MATH45061: SOLUTION SHEET[1] V

1.) a.) The faces of the cube remain aligned with the same coordinate planes. We assign Cartesian coordinates aligned with the original cube (x, y, z), where $0 \le x, y, z \le 1$. The stretched lengths of the three sets of parallel sides are λ_1, λ_2 and λ_3 and the cube does not rotate, so provided that we choose the axis appropriately the deformed position is

$$X = \lambda_1 x, \quad Y = \lambda_2 y, \quad Z = \lambda_3 z,$$

as required.

b.) In order to find the strain invariants, we must compute the deformed and undeformed metric tensors. We have a Cartesian coordinate system so the undeformed metric is just the identity

$$g_{ij} = g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The deformed covariant base vectors are given by

$$\boldsymbol{G}_1 = (\lambda_1, 0, 0)^T, \quad \boldsymbol{G}_2 = (0, \lambda_2, 0)^T, \quad \boldsymbol{G}_3 = (0, 0, \lambda_3)^T,$$

which gives the metric tensors

$$G_{ij} = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix}, \qquad G^{ij} = \begin{pmatrix} \frac{1}{\lambda_1^2} & 0 & 0 \\ 0 & \frac{1}{\lambda_2^2} & 0 \\ 0 & 0 & \frac{1}{\lambda_3^2} \end{pmatrix}.$$

Thus the strain invariants are

$$I_1 = g^{ij} G_{ij} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,$$

$$I_2 = \frac{1}{2}\left[I_1^2 - g^{ik} G_{jk} g^{jl} G_{il}\right] = \frac{1}{2}\left[I_1^2 - \delta^{ik} G_{jk} \delta^{jl} G_{il}\right] = \frac{1}{2} = \left[I_1^2 - G_{ji} G_{ij}\right],$$

$$= \frac{1}{2}\left[(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^2 - (\lambda_1^4 + \lambda_2^4 + \lambda_3^4\right] = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2,$$

$$I_3 = G/g = \lambda_1^2\lambda_2^2\lambda_3^2.$$

c.) The general constitutive law is

$$T^{ij} = A g^{ij} + B B^{ij} + P G^{ij},$$

where $A = \frac{2}{\sqrt{I_3}}\frac{\partial \mathcal{W}}{\partial I_1}$, $B = \frac{2}{\sqrt{I_3}}\frac{\partial \mathcal{W}}{\partial I_2}$, $P = 2\sqrt{I_3}\frac{\partial \mathcal{W}}{\partial I_3}$, and $B^{ij} = I_1 g^{ij} - g^{ik} g^{jl} G_{kl}$, so

$$B^{ij} = \begin{pmatrix} I_1 - \lambda_1^2 & 0 & 0 \\ 0 & I_1 - \lambda_2^2 & 0 \\ 0 & 0 & I_1 - \lambda_3^2 \end{pmatrix} = \begin{pmatrix} \lambda_2^2 + \lambda_3^2 & 0 & 0 \\ 0 & \lambda_1^2 + \lambda_3^2 & 0 \\ 0 & 0 & \lambda_1^2 + \lambda_2^2 \end{pmatrix}.$$

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The strain invariants do not depend on position and so A, B and P are constants. It follows that the only non-zero stress components are the diagonal entries

$$
\begin{aligned}
T^{11} &= A + B(\lambda_2^2 + \lambda_3^2) + P/\lambda_1^2,\\
T^{22} &= A + B(\lambda_1^2 + \lambda_3^2) + P/\lambda_2^2,\\
T^{33} &= A + B(\lambda_1^2 + \lambda_2^2) + P/\lambda_3^2.
\end{aligned}
$$

All the stress components are constant in space, which means that the equilibrium equations $T^{ij}||_j = 0$ are trivially satisfied.

d.) We assume that the stretches in the y and z directions are the same $\lambda_2 = \lambda_3$ The stress components are therefore

$$
\begin{aligned}
T^{11} &= A + 2B\lambda_2^2 + P/\lambda_1^2,\\
T^{22} = T^{33} &= A + B(\lambda_1^2 + \lambda_2^2) + P/\lambda_2^2.
\end{aligned}
$$

The in-plane stresses are zero, so $T^{22} = T^{33} = 0$, which means that

$$P = -A\lambda_2^2 - B\lambda_2^2(\lambda_1^2 + \lambda_2^2),$$

and therefore

$$T^{11} = A + 2B\lambda_2^2 - A\frac{\lambda_2^2}{\lambda_1^2} - B\lambda_2^2\left(1 + \frac{\lambda_2^2}{\lambda_1^2}\right) = (A + B\lambda_2^2)\left(1 - \frac{\lambda_2^2}{\lambda_1^2}\right).$$

