MATH45061: SOLUTION SHEET[1] III

1.) Balance of angular momentum about a particular point $\boldsymbol{Z}$ requires that

$$\int_\Omega (\boldsymbol{R} - \boldsymbol{Z}) \times \rho \dot{\boldsymbol{V}} \, \mathrm{d}\mathcal{V} = \boldsymbol{\mathcal{L}}_{\boldsymbol{Z}}.$$

The moment about the point $\boldsymbol{Z}$ can be decomposed into torque due to the body force, $\boldsymbol{F}$, and surface traction $\boldsymbol{T}$:

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{Z}} = \int_\Omega (\boldsymbol{R} - \boldsymbol{Z}) \times \rho \boldsymbol{F} \, \mathrm{d}\mathcal{V} + \int_{\partial\Omega} (\boldsymbol{R} - \boldsymbol{Z}) \times \boldsymbol{T} \, \mathrm{d}\mathcal{S};$$

and so

$$\int_\Omega (\boldsymbol{R} - \boldsymbol{Z}) \times \rho \dot{\boldsymbol{V}} \, \mathrm{d}\mathcal{V} = \int_\Omega (\boldsymbol{R} - \boldsymbol{Z}) \times \rho \boldsymbol{F} \, \mathrm{d}\mathcal{V} + \int_{\partial\Omega} (\boldsymbol{R} - \boldsymbol{Z}) \times \boldsymbol{T} \, \mathrm{d}\mathcal{S}. \tag{1}$$

If linear momentum is in balance then

$$\int_\Omega \rho \dot{\boldsymbol{V}} \, \mathrm{d}\mathcal{V} = \int_\Omega \rho \boldsymbol{F} \, \mathrm{d}\mathcal{V} + \int_{\partial\Omega} \boldsymbol{T} \, \mathrm{d}\mathcal{S};$$

and taking the cross product with the constant vector $\boldsymbol{Z} - \boldsymbol{Y}$, which can be brought inside the integral,

$$\int_\Omega (\boldsymbol{Z} - \boldsymbol{Y}) \times \rho \dot{\boldsymbol{V}} \, \mathrm{d}\mathcal{V} = \int_\Omega (\boldsymbol{Z} - \boldsymbol{Y}) \times \rho \boldsymbol{F} \, \mathrm{d}\mathcal{V} + \int_{\partial\Omega} (\boldsymbol{Z} - \boldsymbol{Y}) \times \boldsymbol{T} \, \mathrm{d}\mathcal{S}. \tag{2}$$

Adding equation (2) to equation (1) and simplifying

$$\int_\Omega (\boldsymbol{R} - \boldsymbol{Y}) \times \rho \dot{\boldsymbol{V}} \, \mathrm{d}\mathcal{V} = \int_\Omega (\boldsymbol{R} - \boldsymbol{Y}) \times \rho \boldsymbol{F} \, \mathrm{d}\mathcal{V} + \int_{\partial\Omega} (\boldsymbol{R} - \boldsymbol{Y}) \times \boldsymbol{T} \, \mathrm{d}\mathcal{S}.$$

Thus, the angular momentum about any point $\boldsymbol{Y}$ is in balance.

If body couples, $\boldsymbol{L}$, are present then they need to be included in the angular momentum balance as an additional term on the right-hand side

$$\int_\Omega \boldsymbol{L} \, \mathrm{d}\mathcal{V}.$$

This term will be unaffected by the argument above, so we need to demonstrate that the body couple distribution is independent of the axis about which the angular momentum balance is taken. For a body couple to be independent of the body force, the forces that constitute the couple must be in balance because otherwise an additional contribution to the body force would result. We can write that at a given point $\boldsymbol{R}$, the body couple referred to a given axis is

$$\boldsymbol{L}_{\boldsymbol{Z}}(\boldsymbol{R}) = \lim_{|\boldsymbol{\delta}_{(i)}| \to 0} \sum_i (\boldsymbol{R} + \boldsymbol{\delta}_{(i)} - \boldsymbol{Z}) \times \boldsymbol{F}_{(i)},$$

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the limit of the sum of torques acting at slightly different points but the distance between the points is tending to zero. We assume that such a limit exists and is finite (which actually requires unbounded forces, because otherwise the forces will simply cancel). The argument is now essentially the same as above, because the forces must be in balance then $\sum_i \boldsymbol{F}_{(i)} = \boldsymbol{0}$, and so

$$(\boldsymbol{Z} - \boldsymbol{Y}) \times \sum_i \boldsymbol{F}_{(i)} = \sum_i (\boldsymbol{Z} - \boldsymbol{Y}) \times \boldsymbol{F}_{(i)} = 0,$$

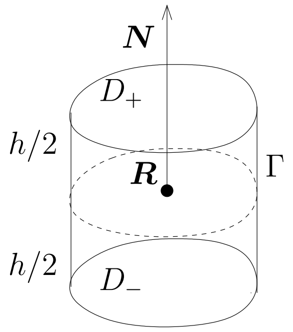
which means that

$$\boldsymbol{L}_{\boldsymbol{Z}} = \lim_{|\boldsymbol{\delta}_{(i)}|\to 0} \sum_i \left[(\boldsymbol{R} + \boldsymbol{\delta}_{(i)} - \boldsymbol{Z}) \times \boldsymbol{F}_{(i)} + (\boldsymbol{Z} - \boldsymbol{Y}) \times \boldsymbol{F}_{(i)} \right]$$

$$= \lim_{|\boldsymbol{\delta}_{(i)}|\to 0} \sum_i (\boldsymbol{R} + \boldsymbol{\delta}_{(i)} - \boldsymbol{Y}) \times \boldsymbol{F}_{(i)} = \boldsymbol{L}_{\boldsymbol{Y}},$$

so the body couple distribution is a free vector distribution — it is independent of the axis about which the angular momentum balance is taken. Thus, the result is still true if body couples are included, provided that our body couple distribution is defined in a manner consistent with the behaviour of couples that act over finite distances.

