MATH45061: SOLUTION SHEET[1] II

1.) The deformation gradient tensor has Cartesian components given by $F_{IJ} = \partial R_I/\partial r_J$; and so

$$F_{11} = \frac{\partial R_1}{\partial r_1} = e^{x_1}, \quad F_{12} = \frac{\partial R_1}{\partial r_2} = 0, \quad F_{13} = \frac{\partial R_1}{\partial r_3} = 0,$$
$$F_{21} = \frac{\partial R_2}{\partial r_1} = 0, \quad F_{22} = \frac{\partial R_2}{\partial r_2} = 2, \quad F_{23} = \frac{\partial R_2}{\partial r_3} = -1,$$
$$F_{31} = \frac{\partial R_3}{\partial r_1} = 0, \quad F_{32} = \frac{\partial R_3}{\partial r_2} = 1, \quad F_{33} = \frac{\partial R_3}{\partial r_3} = 2.$$

Writing the components in matrix form, we have

$$\mathsf{F} = \begin{pmatrix} e^{x_1} & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix},$$

which has determinant given by

$$\det \mathsf{F} = e^{x_1}(2 \times 2 + 1 \times 1) = 5e^{x_1}.$$

The deformation is physically admissible provided that $\det \mathsf{F} > 0$, which is true provided that x_1 does not extend to $-\infty$.

For the two-dimensional lamina, $x_3 = 0$, so the deformed position is given by

$$X_1 = e^{x_1}, \quad X_2 = 2x_2, \quad X_3 = x_2,$$

which means that the corners map as follows

$$(0,0,0) \to (1,0,0); \quad (1,0,0) \to (e,0,0); \quad (0,1,0) \to (1,2,1); \quad (1,1,0) \to (e,2,1),$$

and the deformed lamina moves out of the $x_1 - x_2$ plane and is also stretched, see Figure 1

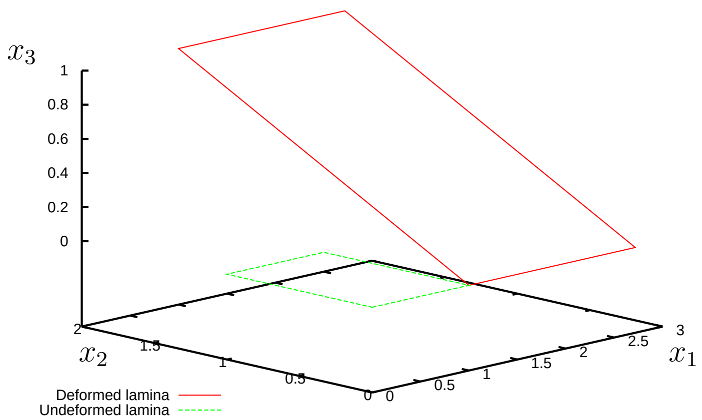

Figure 1: Sketch of the undeformed and deformed laminæ.

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2.) a.) The "easy to compute" base vectors (in component form in the global Cartesian basis) are given by

$$\boldsymbol{g}_1 = \frac{\partial \boldsymbol{r}}{\partial \xi^1} = \frac{\partial}{\partial \xi^1}\begin{pmatrix} \xi^1\xi^2 \\ \xi^2 \end{pmatrix} = \begin{pmatrix} \xi^2 \\ 0 \end{pmatrix}, \quad \boldsymbol{g}_2 = \frac{\partial \boldsymbol{r}}{\partial \xi^2} = \frac{\partial}{\partial \xi^2}\begin{pmatrix} \xi^1\xi^2 \\ \xi^2 \end{pmatrix} = \begin{pmatrix} \xi^1 \\ 1 \end{pmatrix};$$

$$\boldsymbol{G}_{\bar{1}} = \frac{\partial \boldsymbol{R}}{\partial \chi^{\bar{1}}} = \frac{\partial}{\partial \chi^{\bar{1}}}\begin{pmatrix} \chi^{\bar{1}} \\ \chi^{\bar{1}}\chi^{\bar{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ \chi^{\bar{2}} \end{pmatrix}, \quad \boldsymbol{G}_{\bar{2}} = \frac{\partial \boldsymbol{R}}{\partial \chi^{\bar{2}}} = \frac{\partial}{\partial \chi^{\bar{2}}}\begin{pmatrix} \chi^{\bar{1}} \\ \chi^{\bar{1}}\chi^{\bar{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ \chi^{\bar{1}} \end{pmatrix}.$$