The physical stress component is

$$\sigma_{(11)} = T^{11}\sqrt{G_{11}/G^{11}} = \lambda_1^2 T^{11} = (A + B\lambda_2^2)\left(\lambda_1^2 - \lambda_2^2\right);$$

and if the body is incompressible then

$$I_3 = \lambda_1^2\lambda_2^4 = 1 \quad \Rightarrow \quad \lambda_1 = 1/\lambda_2^2,$$

which yields

$$\sigma_{(11)} = (A + B/\lambda_1)(\lambda_1^2 - 1/\lambda_1).$$

If $\lambda_2 = \lambda_3 = 1$, then incompressibility demands that $\lambda_1 = 1$ and the body does not deform, which means that $\sigma_{(11)} = 0$. The physical interpretation is that we cannot possibly stretch an incompressible body in only one direction: if the volume is to remain constant the body must also deform in other directions.

e.) If we assume that $\lambda_1 = 1 + \epsilon\tilde{\lambda}$, then

$$
\begin{aligned}
\sigma_{(11)} &= (A + B(1+\epsilon\tilde{\lambda})^{-1})\left((1+\epsilon\tilde{\lambda})^2 - (1+\epsilon\tilde{\lambda}^{-1})\right),\\
&= (A + B - B\epsilon\tilde{\lambda})(1 + 2\epsilon\tilde{\lambda} - 1 + \epsilon\tilde{\lambda}) + \mathcal{O}(\epsilon^2),\\
&= (A + B)(3\epsilon\tilde{\lambda}),
\end{aligned}
$$

which is the classic linear result that the stress is proportional to the strain $e_{11} = (\lambda_1 - 1)/1 = \epsilon\tilde{\lambda}$.

2.) The three strain invariants are defined by

$$I_1 = g^{ij}G_{ij}, \quad I_2 = \frac{1}{2}\left[I_1^2 - g^{ik}G_{jk}g^{jl}G_{il}\right], \quad I_3 = G/g,$$

and under infinitesimal deformation (from the lecture notes)

$$G_{ij} = g_{ij} + 2\epsilon\tilde{e}_{ij} + \mathcal{O}(\epsilon^2),$$

so

$$I_1 = g^{ij}(g_{ij} + 2\epsilon\tilde{e}_{ij}) = (\delta^i_i + 2\epsilon\tilde{e}^k_k) = 3 + 2e^k_k.$$

$$\begin{aligned} I_2 &= \frac{1}{2}\left[(3 + 2\epsilon\tilde{e}^k_k)^2 - g^{ik}(g_{jk} + 2\epsilon\tilde{e}_{jk})g^{jl}(g_{il} + 2\epsilon\tilde{e}_{il})\right], \\ &= \frac{1}{2}\left[9 + 12\epsilon\tilde{e}^k_k - (\delta^i_j + 2\epsilon\tilde{e}^i_j)(\delta^j_i + 2\epsilon\tilde{e}^j_i)\right] + \mathcal{O}(\epsilon^2), \\ &= \frac{1}{2}\left[9 + 12\epsilon\tilde{e}^k_k - (\delta^i_i + 2\epsilon\tilde{e}^i_i + 2\epsilon\tilde{e}^i_i)\right] + \mathcal{O}(\epsilon^2). \\ &\approx \frac{1}{2}\left[6 + 8\epsilon\tilde{e}^i_i\right] = 3 + 4e^k_k. \end{aligned}$$

Now, in Cartesians $g_{IJ} = \delta_{IJ}$ and $g = 1$, so

$$I_3 = G/g = e^{IJK}G_{1I}G_{2J}G_{3K},$$

after expanding out the determinant using the alternating symbol e^{IJK}. We can now use our approximation to write

$$\begin{aligned} I_3 &= e^{IJK}(\delta_{1I} + 2\epsilon\tilde{e}_{1I})(\delta_{2J} + 2\epsilon\tilde{e}_{2J})(\delta_{3K} + 2\epsilon\tilde{e}_{3K}), \\ &= e^{IJK}\left[\delta_{1I}\delta_{2J}\delta_{3K} + 2\epsilon(\delta_{1I}\delta_{2J}\tilde{e}_{3K} + \delta_{1I}\tilde{e}_{2J}\delta_{3K} + \tilde{e}_{1I}\delta_{2J}\delta_{3K})\right]. \end{aligned}$$

Using the fact that the only non-zero terms occur when $I = 1$, $J = 2$ and $K = 3$, we have

$$I_3 = 1 + 2\epsilon\left[\tilde{e}_{33} + \tilde{e}_{22} + \tilde{e}_{33}\right],$$

Converting back to general coordinates gives

$$I_3 = 1 + 2e^k_k.$$

3.) The St. Venant–Kirchhoff strain energy function is

$$\mathcal{W} = \frac{\lambda}{2}\gamma^i_i\gamma^j_j + \mu(\gamma^i_j\gamma^j_i) = \frac{\lambda}{2}g^{ik}\gamma_{ki}g^{jl}\gamma_{lj} + \mu\left(g^{ik}\gamma_{kj}g^{jl}\gamma_{li}\right).$$