2.) The difficulty of the argument depends on how rigorous you want to be. Here I shall adopt a middle way. The starting point is to consider a short cylinder (pillbox) that is formed by extending a disk D of fixed radius about the point $\boldsymbol{R}$ a distance $h/2$ in both the positive and negative normal directions to the disk.



The surface of the cylinder consists of two disks D_+, D_- and the curved surface Γ_h. We assume that the total force on the cylinder tends to zero as the volume of the cylinder tends to zero, *i.e.* the forces remain bounded, so

$$\lim_{h\to 0} \int_{\Gamma_h} \boldsymbol{T} \, \mathrm{d}\mathcal{S} + \int_{D_+} \boldsymbol{T} \, \mathrm{d}\mathcal{S} + \int_{D_-} \boldsymbol{T} \, \mathrm{d}\mathcal{S} = \boldsymbol{0}.$$

If the forces remain bounded (which follows from continuity of $\boldsymbol{T}$) then because the area of the curved surface $\Gamma_h \to 0$ as $h \to 0$ the first integral vanishes and we have that

$$\int_{D_+} \boldsymbol{T} \, \mathrm{d}\mathcal{S} + \int_{D_-} \boldsymbol{T} \, \mathrm{d}\mathcal{S} = \boldsymbol{0}.$$

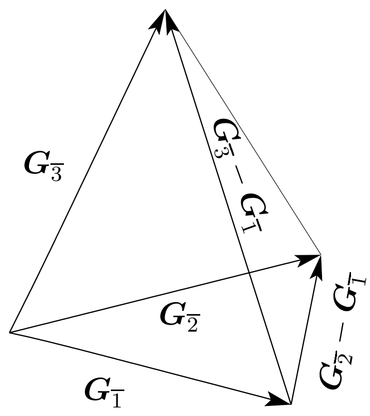
(If you want to make this really rigorous, you need to use the mean value theorem for integration.) In this limit the two disks are the same and coincide with the original disk D, but the normals equal an opposite, so

$$\int_D \boldsymbol{T}(\boldsymbol{N}(\boldsymbol{R}), \boldsymbol{R})\,\mathrm{d}\mathcal{S} + \int_D \boldsymbol{T}(-\boldsymbol{N}(\boldsymbol{R}), \boldsymbol{R})\,\mathrm{d}\mathcal{S} = \boldsymbol{0},$$

which is true for any size of disk so

$$\boldsymbol{T}(\boldsymbol{N}(\boldsymbol{R}), \boldsymbol{R}) + \int_D \boldsymbol{T}(-\boldsymbol{N}(\boldsymbol{R}), \boldsymbol{R}) = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{T}(\boldsymbol{N}(\boldsymbol{R}), \boldsymbol{R}) = -\boldsymbol{T}(-\boldsymbol{N}(\boldsymbol{R}), \boldsymbol{R}).$$

3.) This is a straightforward exercise in vector geometry. The key point is that in the infinitesimal limit, the tetrahedron is straight-sided with three edges given by $\boldsymbol{G}_{\bar{i}}$.



We now use the standard result that the cross product of two vectors gives a vector normal to both (with direction given by the right-hand rule) and with magnitude equal to twice the area of the triangle defined by the two vectors. Thus the vector area of the face whose edges are not aligned with a coordinate direction is

$$\mathbf{d}\boldsymbol{S} = \boldsymbol{N}\,\mathrm{d}S = \frac{1}{2}(\boldsymbol{G}_{\bar{2}} - \boldsymbol{G}_{\bar{1}}) \times (\boldsymbol{G}_{\bar{3}} - \boldsymbol{G}_{\bar{1}}),$$

where the direction of the normal is out of the tetrahedron. Hence,

$$\boldsymbol{N}\,\mathrm{d}S = \frac{1}{2}\left(\boldsymbol{G}_{\bar{2}} \times \boldsymbol{G}_{\bar{3}} - \boldsymbol{G}_{\bar{2}} \times \boldsymbol{G}_{\bar{1}} - \boldsymbol{G}_{\bar{1}} \times \boldsymbol{G}_{\bar{3}} + \boldsymbol{G}_{\bar{1}} \times \boldsymbol{G}_{\bar{1}}\right),$$

the final term is zero due to the properties of the cross product, so

$$\boldsymbol{N}\,\mathrm{d}S = \frac{1}{2}\boldsymbol{G}_{\bar{2}} \times \boldsymbol{G}_{\bar{3}} + \frac{1}{2}\boldsymbol{G}_{\bar{1}} \times \boldsymbol{G}_{\bar{2}} + \frac{1}{2}\boldsymbol{G}_{\bar{1}} \times \boldsymbol{G}_{\bar{3}}.$$

after using the antisymmetry property of the cross product.