These are the natural choices for the (covariant) base vectors in the original and deformed positions. The other base vectors are "unnatural" in the sense that we are using a coordinate system associated with the other position and so we must make use of the mapping between the two.

$$\boldsymbol{R} = 2\boldsymbol{r} = \begin{pmatrix} 2\xi^1\xi^2 \\ 2\xi^2 \end{pmatrix},$$

so that

$$\boldsymbol{G}_1 = \frac{\partial}{\partial \xi^1}\begin{pmatrix} 2\xi^1\xi^2 \\ 2\xi^2 \end{pmatrix} = \begin{pmatrix} 2\xi^2 \\ 0 \end{pmatrix}, \quad \boldsymbol{G}_2 = \frac{\partial}{\partial \xi^2}\begin{pmatrix} 2\xi^1\xi^2 \\ 2\xi^2 \end{pmatrix} = \begin{pmatrix} 2\xi^1 \\ 2 \end{pmatrix};$$

and

$$\boldsymbol{r} = \frac{1}{2}\boldsymbol{R} = \frac{1}{2}\begin{pmatrix} \chi^{\bar{1}} \\ \chi^{\bar{1}}\chi^{\bar{2}} \end{pmatrix},$$

leading to

$$\boldsymbol{g}_{\bar{1}} = \frac{\partial}{\partial \chi^{\bar{1}}}\begin{pmatrix} \frac{1}{2}\chi^{\bar{1}} \\ \frac{1}{2}\chi^{\bar{1}}\chi^{\bar{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2}\chi^{\bar{2}} \end{pmatrix}, \quad \boldsymbol{g}_{\bar{2}} = \frac{\partial}{\partial \chi^{\bar{2}}}\begin{pmatrix} \frac{1}{2}\chi^{\bar{1}} \\ \frac{1}{2}\chi^{\bar{1}}\chi^{\bar{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2}\chi^{\bar{1}} \end{pmatrix}.$$

b.) The associated metric tensors are:

$$g_{ij} = \begin{pmatrix} \boldsymbol{g}_1\cdot\boldsymbol{g}_1 & \boldsymbol{g}_1\cdot\boldsymbol{g}_2 \\ \boldsymbol{g}_2\cdot\boldsymbol{g}_1 & \boldsymbol{g}_2\cdot\boldsymbol{g}_2 \end{pmatrix} = \begin{pmatrix} \xi^2\xi^2 & \xi^2\xi^1 \\ \xi^2\xi^1 & 1+\xi^1\xi^1 \end{pmatrix};$$

$$G_{ij} = \begin{pmatrix} \boldsymbol{G}_1\cdot\boldsymbol{G}_1 & \boldsymbol{G}_1\cdot\boldsymbol{G}_2 \\ \boldsymbol{G}_2\cdot\boldsymbol{G}_1 & \boldsymbol{G}_2\cdot\boldsymbol{G}_2 \end{pmatrix} = \begin{pmatrix} 4\xi^2\xi^2 & 4\xi^1\xi^2 \\ 4\xi^1\xi^2 & 4+4\xi^1\xi^1 \end{pmatrix},$$

$$g_{\bar{i}\bar{j}} = \begin{pmatrix} \boldsymbol{g}_{\bar{1}}\cdot\boldsymbol{g}_{\bar{1}} & \boldsymbol{g}_{\bar{1}}\cdot\boldsymbol{g}_{\bar{2}} \\ \boldsymbol{g}_{\bar{2}}\cdot\boldsymbol{g}_{\bar{1}} & \boldsymbol{g}_{\bar{2}}\cdot\boldsymbol{g}_{\bar{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}+\frac{1}{4}\chi^{\bar{2}}\chi^{\bar{2}} & \frac{1}{4}\chi^{\bar{1}}\chi^{\bar{2}} \\ \frac{1}{4}\chi^{\bar{1}}\chi^{\bar{2}} & \frac{1}{4}\chi^{\bar{1}}\chi^{\bar{1}} \end{pmatrix},$$