The second Piola–Kirchhoff stress tensor is given by $s^{nm} = \frac{\partial\mathcal{W}}{\partial\gamma_{nm}}$, so

$$\begin{aligned} s^{nm} &= \frac{\lambda}{2}\left(g^{ik}\delta_{kn}\delta_{im}g^{jl}\gamma_{lj} + g^{ik}\gamma_{ki}g^{jl}\delta_{ln}\delta jm\right) + \mu\left(g^{ik}\delta_{nk}\delta_{mj}g^{jl}\gamma_{li} + g^{ik}\gamma_{kj}g^{jl}\delta_{ln}\delta_{im}\right), \\ &= \frac{\lambda}{2}\left(g^{nm}\gamma^j_j + g^{nm}\gamma^i_i\right) + \mu\left(g^{in}g^{ml}\gamma_{li} + g^{mk}\gamma_{kj}g^{jn}\right), \\ \Rightarrow \quad s^{nm} &= \lambda g^{nm}\gamma^k_k + \mu(g^{ni}g^{mj}\gamma_{ij} + g^{nj}g^{mi}\gamma_{ij}) = \lambda g^{nm}\gamma^k_k + 2\mu\gamma^{nm}, \end{aligned}$$

from the symmetry properties of γ_{ij}. In the infinitesimal limit, $s^{ij} \approx \tau^{ij}$ and $\gamma_{ij} \approx e_{ij}$, so we recover the linear, homogeneous, isotropic behaviour.

4.) a.) The deformed position is represented by standard cylindrical polar coordinates,

$$X_1 = r\cos\theta, \quad X_2 = r\sin\theta, \quad X_3 = z.$$

Assuming uniform stretch μ along the axis of the cylinder $X_3 = \mu x_3$. In addition, the deformation is incompressible which means that the volume remains constant, so if the undeformed radius is r_0 then $\pi r_0^2 l = \pi r^2 \mu l$, so $r_0 = r\sqrt{\mu}$. Hence, the undeformed position is given by

$$x_1 = r\sqrt{\mu}\cos\theta, \quad x_2 = r\sqrt{\mu}\sin\theta, \quad x_3 = z/\mu.$$

b.) The undeformed covariant base vectors are

$$\boldsymbol{g}_1 = (\sqrt{\mu}\cos\theta, \sqrt{\mu}\sin\theta, 0)^T, \quad \boldsymbol{g}_2 = (-\sqrt{\mu}r\sin\theta, \sqrt{\mu}r\cos\theta, 0)^T, \quad \boldsymbol{g}_3 = (0, 0, 1/\mu)^T,$$

so the undeformed metric tensors are

$$g_{ij} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu r^2 & 0 \\ 0 & 0 & \mu^{-2} \end{pmatrix}, \quad \text{and} \quad g^{ij} = \begin{pmatrix} \mu^{-1} & 0 & 0 \\ 0 & \mu^{-1} r^{-2} & 0 \\ 0 & 0 & \mu^2 \end{pmatrix}.$$

The deformed metric tensor is simply the standard cylindrical polar metric

$$G_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad G^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence the strain invariants are

$$I_1 = g^{ij} G_{ij} = 2\mu^{-1} + \mu^2,$$

$$I_2 = \frac{1}{2}\left(I_1^2 - g^{ik} G_{kj} g^{jl} G_{li}\right) = \frac{1}{2}\left(4\mu^{-2} + 4\mu + \mu^4 - (\mu^{-2} + \mu^{-2} + \mu^4)\right)$$
$$= \frac{1}{2}\left(2\mu^{-2} + 4\mu\right) = 2\mu + \mu^{-2}.$$

and $I_3 = 1$, by construction.
In addition

$$B^{ij} = I_1 g^{ij} - g^{ik} g^{jl} G_{kl}$$
$$= (2\mu^{-1} + \mu^2) \begin{pmatrix} \mu^{-1} & 0 & 0 \\ 0 & \mu^{-1} r^{-2} & 0 \\ 0 & 0 & \mu^2 \end{pmatrix}$$
$$- \begin{pmatrix} \mu^{-1} & 0 & 0 \\ 0 & \mu^{-1} r^{-2} & 0 \\ 0 & 0 & \mu^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu^{-1} & 0 & 0 \\ 0 & \mu^{-1} r^{-2} & 0 \\ 0 & 0 & \mu^2 \end{pmatrix},$$
$$= \begin{pmatrix} 2\mu^{-2} + \mu & 0 & 0 \\ 0 & (2\mu^{-2} + \mu) r^{-2} & 0 \\ 0 & 0 & (2\mu + \mu^4) \end{pmatrix} - \begin{pmatrix} \mu^{-2} & 0 & 0 \\ 0 & \mu^{-2} r^{-2} & 0 \\ 0 & 0 & \mu^4 \end{pmatrix},$$

$$= \begin{pmatrix} \mu+\mu^{-2} & 0 & 0 \\ 0 & (\mu+\mu^{-2})r^{-2} & 0 \\ 0 & 0 & 2\mu \end{pmatrix}.$$

