n-1},$$

for some constant α, and so

$$T^{12} = \mu(F/R)' = \alpha R^{n-1}\left[\frac{\partial(F/R)}{\partial R}\right]^n.$$

Using the results from (c.) and (d.) we have that

$$T^{12} = \frac{T}{2\pi}\frac{1}{R^3} = \alpha R^{n-1}\left[\frac{\partial(F/R)}{\partial R}\right]^n,$$

$$\Rightarrow \quad \left(\frac{T}{2\pi\alpha}\right)^{\frac{1}{n}} R^{-\left(1+\frac{2}{n}\right)} = \frac{\partial(F/R)}{\partial R};$$

and integrating with respect to R gives

$$V^2 = \frac{F}{R} = -\frac{n}{2}\left(\frac{T}{2\pi\alpha}\right)^{\frac{1}{n}} R^{-\frac{2}{n}} + C,$$

applying the boundary conditions $V^2 = 0$ at $R = a$, we obtain

$$V^2 = \frac{n}{2}\left(\frac{T}{2\pi\alpha}\right)^{\frac{1}{n}}\left(a^{-\frac{2}{n}} - R^{-\frac{2}{n}}\right),$$

and from the second boundary condition $V^2 = \Omega$ at $R = b$, we deduce that

$$\Omega = \frac{n}{2}\left(\frac{T}{2\pi\alpha}\right)^{\frac{1}{n}}\left(a^{-\frac{2}{n}} - b^{-\frac{2}{n}}\right),$$

5.) a.) This is a simple matter of substituting $\Phi = c_p\Theta$ into the energy equation (7.16c) in the lecture notes to obtain

$$\rho c_p \frac{D\Theta}{Dt} = 2\mu \mathsf{D} : \mathsf{D} + \kappa \boldsymbol{\nabla}_{\boldsymbol{R}}^2\Theta + \rho B.$$

We then convert to Cartesian components to obtain

$$\rho c_p\left[\frac{\partial\Theta}{\partial t} + V_I\Theta_{,I}\right] = 2\mu\frac{1}{4}\left(V_{I,J} + V_{J,I}\right)\left(V_{I,J} + V_{J,I}\right) + \kappa\Theta_{,KK} + \rho B,$$

and expanding the velocity gradient terms and relabelling the indices we obtain the result

$$\rho c_p\left[\frac{\partial\Theta}{\partial t} + V_I\Theta_{,I}\right] = \mu\left(V_{I,J}V_{I,J} + V_{J,I}V_{I,J}\right) + \kappa\Theta_{,KK} + \rho B.$$

b.) If we choose typical velocity, length, time and temperature scales by

$$V_I = \mathcal{U}V_I^*, \quad X_I = \mathcal{L}X_I^*, \quad t = \mathcal{T}t^*, \quad \Theta = \Delta\Theta\Theta^*,$$

then the equations become

$$\rho c_p\left[\frac{\Delta\Theta}{\mathcal{T}}\frac{\partial\Theta^*}{\partial t^*} + \frac{\mathcal{U}\Delta\Theta}{\mathcal{L}}V_I^*\Theta_{,I}^*\right] = \frac{\kappa\Delta\Theta}{\mathcal{L}^2}\Theta_{,KK}^* + \frac{\mu\mathcal{U}^2}{\mathcal{L}^2}\left(V_{I,J}^*V_{I,J}^* + V_{I,J}^*V_{J,I}^*\right).$$

If we choose the natural timescale $\mathcal{T} = \mathcal{L}/\mathcal{U}$ and divide by $\kappa\Delta\Theta/\mathcal{L}^2$ then we have

$$\frac{\rho c_p \mathcal{U}\mathcal{L}}{\kappa}\left[\frac{\partial \Theta^*}{\partial t^*} + V_I^* \Theta^*_{,I}\right] = \Theta^*_{,KK} + \frac{\mu \mathcal{U}^2}{\kappa \Delta\Theta}\left(V^*_{I,J}V^*_{I,J} + V^*_{I,J}V^*_{J,I}\right).$$