$$G_{\bar{i}\bar{j}} = \begin{pmatrix} \boldsymbol{G}_{\bar{1}}\cdot\boldsymbol{G}_{\bar{1}} & \boldsymbol{G}_{\bar{1}}\cdot\boldsymbol{G}_{\bar{2}} \\ \boldsymbol{G}_{\bar{2}}\cdot\boldsymbol{G}_{\bar{1}} & \boldsymbol{G}_{\bar{2}}\cdot\boldsymbol{G}_{\bar{2}} \end{pmatrix} = \begin{pmatrix} 1+\chi^{\bar{2}}\chi^{\bar{2}} & \chi^{\bar{1}}\chi^{\bar{2}} \\ \chi^{\bar{1}}\chi^{\bar{2}} & \chi^{\bar{1}}\chi^{\bar{1}} \end{pmatrix};$$

and the differences are then

$$A_{ij} = \begin{pmatrix} 3\xi^2\xi^2 & 3\xi^1\xi^2 \\ 3\xi^1\xi^2 & 3+3\xi^1\xi^1 \end{pmatrix} \quad \text{and} \quad A_{\bar{i}\bar{j}} = \begin{pmatrix} \frac{3}{4}+\frac{3}{4}\chi^{\bar{2}}\chi^{\bar{2}} & \frac{3}{4}\chi^{\bar{1}}\chi^{\bar{2}} \\ \frac{3}{4}\chi^{\bar{1}}\chi^{\bar{2}} & \frac{3}{4}\chi^{\bar{1}}\chi^{\bar{1}} \end{pmatrix}$$

The two tensors should be related by covariant transformation between the Eulerian and Lagrangian coordinates:

$$A_{ij} = \frac{\partial \chi^{\overline{k}}}{\partial \xi^i} \frac{\partial \chi^{\overline{l}}}{\partial \xi^j} A_{\overline{k}\,\overline{l}}. \tag{1}$$

Confirming this is a little painful algebraically (as with many explicit tensor calculations), but perfectly straightforward.
From the deformation

$$\boldsymbol{R} = \begin{pmatrix} \chi^{\overline{1}} \\ \chi^{\overline{1}}\chi^{\overline{2}} \end{pmatrix} = 2\boldsymbol{r} = \begin{pmatrix} 2\xi^1\xi^2 \\ 2\xi^2 \end{pmatrix},$$

and so

$$\chi^{\overline{1}} = 2\xi^1\xi^2, \quad \text{and} \quad \chi^{\overline{2}} = 2\xi^2/\chi^{\overline{1}} = 1/\xi^1.$$

Thus coordinate transformation is given by

$$\begin{pmatrix} \frac{\partial \chi^{\overline{1}}}{\partial \xi^1} & \frac{\partial \chi^{\overline{1}}}{\partial \xi^2} \\ \frac{\partial \chi^{\overline{2}}}{\partial \xi^1} & \frac{\partial \chi^{\overline{2}}}{\partial \xi^2} \end{pmatrix} = \begin{pmatrix} 2\xi^2 & 2\xi^1 \\ -1/(\xi^1\xi^1) & 0 \end{pmatrix};$$

and we can write the transformation (1) in the form of a matrix product

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 2\xi^2 & 2\xi^1 \\ -1/(\xi^1\xi^1) & 0 \end{pmatrix}^T \begin{pmatrix} A_{\overline{1}\,\overline{1}} & A_{\overline{1}\,\overline{2}} \\ A_{\overline{2}\,\overline{1}} & A_{\overline{2}\,\overline{2}} \end{pmatrix} \begin{pmatrix} 2\xi^2 & 2\xi^1 \\ -1/(\xi^1\xi^1) & 0 \end{pmatrix},$$

$$= \begin{pmatrix} 2\xi^2 & -1/(\xi^1\xi^1) \\ 2\xi^1 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{4} + \frac{3}{4}\chi^{\overline{2}}\chi^{\overline{2}} & \frac{3}{4}\chi^{\overline{1}}\chi^{\overline{2}} \\ \frac{3}{4}\chi^{\overline{1}}\chi^{\overline{2}} & \frac{3}{4}\chi^{\overline{1}}\chi^{\overline{1}} \end{pmatrix} \begin{pmatrix} 2\xi^2 & 2\xi^1 \\ -1/(\xi^1\xi^1) & 0 \end{pmatrix}.$$