Now from the constitutive law

$$T^{ij} = Ag^{ij} + BB^{ij} + PG^{ij},$$

and because I_1 and I_2 are constant it follows that A and B are independent of position. The equilibrium equation is

$$T^{ij}||_j + \rho F^i = 0,$$

where $\boldsymbol{F} = (r\omega^2, 0, 0)^T$. We could just plough ahead and compute the covariant derivatives without further thought, but there is a little trick we can use to save some work. For this deformation, there is a simple relationship between the tensors g^{ij}, G^{ij} and B^{ij}. We note that

$$g^{ij} - G^{ij}\mu^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu^2-\mu^{-1} \end{pmatrix},$$

which is constant, so taking the covariant derivative gives

$$g^{ij}||_i - \mu^{-1}\, G^{ij}||_i = 0.$$

However, transforming $G^{ij}||_i$ back to Cartesian coordinates gives $\delta_{IJ,I} = 0$, so we have that $G^{ij}||_i = 0$ (a general result) and hence $g^{ij}||_i = 0$. Similarly, by taking the covariant derivative of

$$B^{ij} - (\mu+\mu^{-2})G^{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu-\mu^{-2} \end{pmatrix},$$

we also deduce that $B^{ij}||_i = 0$.

We cannot assume anything about the spatial dependence of P, so the equilibrium equation becomes

$$G^{ij}P||_i + \rho F^i = 0 \quad \Rightarrow \quad G^{ij}P_{,i} + \rho F^i = 0,$$

because P is a scalar. Thus the governing equations are

$$\frac{\partial P}{\partial r} = -\rho r\omega^2, \quad \frac{\partial P}{\partial \theta} = \frac{\partial P}{\partial z} = 0,$$

which implies that

$$P = -\frac{1}{2}\rho r^2\omega^2 + P_0,$$

where P_0 is a constant.

It follows that

$$T^{11} = A\mu^{-1} + B(\mu+\mu^{-2}) + P = A\mu^{-1} + B(\mu+\mu^{-2}) - \frac{1}{2}\rho r^2\omega^2 + P_0.$$

If the curved surface of the cylinder is traction free than

$$T^{11} = 0 \quad \text{at} \quad r\sqrt{\mu} = a \quad \Rightarrow \quad r = a/\sqrt{\mu},$$

so

$$0 = A\mu^{-1} + B(\mu + \mu^{-2}) - \frac{1}{2}\rho a^2\omega^2\mu^{-1} + P_0,$$

$$\Rightarrow \quad P_0 = \frac{1}{2}\rho a^2\omega^2\mu^{-1} - A\mu^{-1} - B(\mu + \mu^{-2}).$$

Thus, finally, we obtain

$$T^{11} = \frac{1}{2}\rho\omega^2\left(\frac{a^2}{\mu} - r^2\right).$$

and the result does not depend on the strain energy function because the radial stress must balance the body force which is independent of constitutive law.

5.) a.) Our starting point is simply that

$$\int_{\partial\Omega_t} \widehat{T}^{ij} N_i R_j^* \, \mathrm{d}\mathcal{S}_t = \int_{\partial\Omega_t^r} \widehat{T}^{ij} N_i R_j^* \, \mathrm{d}\mathcal{S}_t + \int_{\partial\Omega_t^s} \widehat{T}^{ij} N_i R_j^* \, \mathrm{d}\mathcal{S}_t,$$

after dividing the boundary into the two sections. From the boundary conditions we have that

$$R_j^* = X_j, \quad \text{on} \quad \partial\Omega_t^r;$$

and

$$\widehat{T}^{ij} N_j = \hat{t}^i, \quad \text{on} \quad \partial\Omega_t^s,$$

which gives the result that

$$\int_{\partial\Omega_t} \widehat{T}^{ij} N_i R_j^* \, \mathrm{d}\mathcal{S}_t = \int_{\partial\Omega_t^r} \widehat{T}^{ij} N_i X_j \, \mathrm{d}\mathcal{S}_t + \int_{\partial\Omega_t^s} \hat{t}^j R_j^* \, \mathrm{d}\mathcal{S}_t.$$

For the remainder of the equality we can use the divergence theorem to write

$$\int_{\partial\Omega_t} \widehat{T}^{ij} N_i R_j^* \, \mathrm{d}\mathcal{S}_t = \int_{\Omega_t} (\widehat{T}^{ij} R_j^*)_{,i} \, \mathrm{d}\mathcal{V}_t,$$

and expanding out the derivative gives

$$\int_{\partial\Omega_t} \widehat{T}^{ij} N_i R_j^* \, \mathrm{d}\mathcal{S}_t = \int_{\Omega_t} \widehat{T}^{ij}||_i R_j^* + T^{ij} R_{j,i}^* \, \mathrm{d}\mathcal{V}_t,$$

and from the equations of static equilibrium, we have that

$$\widehat{T}^{ij}||_i = -\rho\widehat{F}^j,$$

so

$$\int_{\partial\Omega_t} \widehat{T}^{ij} N_i R_j^* \, \mathrm{d}\mathcal{S}_t = \int_{\partial\Omega_t^r} \widehat{T}^{ij} N_i X_j \, \mathrm{d}\mathcal{S}_t + \int_{\partial\Omega_t^s} \hat{t}^j R_j^* \, \mathrm{d}\mathcal{S}_t.$$