The three terms on the right-hand side are simply the vector areas of the other three faces of the tetrahedron; and the outer normals to these faces are given by the contravariant base vectors $\boldsymbol{G}^{\bar{i}}$. Thus, outer unit normals are given by $\boldsymbol{G}^{\bar{i}}/\sqrt{G^{\bar{i}\bar{i}}}$ (no summation) and we have the required result

$$\boldsymbol{N}\,\mathrm{d}S = \sum_{\bar{i}} \frac{\boldsymbol{G}^{\bar{i}}\mathrm{d}S_{(\bar{i})}}{\sqrt{G^{\bar{i}\bar{i}}}}.$$

4.) Here we need only use the transformation properties of the two vectors. We know that the vector components transform as

$$N_{\hat{I}} = \frac{\partial X_{\hat{I}}}{\partial X_J} N_J = Q_{\hat{I}J} N_J.$$

and so multiplying by the inverse of the matrix gives

$$N_{\hat{I}} Q^{-1}_{K\hat{I}} = Q^{-1}_{K\hat{I}} Q_{\hat{I}J} N_J = \delta_{KJ} N_J = N_K.$$

We use these vector transformation properties to determine that

$$T_I = T_{JI} N_J \quad \Rightarrow \quad T_{\widehat{K}}\, Q^{-1}_{I\widehat{K}} = T_{JI} N_{\hat{L}} Q^{-1}_{J\hat{L}};$$

and then multiplying both sides by $Q_{\widehat{M}I}$

$$T_{\widehat{M}} = Q_{\widehat{M}I} T_{JI} N_{\hat{L}} Q^{-1}_{J\hat{L}} \quad \Rightarrow \quad T_{\widehat{M}} = Q_{\widehat{M}I} T_{JI} Q^{-1}_{J\hat{L}} N_{\hat{L}},$$

which means that if in the transformed coordinates $T_{\widehat{M}} = T_{\hat{L}\widehat{M}} N_{\hat{L}}$, then

$$T_{\hat{L}\widehat{M}} = Q_{\widehat{M}I} T_{JI} Q^{-1}_{J\hat{L}} = Q^{-T}_{\hat{L}J} T_{JI} Q^{T}_{I\widehat{M}} = Q_{\hat{L}J} T_{JI} Q^{T}_{I\widehat{M}},$$

because Q is orthogonal. Thus the Cauchy stress tensor does satisfy Cartesian tensor transformation properties.

5.) **a.)** The total surface force on the body is given by integrating the traction over the surface

$$\boldsymbol{F}_S = \int_{\partial\Omega} \boldsymbol{T}\,\mathrm{d}\mathcal{S} = \int_{\partial\Omega} -p\boldsymbol{N}\,\mathrm{d}\mathcal{S} = \int_{\partial\Omega} \rho_f g X_3 \boldsymbol{N}\,\mathrm{d}\mathcal{S}.$$

We would like to apply the divergence theorem, but the integrand is not of the correct form. In the examples class, I took the following approach: Consider the three separate integrals,

$$\int_{\partial\Omega} \rho_f g X_3 N_I\,\mathrm{d}\mathcal{S} = \int_{\partial\Omega} \rho_f g X_3 \boldsymbol{e}_I \boldsymbol{\cdot} \boldsymbol{N}\,\mathrm{d}\mathcal{S}, \quad I = 1, 2, 3.$$

Then applying the divergence theorem to each integral, we obtain

$$\int_{\partial\Omega} \rho_f g X_3 N_I\,\mathrm{d}\mathcal{S} = \int_{\Omega} \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} (\rho_f g X_3 \boldsymbol{e}_I)\,\mathrm{d}\mathcal{V} = \rho_f g \delta_{I3} \int_{\Omega} \mathrm{d}\mathcal{V},$$

which means that

$$\boldsymbol{F}_s = \rho_f g V \boldsymbol{e}_3.$$

Alternatively, a more sophisticated method is to take the inner product with a constant vector $\boldsymbol{a}$ so that

$$\boldsymbol{F}_S \boldsymbol{\cdot} \boldsymbol{a} = \int_{\partial\Omega} \rho_f g X_3 a_I N_I\,\mathrm{d}\mathcal{S}.$$

Now we can use the divergence theorem to write

$$\boldsymbol{F}_S \boldsymbol{\cdot} \boldsymbol{a} = \int_{\Omega} \frac{\partial}{\partial X_I} (\rho_f g X_3 a_I)\,\mathrm{d}\mathcal{V} = \rho_f g a_3 \int_{\Omega} \mathrm{d}\mathcal{V}.$$

Hence, because $\boldsymbol{a}$ is arbitrary it follows that

$$\boldsymbol{F}_s = \rho_f g \int_\Omega \mathrm{d}\mathcal{V} \boldsymbol{e}_3 = \rho_f g V \boldsymbol{e}_3,$$

where V is the volume of the body. Hence the resultant surface force is the mass of water displaced $\rho_f V$ multiplied by g, which gives the weight of water displaced and acts in the positive X_3 direction.

b.) If we let $\langle \boldsymbol{R} \rangle$ be the centre of volume then the resultant torque about the centre of volume is

$$\boldsymbol{L} = \int_{\partial\Omega} (\boldsymbol{R} - \langle \boldsymbol{R} \rangle) \times \rho_f g X_3 \boldsymbol{N} \, \mathrm{d}\mathcal{S}.$$

In a Cartesian coordinate system, the component in the direction $\boldsymbol{e}_I$ is

$$L_I = \rho_f g \int_{\partial\Omega} X_3 \, e_{IJK} (X_J - \langle X_J \rangle) N_K \, \mathrm{d}\mathcal{S}.$$