We define Pe, the Péclet number, to be $Pe = \rho c_p \mathcal{U}\mathcal{L}/\kappa$, which reflects the ratio of transport of temperature due to convection by the fluid to that due to diffusion, and then

$$Pe\left[\frac{\partial \Theta^*}{\partial t^*} + V_I^* \Theta^*_{,I}\right] = \Theta^*_{,KK} + Pe\frac{\mu \mathcal{U}}{\rho c_p \mathcal{L}\Delta\Theta}\left(V^*_{I,J}V^*_{I,J} + V^*_{I,J}V^*_{J,I}\right),$$

so $A = \mu\mathcal{U}/(\rho c_p \mathcal{L}\Delta\Theta)$, which represents the ratio of work done by the viscous forces to the change in internal energy caused by the typical temperature change $\Delta\Theta$.

c.) For $C = 0$ and a steady shear flow, the dimensionless energy equation becomes

$$Pe\gamma Y^* \frac{\partial \Theta^*}{\partial X^*} = \frac{\partial^2 \Theta^*}{\partial^2 X^*} + \frac{\partial^2 \Theta^*}{\partial^2 Y^*} \quad \Rightarrow \quad \gamma Y^* \frac{\partial \Theta^*}{\partial X^*} = \frac{1}{Pe}\left[\frac{\partial^2 \Theta^*}{\partial^2 X^*} + \frac{\partial^2 \Theta^*}{\partial^2 Y^*}\right].$$

In the limit of large Pe, we would lose the highest derivatives, so this is a singular perturbation. We shall assume the existence of a thin boundary layer of thickness δ adjacent to the plane wall and rescale into boundary-layer coordinates:

$$X^* = \tilde{X}, \quad Y^* = \delta\tilde{Y}, \quad \Theta^* = \tilde{\Theta},$$

so that

$$\gamma\,\delta\,\tilde{Y}\frac{\partial\tilde{\Theta}}{\partial\tilde{X}} = \frac{1}{Pe}\frac{\partial^2\tilde{\Theta}}{\partial^2\tilde{X}} + \frac{1}{Pe\delta^2}\frac{\partial^2\tilde{\Theta}}{\partial^2\tilde{Y}}.$$

Now as $Pe \to \infty$, the first term on the right-hand side will tend to zero, but the second term can remain in balance with the left-hand side provided that $\delta \sim 1/(Pe\delta^2)$. (Note that this is different from the viscous boundary layer because in that case the advective terms are $\mathcal{O}(1)$, rather than the $\mathcal{O}(\delta)$ here). Thus, $\delta^3 \sim Pe^{-1}$, which give the required boundary-layer scaling $\delta \sim Pe^{-1/3}$ and then the governing equation becomes

$$\gamma\,\tilde{Y}\frac{\partial\tilde{\Theta}}{\partial\tilde{X}} = \frac{\partial^2\tilde{\Theta}}{\partial^2\tilde{Y}},$$

as required.

You may be interested to know that a similarity solution technique can be used to solve this equation, by letting $\eta = \tilde{Y}\tilde{X}^{-1/3}$, then

$$\frac{\partial}{\partial\tilde{X}} = \frac{\partial\eta}{\partial\tilde{X}}\frac{\partial}{\partial\eta} = -\frac{\eta}{3}\tilde{X}^{-1}\frac{\partial}{\partial\eta}, \quad \frac{\partial}{\partial\tilde{Y}} = \frac{\partial\eta}{\partial\tilde{Y}}\frac{\partial}{\partial\eta} = \tilde{X}^{-1/3}\frac{\partial}{\partial\eta},$$

which means that

$$\gamma\tilde{X}^{1/3}\eta\left(-\frac{\eta}{3}\tilde{X}^{-1}\right)\frac{\partial\tilde{\Theta}}{\partial\eta} = \tilde{X}^{-2/3}\frac{\partial^2\tilde{\Theta}}{\partial\eta^2},$$

and

$$\frac{\partial^2\tilde{\Theta}}{\partial\eta^2} + \gamma\eta^2\frac{\partial\tilde{\Theta}}{\partial\eta} = 0,$$

which can be integrated to give a solution of the form

$$\tilde{\Theta} = \int^{\eta} \mathrm{e}^{-\frac{\gamma}{3}s^3}\,\mathrm{d}s.$$