$$= \int_{\Omega_t} \widehat{T}^{ij} R_{j,i}^* \, \mathrm{d}\mathcal{V}_t - \int_{\Omega_t} \rho\widehat{F}^j R_j^* \, \mathrm{d}\mathcal{V}_t. \tag{1}$$

b.) If $\widehat{\boldsymbol{R}} = \boldsymbol{R}$ and $\widehat{\mathsf{T}} = \mathsf{T}$ corresponds to an equilibrium configuration, then

$$\int_{\partial\Omega_t^r} T^{ij} N_i X_j \,\mathrm{d}\mathcal{S}_t + \int_{\partial\Omega_t^s} t^j R_j^* \,\mathrm{d}\mathcal{S}_t = \int_{\Omega_t} T^{ij} R_{j,i}^* \,\mathrm{d}\mathcal{V}_t - \int_{\Omega_t} \rho F^j R_j^* \,\mathrm{d}\mathcal{V}_t.$$

There is no restriction on $\boldsymbol{R}^*$ in the above other than the fact that it is consistent with the displacement boundary conditions, so we could write an exactly similar equation with $\boldsymbol{R}^* = \boldsymbol{R}$. Subtracting these two equations gives

$$\int_{\partial\Omega_t^r} T^{ij} N_i (X_j - X_j) \,\mathrm{d}\mathcal{S}_t + \int_{\partial\Omega_t^s} t^j (R_j^* - R_j) \,\mathrm{d}\mathcal{S}_t$$
$$= \int_{\Omega_t} T^{ij} (R_{j,i}^* - R_{j,i}) \,\mathrm{d}\mathcal{V}_t - \int_{\Omega_t} \rho F^j (R_j^* - R_j) \,\mathrm{d}\mathcal{V}_t.$$

Thus, the principle of virtual work is

$$\int_{\partial\Omega_t^s} t^j \delta R_j \,\mathrm{d}\mathcal{S}_t = \int_{\Omega_t} T^{ij} \delta R_{j,i} \,\mathrm{d}\mathcal{V}_t - \int_{\Omega_t} \rho F^j \delta R_j \,\mathrm{d}\mathcal{V}_t,$$

where $\delta\boldsymbol{R} = \boldsymbol{R}^* - \boldsymbol{R}$. Note that we have shown that the virtual work on the boundary can only be applied on portions of the boundary where displacement boundary conditions are **not** applied.

c.) If we keep the deformation fixed at $\boldsymbol{R}$ in the equation (1),

$$\int_{\partial\Omega_t^r} T^{ij} N_i X_j \,\mathrm{d}\mathcal{S}_t + \int_{\partial\Omega_t^s} t^j R_j \,\mathrm{d}\mathcal{S}_t$$
$$= \int_{\Omega_t} T^{ij} R_{j,i} \,\mathrm{d}\mathcal{V}_t - \int_{\Omega_t} \rho F^j R_j \,\mathrm{d}\mathcal{V}_t.$$

We can replace T by T^* in the above provided that we also replace the corresponding body forces. Once again, we subtract the two equations to obtain

$$\int_{\partial\Omega_t^r} (T^{*ij} - T^{ij}) N_i X_j \,\mathrm{d}\mathcal{S}_t + \int_{\partial\Omega_t^s} (t^j - t^j) R_j \,\mathrm{d}\mathcal{S}_t$$
$$= \int_{\Omega_t} (T^{*ij} - T^{ij}) R_{j,i} \,\mathrm{d}\mathcal{V}_t - \int_{\Omega_t} \rho (F^{*j} - F^j) R_j \,\mathrm{d}\mathcal{V}_t.$$

The integral over the traction boundary vanishes and we have the virtual stress principle

$$\int_{\partial\Omega_t^r} \delta T^{ij} N_i X_j \,\mathrm{d}\mathcal{S}_t = \int_{\Omega_t} \delta T^{ij} R_{j,i} \,\mathrm{d}\mathcal{V}_t - \int_{\Omega_t} \rho \delta F^j R_j \,\mathrm{d}\mathcal{V}_t,$$

which is also called the principle of complementary virtual work. In this case it is only the portion of the boundary where displacement conditions are applied that is important.

6.) a.) The deformation is entirely within the plane, therefore the position off-plane does not vary, which means that

$$\boldsymbol{R}(\xi^1, \xi^2, \xi^3) = \boldsymbol{M}(\xi^1, \xi^2) + \xi^3 \boldsymbol{g}_3.$$

b.) In general

$$T^{ij} = Ag^{ij} + BB^{ij} + PG^{ij}.$$

Given the form of the undeformed and deformed positions

$$\boldsymbol{r}(\xi^1,\xi^2,\xi^3) = \boldsymbol{m}(\xi^1,\xi^2) + \xi^3\boldsymbol{g}_3, \quad \boldsymbol{R}(\xi^1,\xi^2,\xi^3) = \boldsymbol{M}(\xi^1,\xi^2) + \xi^3\boldsymbol{g}_3,$$