We now convert all components $A_{\overline{i}\,\overline{j}}$ into the Lagrangian coordinates ξ^i, so that

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 2\xi^2 & -1/(\xi^1\xi^1) \\ 2\xi^1 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{4} + \frac{3}{4}\frac{1}{(\xi^1\xi^1)} & \frac{3}{2}\xi^2 \\ \frac{3}{2}\xi^2 & 3\xi^1\xi^1\xi^2\xi^2 \end{pmatrix} \begin{pmatrix} 2\xi^2 & 2\xi^1 \\ -1/(\xi^1\xi^1) & 0 \end{pmatrix},$$

$$= \begin{pmatrix} 2\xi^2 & -1/(\xi^1\xi^1) \\ 2\xi^1 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{2}\xi^2 & \frac{3}{2}\xi^1 + \frac{3}{2}(1/\xi^1) \\ 0 & 3\xi^1\xi^2 \end{pmatrix} = \begin{pmatrix} 3\xi^2\xi^2 & 3\xi^1\xi^2 \\ 3\xi^1\xi^2 & 3(\xi^1\xi^1 + 1) \end{pmatrix},$$

as expected (phew!).

3.) a.) The deformation gradient is given by

$$\mathsf{F} = \begin{pmatrix} 1+t & 0 & 0 \\ 0 & 1 & t \\ 0 & -t & 1 \end{pmatrix},$$

which has $\det \mathsf{F} = (1+t)(1+t^2) > 0$, when $t \geq 0$, so the deformation is physically admissible for all $t > 0$. We have that the deformation is given by

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} (1+t) & 0 & 0 \\ 0 & 1 & t \\ 0 & -t & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{2}$$

and in order to find the inverse transform we must simply invert the matrix to obtain

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} (1+t)^{-1} & 0 & 0 \\ 0 & (1+t^2)^{-1} & -t(1+t^2)^{-1} \\ 0 & t(1+t^2)^{-1} & (1+t^2)^{-1} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \tag{3}$$

b.) The Lagrangian velocity field is given by

$$\frac{\partial}{\partial t}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{\partial}{\partial t}\begin{pmatrix} (1+t) & 0 & 0 \\ 0 & 1 & t \\ 0 & -t & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

so $v_1 = x_1$, $v_2 = x_3$, $v_3 = -x_2$.
The Eulerian velocity field is obtained by using the inverse transformation (3) in the expression for the velocity

$$\boldsymbol{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} (1+t)^{-1} & 0 & 0 \\ 0 & (1+t^2)^{-1} & -t(1+t^2)^{-1} \\ 0 & t(1+t^2)^{-1} & (1+t^2)^{-1} \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$$= \begin{pmatrix} (1+t)^{-1} & 0 & 0 \\ 0 & t(1+t^2)^{-1} & (1+t^2)^{-1} \\ 0 & -(1+t^2)^{-1} & t(1+t^2)^{-1} \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix},$$

so $V_1 = X_1/(1+t)$, $V_2 = (tX_2 + X_3)/(1+t^2)$, $V_3 = (tX_3 - X_2)/(1+t^2)$.

c.) The function $\Phi(t, X_I) = tX_1 + X_2$ can be converted to Lagrangian coordinates by using the deformation (2)

$$\phi(t, x_I) = t(1+t)x_1 + x_2 + tx_3.$$

The material derivative is then simply

$$\left.\frac{\partial \phi}{\partial t}\right|_{x_I} = (1+2t)x_1 + x_3 = \frac{1+2t}{1+t}X_1 + \frac{tX_2 + X_3}{1+t^2},$$

after using the inverse deformation (3).

d.) The material derivative in Eulerian coordinates is

$$\frac{D\Phi}{Dt} = \frac{\partial \Phi}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}_{\!R}\Phi = X_1 + \left(\frac{X_1}{1+t}, \frac{tX_2+X_3}{1+t^2}, \frac{tX_3 - X_2}{1+t^2}\right)\cdot(t, 1, 0),$$

$$= X_1 + \frac{tX_1}{1+t} + \frac{tX_2 + X_3}{1+t^2} = \frac{1+2t}{1+t}X_1 + \frac{tX_2+X_3}{1+t^2},$$

which, of course, agrees with the answer in part (c).