Thus, using the divergence theorem gives

$$L_I = \rho_f g \int_\Omega \frac{\partial}{\partial X_K} \left(X_3 \, e_{IJK} (X_J - \langle X_J \rangle \right) \, \mathrm{d}\mathcal{V}$$

$$= \rho_f g \int_\Omega \delta_{K3} \, e_{IJK} (X_J - \langle X_J \rangle) + X_3 \, e_{IJK} \delta_{JK} \, \mathrm{d}\mathcal{V}.$$

Now δ_{JK} is symmetric on interchange of J and K, but e_{IJK} is antisymmetric on interchange of J and K so the last term must be zero. Thus,

$$L_I = \rho_f g \int_\Omega e_{IJ3} (X_J - \langle X_J \rangle) \, \mathrm{d}\mathcal{V},$$

but $\langle X_J \rangle$ is a constant and so we have

$$L_I = \rho_f g e_{IJ3} \left[\int_\Omega X_J \, \mathrm{d}\mathcal{V} - V \langle X_J \rangle \right],$$

where V is the volume of the body. Now by definition of the centre of volume

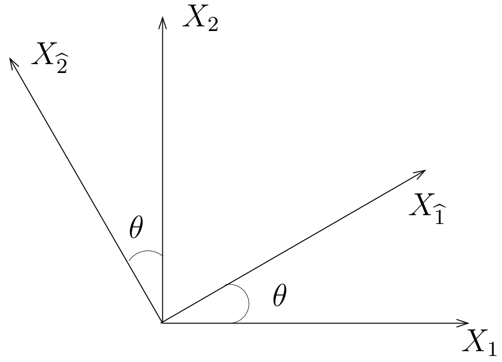
$$\langle X_J \rangle = \frac{1}{V} \int_\Omega X_J \, \mathrm{d}\mathcal{V},$$

which means that

$$L_I = \rho_f \, g \, e_{IJ3} \left[V \langle X_J \rangle - V \langle X_J \rangle \right] = 0 \quad \Rightarrow \quad \boldsymbol{L} = \boldsymbol{0},$$

as required.

6.) a.) A change from one set of Cartesian coordinates to another does not change the angle between the coordinate axes. If the X_3 direction and handedness remain fixed, the only possible family of transformations are rotations about the X_3 axis. These rotations can be described by a single angle θ, between the X_1 and $X_{\hat{1}}$ directions:



b.) The transformation between the two coordinate systems is given by

$$\begin{pmatrix} X_{\widehat{1}} \\ X_{\widehat{2}} \\ X_{\widehat{3}} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}.$$

Thus the transformation rule for the stress tensor is

$$T_{\widehat{I}\widehat{J}} = Q_{\widehat{I}N} T_{NM} Q_{\widehat{J}M}, \quad \text{where} \quad Q_{\widehat{I}J} = \frac{\partial X_{\widehat{I}}}{\partial X_J},$$

so

$$\begin{aligned} \widehat{\mathsf{T}} &= \begin{pmatrix} T_{\widehat{1}\widehat{1}} & T_{\widehat{1}\widehat{2}} & T_{\widehat{1}\widehat{3}} \\ T_{\widehat{1}\widehat{2}} & T_{\widehat{2}\widehat{2}} & T_{\widehat{2}\widehat{3}} \\ T_{\widehat{1}\widehat{3}} & T_{\widehat{2}\widehat{3}} & T_{\widehat{3}\widehat{3}} \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_{11}\cos\theta + T_{12}\sin\theta & -T_{11}\sin\theta + T_{12}\cos\theta & 0 \\ T_{12}\cos\theta + T_{22}\sin\theta & -T_{12}\sin\theta + T_{22}\cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}; \end{aligned}$$

and so

$$\begin{aligned} T_{\widehat{1}\widehat{1}} &= T_{11}\cos^2\theta + T_{22}\sin^2\theta + 2T_{12}\cos\theta\sin\theta, \\ T_{\widehat{2}\widehat{2}} &= T_{11}\sin^2\theta + T_{22}\cos^2\theta - 2T_{12}\cos\theta\sin\theta, \\ T_{\widehat{1}\widehat{2}} &= (T_{22} - T_{11})\sin\theta\cos\theta + T_{12}(\cos^2\theta - \sin^2\theta). \end{aligned}$$

We can use the double angle formulæ to write

$$\begin{aligned} T_{\widehat{1}\widehat{1}} &= T_{11}\frac{1}{2}(1 + \cos 2\theta) + T_{22}\frac{1}{2}(1 - \cos 2\theta) + T_{12}\sin 2\theta, \\ &= \frac{(T_{11} + T_{22})}{2} + \frac{(T_{11} - T_{22})}{2}\cos 2\theta + T_{12}\sin 2\theta \\ T_{\widehat{2}\widehat{2}} &= T_{11}\frac{1}{2}(1 - \cos 2\theta) + T_{22}\frac{1}{2}(1 + \cos 2\theta) - T_{12}\sin 2\theta, \\ &= \frac{(T_{11} + T_{22})}{2} + \frac{(T_{22} - T_{11})}{2}\cos 2\theta - T_{12}\sin 2\theta, \\ T_{\widehat{1}\widehat{2}} &= \frac{(T_{22} - T_{11})}{2}\sin 2\theta + T_{12}\cos 2\theta. \end{aligned}$$

c.) If $T_{\hat{1}\hat{2}} = 0$, then

$$\frac{(T_{22} - T_{11})}{2} \sin 2\theta + T_{12} \cos 2\theta = 0 \quad \Rightarrow \quad \tan 2\theta = \frac{2T_{12}}{(T_{11} - T_{22})}.$$