the undeformed and deformed metric tensors must take the form

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} g^{11} & g^{12} & 0 \\ g^{12} & g^{22} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$G_{ij} = \begin{pmatrix} G_{11} & G_{12} & 0 \\ G_{12} & G_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G^{ij} = \begin{pmatrix} G^{11} & G^{12} & 0 \\ G^{12} & G^{22} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which means that

$$B^{ij} = \frac{1}{2}\left[I_1 g^{ij} - g^{ik}g^{jl}G_{kl}\right] = \begin{pmatrix} B^{11} & B^{12} & 0 \\ B^{12} & B^{22} & 0 \\ 0 & 0 & (I_1-1)/2 \end{pmatrix},$$

and all quantities are functions only of ξ^1 and ξ^2. Thus, the stress has the form

$$T^{ij}(\xi^1,\xi^2) = \begin{pmatrix} T^{11} & T^{12} & 0 \\ T^{12} & T^{22} & 0 \\ 0 & 0 & T^{33} \end{pmatrix},$$

as required.

c.) The equation of equilibrium is

$$T^{ij}||_j + \rho F^i = 0,$$

but T^{ij} does not vary with ξ^3, and $T^{3\alpha} = 0$, so the third equation, when $i = 3$ is automatically satisfied, provided that the body force does not act out of the plane. (If the body force does act out of the plane, then we cannot have plane strain.)

Thus, if $\boldsymbol{F} = -U_{,\alpha}\boldsymbol{G}^\alpha$, the equation of equilibrium in terms of the stress vectors is

$$\frac{1}{\sqrt{G}}\boldsymbol{T}^\alpha_{,\alpha} - \rho U_{,\alpha}\boldsymbol{G}^\alpha = \boldsymbol{0},$$

$$\Rightarrow \quad \boldsymbol{T}^\alpha_{,\alpha} - \sqrt{G}\rho U_{,\alpha}\boldsymbol{G}^\alpha = \boldsymbol{0},$$

$$\Rightarrow \quad \left[\boldsymbol{T}^\alpha - \rho\sqrt{G}U\boldsymbol{G}^\alpha\right]_{,\alpha} = \boldsymbol{0}.$$

The result follows because

$$\left[\sqrt{G}U\boldsymbol{G}^\alpha\right]_{,\alpha} = \frac{\partial\sqrt{G}}{\partial\xi^\alpha}U\boldsymbol{G}^\alpha + \sqrt{G}U_{,\alpha}\boldsymbol{G}^\alpha - \sqrt{G}U\Gamma^\alpha_{\beta\alpha}\boldsymbol{G}^\beta,$$

and $\frac{\partial\sqrt{G}}{\partial\xi^\alpha} = \sqrt{G}\Gamma^j_{j\alpha}$, lecture notes equation (1.54). The normal vector $\boldsymbol{g}_3$ is constant because we are dealing with a plane. Therefore $\boldsymbol{g}_{3,\alpha} = \boldsymbol{0}$, and so $\Gamma^3_{3\alpha} = 0$, which means that

$$\left[\sqrt{G}U\boldsymbol{G}^\alpha\right]_{,\alpha} = \sqrt{G}U\boldsymbol{G}^\alpha\Gamma^\beta_{\beta\alpha} + \sqrt{G}U_{,\alpha}\boldsymbol{G}^\alpha - \sqrt{G}U\Gamma^\alpha_{\beta\alpha}\boldsymbol{G}^\beta = \sqrt{G}U_{,\alpha}\boldsymbol{G}^\alpha.$$

d.) Starting from the given expression

$$\boldsymbol{T}^\alpha = \sqrt{G}\epsilon^{\gamma\alpha}\boldsymbol{\chi}_{,\gamma} + \rho\sqrt{G}U\boldsymbol{G}^\alpha,$$

it follows that

$$\boldsymbol{T}^\alpha - \rho\sqrt{G}U\boldsymbol{G}^\alpha = \sqrt{G}\epsilon^{\gamma\alpha}\boldsymbol{\chi}_{,\gamma},$$

and differentiating with respect to ξ^α we obtain

$$\left[\boldsymbol{T}^\alpha - \rho\sqrt{G}U\boldsymbol{G}^\alpha\right]_{,\alpha} = \sqrt{G}\epsilon^{\gamma\alpha}\boldsymbol{\chi}_{,\gamma\alpha},$$

because $\sqrt{G}\epsilon^{\gamma\alpha}$ only takes the values ± 1 and is a constant. The term in involving $\epsilon^{\gamma\alpha}$ is antisymmetric, but $\boldsymbol{\chi}_{,\gamma\alpha}$ is symmetric, so their product must be zero, providing a solution to the equilibrium equation.

e.) The stress vector is given by

$$\boldsymbol{T}^\alpha = \sqrt{G}T^{\alpha\beta}\boldsymbol{G}_\beta,$$

which means that

$$\sqrt{G}T^{\alpha\delta}\boldsymbol{G}_\delta = \sqrt{G}\epsilon^{\gamma\alpha}\boldsymbol{\chi}_{,\gamma} + \rho\sqrt{G}U\boldsymbol{G}^\alpha.$$