4.) This proof does rely on some previous knowledge of vector calculus and differential equations so don't worry too much if you struggled.

The key piece of information is that the normal to a surface $f(\chi^{\bar{i}}, t) = 0$ (or any constant in fact) is given by $\boldsymbol{\nabla} f$, so

$$\tilde{\boldsymbol{n}} = \boldsymbol{G}^{\bar{i}} f_{,\bar{i}}, \quad \text{with length} \quad |\tilde{\boldsymbol{n}}|^2 = \boldsymbol{G}^{\bar{i}} f_{,\bar{i}} \boldsymbol{\cdot} \boldsymbol{G}^{\bar{j}} f_{,\bar{j}} = G^{\bar{i}\bar{j}} f_{,\bar{i}} f_{,\bar{j}},$$

where a comma $,\bar{i}$ denotes $\partial/\partial\chi^{\bar{i}}$ Thus, the covariant components of a unit normal to the surface are given by

$$n_{\bar{k}} = \frac{f_{,\bar{k}}}{\sqrt{G^{\bar{i}\bar{j}} f_{,\bar{i}} f_{,\bar{j}}}}.$$

We differentiate the defining expression of the surface with respect to time:

$$\dot{f} = \frac{\partial f}{\partial t} + \frac{\partial \chi^{\bar{k}}}{\partial t} \frac{\partial f}{\partial \chi^{\bar{k}}} = 0,$$

where $\partial\chi^{\bar{k}}/\partial t$ is for a given particle within the surface. We suppose that the particles in the surface are moving with an arbitrary velocity field $\widehat{\boldsymbol{V}}(\chi^{\bar{i}}, t)$. Thus, for a given particle

$$\frac{\partial \boldsymbol{R}}{\partial t} = \widehat{\boldsymbol{V}} \quad \Rightarrow \quad \frac{\partial X_I}{\partial t} = \widehat{V}_I \quad \Rightarrow \quad \frac{\partial \chi^{\bar{i}}}{\partial t} \frac{\partial X_I}{\partial \chi^{\bar{i}}} = \widehat{V}_I \quad \Rightarrow \quad \frac{\partial \chi^{\bar{i}}}{\partial t} = \frac{\partial \chi^{\bar{i}}}{\partial X_I} \widehat{V}_I = \widehat{V}^{\bar{i}},$$

and so

$$\dot{f} = \frac{\partial f}{\partial t} + \widehat{V}^{\bar{k}} \frac{\partial f}{\partial \chi^{\bar{k}}} = \frac{\partial f}{\partial t} + \widehat{V}^{\bar{k}} n_{\bar{k}} \sqrt{G^{\bar{i}\bar{j}} f_{,\bar{i}} f_{,\bar{j}}} = 0,$$

which means that

$$\frac{\partial f}{\partial t} = -\widehat{V}^{\bar{k}} n_{\bar{k}} \sqrt{G^{\bar{i}\bar{j}} f_{,\bar{i}} f_{,\bar{j}}};$$

in other words the normal speed of the particles must be the same as the speed of movement of the surface in order to remain within the surface.

The material derivative of the surface function $f(\chi^{\bar{i}}, t)$ is

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + V^{\bar{i}} \frac{\partial f}{\partial \chi^{\bar{i}}},$$

where $V^{\bar{i}}$ are the components of the underlying velocity of the continua. Using the above expression we have

$$\frac{Df}{Dt} = \left(V^{\bar{i}} - \widehat{V}^{\bar{i}}\right) n_{\bar{k}} \sqrt{G^{\bar{i}\bar{j}} f_{,\bar{i}} f_{,\bar{j}}}.$$

If the particles are material then their velocity $\widehat{\boldsymbol{V}}$ must coincide with the material velocity $\boldsymbol{V}$, which implies that $Df/Dt = 0$ from the above equation.