The maximum and minimum values of $T_{\hat{1}\hat{1}}$ and $T_{\hat{2}\hat{2}}$ occur when $\partial T_{\hat{1}\hat{1}}/\partial\theta = 0$ and $\partial T_{\hat{2}\hat{2}}/\partial\theta = 0$, respectively; and

$$\frac{\partial T_{\hat{1}\hat{1}}}{\partial\theta} = \sin 2\theta (T_{22} - T_{11}) + 2T_{12} \cos 2\theta = 2T_{\hat{1}\hat{2}},$$

$$\frac{\partial T_{\hat{2}\hat{2}}}{\partial\theta} = \sin 2\theta (T_{11} - T_{22}) - 2T_{12} \cos 2\theta = -2T_{\hat{1}\hat{2}}.$$

Hence, when $T_{\hat{1}\hat{2}} = 0$, the values of $T_{\hat{1}\hat{1}}$ and $T_{\hat{2}\hat{2}}$ are either at their maximum or minimum. In other words, these are the principal stresses.

d.) The maximum value of $T_{\hat{1}\hat{2}}$ occurs when $\partial T_{\hat{1}\hat{2}}/\partial\theta = 0$, *i. e.*

$$(T_{22} - T_{11}) \cos 2\theta - 2T_{12} \sin 2\theta = 0 \quad \Rightarrow \quad \tan 2\theta = \frac{(T_{22} - T_{11})}{2T_{12}}.$$

If we indicate the value of θ that extremises the normal stress by θ_n and that corresponding to the maximum value of $T_{\hat{1}\hat{2}}$ by θ_s, then

$$\tan 2\theta_n = -\frac{1}{\tan 2\theta_s} \quad \Rightarrow \quad \frac{\sin 2\theta_n}{\cos 2\theta_n} = -\frac{\cos 2\theta_s}{\sin 2\theta_s},$$

$$\Rightarrow \quad \sin 2\theta_n \sin 2\theta_s + \cos 2\theta_n \cos 2\theta_s = 0 \Rightarrow \quad \cos 2(\theta_n - \theta_s) = 0;$$

and therefore

$$2(\theta_n - \theta_s) = \pm\frac{\pi}{2} \quad \Rightarrow \quad \theta_n = \theta_s \pm \frac{\pi}{4}.$$

Thus the angle at which $T_{\hat{1}\hat{2}}$ gains its maximum value is $\pm 45°$ from the angle at which the $T_{\hat{1}\hat{1}}$ is maximised or minimised.

7.) Note that we shall drop the overbars for convenience in the working here, but we are working in the Eulerian coordinate system. The principal invariants of the Cauchy stress tensor are given by

$$|T^i_j - \mu\delta^i_j| = -\mu^3 + I_1\,\mu^2 - I_2\,\mu + I_3,$$

so

$$I_1 = T^i_i,$$
$$I_2 = \frac{1}{2}\left(T^i_i T^j_j - T^i_j T^j_i\right),$$
$$I_3 = |T^i_j|.$$

8.) The principal invariants of the stress deviation tensor are given by

$$|\widetilde{T}^i_j - \mu\delta^i_j| = |T^i_j - \frac{1}{3}T^k_k\delta^i_j - \mu\delta^i_j| = -\mu^3 + J_1\,\mu^2 - J_2\,\mu + J_3.$$

Thus,

$$J_1 = T^i_i - \frac{1}{3}T^k_k\delta^i_i = T^i_i - \frac{1}{3}T^k_k \times 3 = 0.$$

$$J_2 = \frac{1}{2}\left[\left(T^i_i - T^k_k\delta^i_i/3\right)\left(T^j_j - T^m_m\delta^j_j/3\right) - \left(T^i_j - T^k_k\delta^i_j/3\right)\left(T^j_i - T^m_m\delta^j_i/3\right)\right]$$
$$= \frac{1}{2}\left[J_1\, J_1 - T^i_jT^j_i + T^k_kT^j_j/3 + T^i_iT^m_m/3 - T^k_kT^m_m\delta^i_j\delta^j_i/9\right].$$

We know that $J_1 = 0$ and $\delta^i_j\,\delta^j_i = \delta^j_j = 3$, so

$$J_2 = \frac{1}{2}\left[\frac{1}{3}T^i_i\,T^j_j - T^i_j\,T^j_i\right] = \frac{1}{2}\left[T^i_i\,T^j_j - T^i_j\,T^j_i\right] - \frac{1}{3}T^i_iT^j_j = I_2 - (I_1)^2/3,$$

$$J_3 = \left|T^i_j - \frac{1}{3}T^k_k\,\delta^i_j\right|$$

9.) a.) From the definition

$$\boldsymbol{T} = T_{\boldsymbol{N}}\boldsymbol{N} + \boldsymbol{T}_S,$$

and taking the dot product with the unit normal gives

$$\boldsymbol{T}\cdot\boldsymbol{N} = T_{\boldsymbol{N}};$$

and by the definition of the Cauchy stress tensor

$$T_{\boldsymbol{N}} = T_{JI}N_IN_J = T_{IJ}N_IN_J,$$

after using the symmetry property of the Cauchy stress tensor as well.