If we decompose $\boldsymbol{\chi}$ into components in the plane

$$\boldsymbol{\chi} = \chi^\delta\boldsymbol{G}_\delta \quad \text{and} \quad \boldsymbol{\chi}_{,\gamma} = \chi^\delta|_\gamma\boldsymbol{G}_\delta,$$

so

$$T^{\alpha\delta}\boldsymbol{G}_\delta = \epsilon^{\gamma\alpha}\chi^\delta|_\gamma\boldsymbol{G}_\delta + \rho U\boldsymbol{G}^\alpha,$$

and taking the dot product with $\boldsymbol{G}^\beta$ gives the result

$$T^{\alpha\beta} = \epsilon^{\gamma\alpha}\chi^\beta|_\gamma + \rho U G^{\alpha\beta}.$$

The tensors $T^{\alpha\beta}$ and $G^{\alpha\beta}$ are symmetric which means that

$$\epsilon^{\gamma\alpha}\chi^\beta|_\gamma = \epsilon^{\gamma\beta}\chi^\alpha|_\gamma.$$

Hence if we set

$$\chi^\beta = \epsilon^{\delta\beta}\phi_{,\delta},$$

then the symmetry is achieved because

$$\epsilon^{\gamma\alpha}\chi^\beta|_\gamma = \epsilon^{\gamma\alpha}\epsilon^{\delta\beta}\phi_{,\delta\gamma} = \epsilon^{\gamma\beta}\epsilon^{\delta\alpha}\phi_{,\delta\gamma} = \epsilon^{\gamma\beta}\chi^\alpha|_\gamma,$$

where we have used the symmetry of the partial derivative and the fact that the covariant derivative and partial derivative coincide for scalar functions.

f.) Using the result from part (e) directly in the expression for the stress components, we obtain

$$T^{\alpha\beta} = \epsilon^{\gamma\alpha}\epsilon^{\delta\beta}\phi|_{\delta\gamma} + \rho U G^{\alpha\beta}.$$

In the absence of body forces $U = 0$ and in Cartesian coordinates we have

$$T_{ab} = e_{ga}e_{db}\phi_{,dg},$$

The alternating symbol is only non-zero if $g \neq a$ and $d \neq b$, so we have

$$T_{11} = \phi_{,22}, \quad T_{22} = \phi_{,11}, \quad T_{12} = -\phi_{,12},$$

as needed.

7.) a.) We simply substitute the assumed form into the governing equations and cancel the exponential terms. We find that

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = -\omega^2 \boldsymbol{U} \mathrm{e}^{\mathrm{i}(n\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)},$$

$$\boldsymbol{\nabla}^2 \boldsymbol{u} = -n^2 \boldsymbol{U} \mathrm{e}^{\mathrm{i}(n\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)},$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \mathrm{i}n(\boldsymbol{k}\cdot\boldsymbol{U})\mathrm{e}^{\mathrm{i}(n\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)},$$

so that

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) = -n^2(\boldsymbol{k}\cdot\boldsymbol{U})\boldsymbol{k}\mathrm{e}^{\mathrm{i}(n\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}.$$

and

$$\boldsymbol{\nabla}\cdot\dot{\boldsymbol{u}} = n\omega(\boldsymbol{k}\cdot\boldsymbol{U})\mathrm{e}^{\mathrm{i}(n\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}.$$

(If you are unhappy with these solutions, check by using index notation. Remember that $\boldsymbol{k}$ is a unit vector.)

We also find that

$$\boldsymbol{\nabla}\theta = \mathrm{i}n\Theta\boldsymbol{k}\mathrm{e}^{\mathrm{i}(n\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)},$$

$$\boldsymbol{\nabla}^2\theta = -n^2\Theta\mathrm{e}^{\mathrm{i}(n\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)},$$

and

$$\frac{\partial\theta}{\partial t} = -\mathrm{i}\omega\Theta\mathrm{e}^{\mathrm{i}(n\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}.$$

Thus the governing equations become

$$\begin{aligned}
-\omega^2\boldsymbol{U} &= -(\lambda+\mu)n^2(\boldsymbol{k}\cdot\boldsymbol{U})\boldsymbol{k} - n^2\mu\boldsymbol{U} + \mathrm{i}n\Theta\alpha\boldsymbol{k},\\
-\mathrm{i}\omega\Theta &= -\kappa n^2\Theta + \nu n\omega(\boldsymbol{k}\cdot\boldsymbol{U}).
\end{aligned}$$

and rearranging we obtain the desired relationships

$$\omega^2\boldsymbol{U} = n^2\left[\mu\boldsymbol{U} + (\lambda+\mu)\boldsymbol{U}\cdot\boldsymbol{k}\boldsymbol{k}\right] - \mathrm{i}\alpha n\Theta\boldsymbol{k},$$

and

$$(\kappa n^2 - \mathrm{i}\omega)\Theta = \nu n\omega\boldsymbol{U}\cdot\boldsymbol{k}.$$

b.) If $\boldsymbol{U} \cdot \boldsymbol{k} = 0$, then the wave solutions exist if

$$\omega^2 \boldsymbol{U} = n^2 [\mu \boldsymbol{U}] - \mathrm{i}\alpha n \Theta \boldsymbol{k}, \tag{2}$$

and

$$(\kappa n^2 - \mathrm{i}\omega)\Theta = 0. \tag{3}$$

Taking the dot product of (2) with $\boldsymbol{U}$ gives

$$(\omega^2 - n^2\mu)|\boldsymbol{U}| = 0,$$

and taking the dot product with $\boldsymbol{k}$ yields

$$\mathrm{i}\alpha n \Theta = 0,$$

It follows that $\Theta = 0$ and equation (3) is trivially satisfied. Hence, the waves are independent of thermal effects. Non-trivial waves can only exist if $\omega^2 = \mu n^2$.