b.) Taking the dot product of the stress vector with itself and using the fact that $\boldsymbol{N}$ and $\boldsymbol{T}_S$ are orthogonal gives

$$\boldsymbol{T}\cdot\boldsymbol{T} = (T_{\boldsymbol{N}}\boldsymbol{N} + \boldsymbol{T}_S)\cdot(T_{\boldsymbol{N}}\boldsymbol{N} + \boldsymbol{T}_S) = T^2_{\boldsymbol{N}} + \boldsymbol{T}_S\cdot\boldsymbol{T}_S,$$

so

$$|\boldsymbol{T}_S|^2 = |\boldsymbol{T}|^2 - T^2_{\boldsymbol{N}}. \tag{3}$$

If we work in the Cartesian coordinate system given by the principal axes of stress then the Cauchy stress tensor is diagonal with entries corresponding to the principal stresses, say $\sigma_{(1)}$, $\sigma_{(2)}$ and $\sigma_{(3)}$. Hence from the definition $T_I = \sigma_{(I)}N_I$ (no summation), and so

$$|\boldsymbol{T}|^2 = (\sigma_{(1)}N_1)^2 + (\sigma_{(2)}N_2)^2 + (\sigma_{(3)}N_3)^2.$$

In addition,

$$\boldsymbol{T}^2_{\boldsymbol{N}} = (T_{IJ}N_IN_J)^2 = \left[\sigma_{(1)}N_1^2 + \sigma_{(2)}N_2^2 + \sigma_{(3)}N_3^2\right]^2,$$

and so by direct substitution into equation (3)

$$|\boldsymbol{T}_S|^2 = (T_{IJ}N_IN_J)^2 = (\sigma_{(1)}N_1)^2 + (\sigma_{(2)}N_2)^2 + (\sigma_{(3)}N_3)^2 - \left[\sigma_{(1)}N_1^2 + \sigma_{(2)}N_2^2 + \sigma_{(3)}N_3^2\right]^2.$$
$$= \sigma^2_{(I)}(N_I^2 - N_I^4) - 2\left(\sigma_{(1)}\sigma_{(2)}N_1^2N_2^2 + \sigma_{(1)}\sigma_{(3)}N_1^2N_3^2\sigma_{(2)}\sigma_{(3)}N_2^2N_3^2\right).$$

This expression can be simplified by spotting that

$$N_1^2 - N_1^4 = N_1^2(1 - N_1^2) = N_1^2(N_2^2 + N_3^2),$$

because $\boldsymbol{N}$ is a unit vector. Using similar expressions for $N_2^2 - N_2^4$ and $N_3^2 - N_3^4$, we obtain

$$|\boldsymbol{T}_S|^2 = N_1^2 N_2^2(\sigma_{(1)}^2 + \sigma_{(2)}^2 - 2\sigma_{(1)}\sigma_{(2)}) + N_2^2 N_3^2(\sigma_{(2)}^2 + \sigma_{(3)}^2 - 2\sigma_{(2)}\sigma_{(3)})$$

$$+ N_1^2 N_3^2(\sigma_{(1)}^2 + \sigma_{(3)}^2 - 2\sigma_{(1)}\sigma_{(3)})$$

$$\Rightarrow |\boldsymbol{T}_S|^2 = [N_1 N_2(\sigma_{(1)} - \sigma_{(2)})]^2 + [N_2 N_3(\sigma_{(2)} - \sigma_{(3)})]^2 + [N_1 N_3(\sigma_{(1)} - \sigma_{(3)})]^2. \quad (4)$$

c.) We continue to work in the coordinate system given by the principal axes of stress. The cosine of the angle between a plane and the I-th coordinate axis is given by $\cos\theta_{(I)} = N_I = \boldsymbol{N}\boldsymbol{\cdot}\boldsymbol{e}_I$, where $\boldsymbol{N}$ is the unit normal to the pane and $\boldsymbol{e}_I$ is a unit base vector in the coordinate direction. The result follows directly from the formula $\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\cos\phi$, where ϕ is the angle between the two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$.

If a plane makes the same angle with each of the three principal axes of stress then

$$\cos\theta_{(1)} = \cos\theta_{(2)} = \cos\theta_{(3)} \quad \Rightarrow \quad N_1 = N_2 = N_3,$$

but $\boldsymbol{N}$ is a unit vector, so

$$N_1^2 + N_2^2 + N_3^2 = 1 \quad \Rightarrow \quad 3N_1^2 = 1 \quad \Rightarrow N_1^2 = N_2^2 = N_3^2 = \frac{1}{3}.$$

The square of the magnitude of the shear stress on this plane is found from equation (4)

$$|\boldsymbol{T}_S|^2 = \frac{1}{9}\left[(\sigma_{(1)} - \sigma_{(2)})^2 + (\sigma_{(2)} - \sigma_{(3)})^2 + (\sigma_{(1)} - \sigma_{(3)})^2\right].$$