c.) If we take dot product of the equations found in part (a) with $\boldsymbol{k}$, we obtain

$$\omega^2 |\boldsymbol{U}| = n^2 [\mu |\boldsymbol{U}| + (\lambda + \mu)|\boldsymbol{U}|] - \mathrm{i}\alpha n \Theta,$$

and

$$(\kappa n^2 - \mathrm{i}\omega)\Theta = \nu n \omega |\boldsymbol{U}|.$$

Hence we obtain the matrix system

$$\begin{pmatrix} \omega^2 - n^2(\lambda + 2\mu) & \mathrm{i}\alpha n \\ -\nu n \omega & (\kappa n^2 - \mathrm{i}\omega) \end{pmatrix} \begin{pmatrix} |\boldsymbol{U}| \\ \Theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the matrix must have zero determinant in order for there to be non-trivial waves. Provided that there is thermal coupling ($\alpha \neq 0$ and $\nu \neq 0$), then the waves are not independent of thermal effects.

8.) a.) The second Piola–Kirchhoff stress tensor is given by

$$s^{ij} = 2\frac{\partial \mathcal{W}}{\partial I_1} g^{ij} + 2\frac{\partial \mathcal{W}}{\partial I_2} B^{ij} + 2I_3 \frac{\partial \mathcal{W}}{\partial I_3} G^{ij},$$

so in this case

$$s^{ij} = C_1 g^{ij} + C_2 B^{ij} + (c + 2d(I_3 - 1))I_3 G^{ij}.$$

Hence,

$$s^{ij} = C_1 g^{ij} + C_2 \left[I_1 g^{ij} - g^{ik} g^{jl} G_{kl} \right] + (c + 2d(I_3 - 1))I_3 G^{ij}.$$

In the infinitesimal limit, we know that,

$$I_1 = 3 + 2e^k_k, \quad I_3 = 1 + 2e^k_k, \quad G_{ij} = g_{ij} + 2e_{ij},$$

which means that

$$s^{ij} = \tau^{ij} \approx C_1 g^{ij} + C_2 \left[3g^{ij} + 2e^k_k g^{ij} - g^{ik} g^{jl}(g_{ij} + 2e_{ij}) \right] + (c + 4de^k_k)(1 + 2e^l_l)G^{ij}. \tag{4}$$

We need an expression for G^{ij} and we know that

$$G^{ik}G_{kj} = \delta^i_j \quad \Rightarrow \quad G^{ik}\left[g_{kj} + 2e_{kj}\right] = \delta^i_j.$$

In the absence of any deformation $G^{ij} = g^{ij}$ and so we expect that $G^{ij} \approx g^{ij} + \lambda e^{ij}$. Thus,

$$(g^{ik} + \lambda e^{ik})(g_{kj} + 2e_{kj}) = \delta^i_j + \lambda e^i_j + 2e^i_j + \mathcal{O}(\epsilon^2) = \delta^i_j,$$

and so $\lambda + 2 = 0$, giving the desired result

$$G^{ij} \approx g^{ij} - 2e^{ij}.$$

Using this result in equation (4) gives

$$\tau^{ij} \approx C_1 g^{ij} + C_2\left[3g^{ij} + 2e^k_k g^{ij} - g^{ij} - 2e^{ij})\right] + (c + 4de^k_k)(1 + 2e^l_l)(g^{ij} - 2e^{ij}),$$

and so

$$\tau^{ij} = (C_1 + 2C_2 + c)g^{ij} + (2C_2 + 4d + 2c)e^k_k g^{ij} - (2C_2 + 2c)e^{ij},$$

as required.

b.) The linear constitutive law in the absence of heating is

$$\tau^{ij} = \lambda e^k_k g^{ij} + 2\mu e^{ij},$$

so in order for the strain energy function to be consistent with this linear result

$$C_1 + 2C_2 + c = 0, \tag{5}$$
$$2C_2 + 4d + 2c = \lambda, \tag{6}$$
$$-2C_2 - 2c = 2\mu. \tag{7}$$

Thus, adding (5) and (7)/2 gives

$$2C_1 + 2C_2 = 2\mu \quad \Rightarrow C_1 + C_2 = \mu;$$

then from equation (7)

$$c = -C_2 - \mu = C_1 - 2\mu,$$

and from (6)

$$-2\mu + 4d = \lambda \quad \Rightarrow \quad d = (\lambda + 2\mu)/4.$$