In our chosen coordinate system the invariant J_2 is found from question 8 to be

$$J_2 = I_2 - (I_1)^2/3 = \frac{1}{2}(T_{II}T_{JJ} - T_{IJ}T_{IJ}) - (T_{II})^2/3$$

$$= \frac{1}{2}\left[(\sigma_{(1)} + \sigma_{(2)} + \sigma_{(3)})(\sigma_{(1)} + \sigma_{(2)} + \sigma_{(3)}) - \sigma_{(1)}^2 - \sigma_{(2)}^2 - \sigma_{(3)}^2\right]$$

$$-(\sigma_{(1)} + \sigma_{(2)} + \sigma_{(3)})^2/3.$$

$$= \sigma_{(1)}\sigma_{(2)} + \sigma_{(1)}\sigma_{(3)} + \sigma_{(2)}\sigma_{(3)} - \left[\sigma_{(1)}^2 + \sigma_{(2)}^2 + \sigma_{(3)}^2 + 2\sigma_{(1)}\sigma_{(2)} + 2\sigma_{(1)}\sigma_{(3)} + 2\sigma_{(2)}\sigma_{(3)}\right]/3$$

$$= \frac{1}{3}\left[\sigma_{(1)}\sigma_{(2)} + \sigma_{(1)}\sigma_{(3)} + \sigma_{(2)}\sigma_{(3)} - \sigma_{(1)}^2 - \sigma_{(2)}^2 - \sigma_{(3)}^2\right] = -\frac{1}{3}\frac{9}{2}|\boldsymbol{T}_S|^2,$$

which means that the required magnitude is proportional to J_2.

10.) a.) Using the hint, we consider

$$(X_J T_{IK})_{,K} = X_{J,K} T_{IK} + X_J T_{IK,K}, \tag{5}$$

by the product rule. The equations of equilibrium in Cartesian coordinates are

$$T_{JI,J} + F_I = T_{IJ,J} + F_I = 0, \tag{6}$$

because $\boldsymbol{F}$ is a body force per unit volume (not per unit density) and the Cauchy stress tensor is symmetric Using equation (6) in (5) gives

$$(X_J T_{IK})_{,K} = \delta_{JK} T_{IK} + X_J(-F_I),$$

which gives the result in the hint

$$T_{IJ} = (X_J T_{IK})_{,K} + X_J F_I.$$

Taking the volume average of the equation, we have

$$\langle T_{IJ} \rangle = \frac{1}{V} \int_\Omega T_{IJ} \, \mathrm{d}\mathcal{V} = \frac{1}{V} \int_\Omega (X_J T_{IK})_{,K} + X_J F_I \, \mathrm{d}\mathcal{V},$$

and using the divergence theorem on the first integral

$$\langle T_{IJ} \rangle = \frac{1}{V} \left[\int_{\partial\Omega} X_J T_{IK} N_K \, \mathrm{d}\mathcal{S} + \int_\Omega X_J F_I \, \mathrm{d}\mathcal{V} \right],$$

where N_I are the Cartesian components of the outer unit normal to the body. From the definition of the stress tensor $T_{IK} N_K = \tau_I$ on the body's surface and writing the equation in dyadic form gives the result (ahem, actually we also use the symmetry property of the stress tensor)

$$\langle \mathsf{T} \rangle = \frac{1}{V} \left[\int_{\partial\Omega} \boldsymbol{R} \otimes \boldsymbol{\tau} \, \mathrm{d}\mathcal{S} + \int_\Omega \boldsymbol{R} \otimes \boldsymbol{F} \, \mathrm{d}\mathcal{V} \right].$$

This is known as Signorini's theorem and it demonstrates that the average stress in the body can be determined directly from the surface traction $\boldsymbol{\tau}$ and the body force $\boldsymbol{F}$.

b.) If $\boldsymbol{F} = \boldsymbol{0}$ and the surface is loaded by a uniform pressure then $\boldsymbol{\tau} = -p\boldsymbol{N}$, where $\boldsymbol{N}$ is the outer unit normal to the surface. Hence from Signorini's theorem

$$\langle \mathsf{T} \rangle = \frac{1}{V} \int_{\partial\Omega} \boldsymbol{R} \otimes (-p\boldsymbol{N}) \, \mathrm{d}\mathcal{S} = -\frac{p}{V} \int_{\partial\Omega} \boldsymbol{R} \otimes \boldsymbol{N} \, \mathrm{d}\mathcal{S},$$

because the pressure is uniform (and therefore p is constant). Using the divergence theorem and working in component form

$$\int_{\partial\Omega} R_I N_J \, \mathrm{d}\mathcal{S} = \int \Omega R_{I,J} \, \mathrm{d}\mathcal{V} = \delta_{IJ} \int \Omega \, \mathrm{d}\mathcal{V} = \delta_{IJ} V;$$

so back in matrix form

$$\langle \mathsf{T} \rangle = -p \, \mathsf{I}.$$

If we choose $\mathsf{T} = \langle \mathsf{T} \rangle = -p \, \mathsf{I}$, then the equilibrium equations, in the absence of a body force, are satisfied because

$$T_{IJ,J} = -p \, \delta_{IJ,J} = 0.$$

11.) This question is just to check that you understand the colon notation

$$\mathsf{A} : \mathsf{A} = A_{IJ}A_{IJ} \quad \Rightarrow \quad \frac{D}{Dt}(\mathsf{A} : \mathsf{A}) = \frac{D}{Dt}(A_{IJ}A_{IJ}) = A_{IJ}\frac{DA_{IJ}}{Dt} + \frac{DA_{IJ}}{Dt}A_{IJ},$$

so

$$\frac{D}{Dt}(\mathsf{A} : \mathsf{A}) = 2\mathsf{A} : \frac{D\mathsf{A}}{Dt}.$$

12.) The second Piola–Kirchhoff stress tensor is given by

$$\mathsf{s} = J\mathsf{F}^{-1}\sigma\mathsf{F}^{-T}.$$

Hence the material derivative, by the product rule is

$$\dot{\mathsf{s}} = \dot{J}\mathsf{F}^{-1}\sigma\mathsf{F}^{-T} + J\dot{\mathsf{F}}^{-1}\sigma\mathsf{F}^{-T} + J\mathsf{F}^{-1}\dot{\sigma}\mathsf{F}^{-T} + J\mathsf{F}^{-1}\sigma\dot{\mathsf{F}}^{-T}. \tag{7}$$

From the lecture notes (or working it out again),

$$\dot{J} = J\,\mathrm{div}\boldsymbol{V} = J\,\mathrm{tr}(\mathsf{D}), \quad \text{and} \quad \dot{\mathsf{F}} = \mathsf{L}\mathsf{F}.$$

Now because

$$\mathsf{F}\mathsf{F}^{-1} = \mathsf{I} \quad \Rightarrow \quad \dot{\mathsf{F}}\mathsf{F}^{-1} + \mathsf{F}\dot{\mathsf{F}}^{-1} = 0 \quad \Rightarrow \dot{\mathsf{F}}^{-1} = -\mathsf{F}^{-1}\dot{\mathsf{F}}\mathsf{F}^{-1} = -\mathsf{F}^{-1}\mathsf{L},$$

and so

$$\dot{\mathsf{F}}^{-T} = -\mathsf{L}^{T}\mathsf{F}^{-T}.$$

Using these results into the expression (7) we obtain the required stress rate

$$\dot{\mathsf{s}} = J\mathsf{F}^{-1}\left[\dot{\sigma} + \mathrm{tr}(\mathsf{D})\,\sigma - \mathsf{L}\sigma - \sigma\mathsf{L}^{T}\right]\mathsf{F}^{-T}.$$

The quickest way to establish the objectivity, or not of $\dot{\mathsf{s}}$ is to think about s. Under a change in observer

$$\mathsf{s}^* = J^*(\mathsf{F}^*)^{-1}\sigma^*(\mathsf{F}^*)^{-T}.$$

The Cauchy stress tensor is objective by construction (it is a Cartesian tensor), see question 4, so $\sigma^* = \mathsf{Q}\sigma\mathsf{Q}^T$. We can look up the transformation properties of F in the lecture notes to find that $\mathsf{F}^* = \mathsf{Q}\mathsf{F}$. The Jacobian J is a scalar invariant because $\mathsf{F}^* = \mathsf{Q}\mathsf{F}$, which means that

$$J^* = |\mathsf{F}^*| = |\mathsf{Q}\mathsf{F}| = |\mathsf{Q}||\mathsf{F}| = |\mathsf{F}| = J,$$

because $|\mathsf{Q}| = 1$ by the orthogonality of Q. Hence,

$$\mathsf{s}^* = J(\mathsf{Q}\mathsf{F})^{-1}\mathsf{Q}\sigma\mathsf{Q}^T(\mathsf{Q}\mathsf{F})^{-T} = J\mathsf{F}^{-1}\mathsf{Q}^T\mathsf{Q}\sigma\mathsf{Q}^T\mathsf{Q}\mathsf{F}^{-T} = J\mathsf{F}^{-1}\sigma\mathsf{F}^{-T} = \mathsf{s},$$

which means that the second Piola–Kirchhoff stress tensor is invariant under Eulerian observer transformation. This shouldn't be a surprise, it's a Lagrangian measure of the stress. By taking the material derivative of the above equation we have that

$$\dot{\mathsf{s}}^* = \dot{\mathsf{s}},$$

and so $\dot{\mathsf{s}}$ is also invariant under change of observer, so it is not objective.

13.) If we invent the symbol $\overset{\mathcal{T}}{\mathsf{A}}$ to represent the Truesdell rate, then

$$\overset{\mathcal{T}}{\mathsf{A}}{}^* = \frac{D\mathsf{A}^*}{Dt} + \mathrm{tr}(\mathsf{D}^*)\mathsf{A}^* + \mathsf{L}^{*T}\mathsf{A}^* + \mathsf{A}^*\mathsf{L}^*,$$

$$= \dot{\mathsf{Q}}\mathsf{A}\mathsf{Q}^T + \mathsf{Q}\dot{\mathsf{A}}\mathsf{Q}^T + \mathsf{Q}\mathsf{A}\dot{\mathsf{Q}}^T + \mathrm{tr}(\mathsf{D})\mathsf{Q}\mathsf{A}\mathsf{Q}^T + (\mathsf{Q}\mathsf{L}^T\mathsf{Q}^T + \Omega^T)\mathsf{Q}\mathsf{A}\mathsf{Q}^T + \mathsf{Q}\mathsf{A}\mathsf{Q}^T(\mathsf{Q}\mathsf{L}\mathsf{Q}^T + \Omega)$$

$$= \mathsf{Q}\left[\dot{\mathsf{A}} + \mathrm{tr}(\mathsf{D})\mathsf{A} + \mathsf{L}^T\mathsf{A} + \mathsf{A}\mathsf{L}\right]\mathsf{Q}^T + \Omega\mathsf{Q}\mathsf{A}\mathsf{Q}^T - \mathsf{Q}\mathsf{A}\mathsf{Q}^T\Omega + \Omega^T\mathsf{Q}\mathsf{A}\mathsf{Q}^T + \mathsf{Q}\mathsf{A}\mathsf{Q}^T\Omega,$$

and using the fact that $\Omega = -\Omega^T$, we have

$$\overset{\mathcal{T}}{\mathsf{A}}{}^* = \mathsf{Q}\overset{\mathcal{T}}{\mathsf{A}}\mathsf{Q}^T,$$

which proves that the Truesdell rate is objective.