For the sufficiency condition we can use the method of characteristics. We introduce a new variable s and let $\chi^{\bar{i}}(s)$ and $t(s)$, such that

$$\frac{\mathrm{d}\chi^{\bar{i}}}{\mathrm{d}s} = V^{\bar{i}}, \quad \text{and} \quad \frac{\mathrm{d}t}{\mathrm{d}s} = 1;$$

so that

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \frac{\partial f}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}s} + \frac{\partial f}{\partial \chi^{\bar{i}}}\frac{\mathrm{d}\chi^{\bar{i}}}{\mathrm{d}s} = \frac{\partial f}{\partial t} + V^{\bar{i}}\frac{\partial f}{\partial \chi^{\bar{i}}} = \frac{Df}{Dt} = 0.$$

We can find the solution of the equation by solving the two systems:

$$\frac{\mathrm{d}\chi^{\bar{i}}}{\mathrm{d}s} = V^{\bar{i}}, \quad \text{and} \quad \frac{\mathrm{d}t}{\mathrm{d}s} = 1.$$

The second equation implies that $s = t$ because we are free to choose the arbitrary constant and then the remaining equation becomes

$$\frac{\mathrm{d}\chi^{\bar{i}}}{\mathrm{d}t} = V^{\bar{i}},$$

but this is the defining equation for the evolution of a material particle, which proves sufficiency.

5.) a.) We know that planes of constant x_1 become planes of constant R. If R is a function of x_2 then a change in x_2 would cause a change in R, but this is a contradiction because varying x_2, but keeping x_1 constant must not change R. A similar argument for x_3 demonstrates that $R = f(x_1)$ and similarly $\Theta = g(x_2)$ and $Z = h(x_3)$.

b.) If we let $h(x_3) = \lambda x_3$, then we have the that deformed position is given by

$$\boldsymbol{R} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} R\cos\Theta \\ R\sin\Theta \\ Z \end{pmatrix} = \begin{pmatrix} f(x_1)\cos g(x_2) \\ f(x_2)\sin g(x_2) \\ \lambda x_3 \end{pmatrix}.$$

Thus, the covariant base vectors are

$$\boldsymbol{G}_1 = \frac{\partial \boldsymbol{R}}{\partial x_1} = \begin{pmatrix} f'\cos g \\ f'\sin g \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -fg'\sin g \\ fg'\cos g \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix},$$

and the deformed metric tensor G_{ij} has the matrix of components

$$G_{ij} = \boldsymbol{G}_i \boldsymbol{\cdot} \boldsymbol{G}_j = \begin{pmatrix} (f')^2 & 0 & 0 \\ 0 & f^2(g')^2 & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}.$$

c.) If the deformation is of constant volume then $\sqrt{G/g} = 1$. The undeformed state is represented in Cartesians for which $\sqrt{g} = 1$ and so $\sqrt{G} = 1$. Now $G = \det G_{ij} = (f')^2 f^2 (g')^2 \lambda^2$, which means that

$$\sqrt{G} = 1 \quad \Rightarrow \quad \lambda f f' g' = 1.$$

We now proceed to separate the variables

$$\lambda f f' = 1/g'.$$

The left-hand side is a function only of x_1, the right-hand side is a function of x_2 and therefore the only way that they can be equal for all x_1 and x_2 is for both sides to be equal to the same constant, β, say. Hence,

$$ff' = \beta/\lambda \quad \text{and} \quad g' = 1/\beta,$$

and integrating gives

$$\frac{1}{2}f^2 = \frac{\beta x_1}{\lambda} + c_1 \quad \text{and} \quad g = \frac{x_2}{\beta} + c_2,$$

where c_1 and c_2 are constants. It follows that

$$f(x_1) = \sqrt{\frac{2\beta x_1}{\lambda} + 2c_1} \quad \text{and} \quad g(x_2) = \frac{x_2}{\beta} + c_2.$$

For a specific problem the unknown constants will be fixed by the specific boundary conditions.

6.) a.) The deformation is given by $\boldsymbol{R} = \alpha \boldsymbol{r}$, which in Cartesian coordinates can be written as

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{pmatrix}.$$

The deformation gradient tensor is given by

$$\mathsf{F} = \nabla_{\boldsymbol{r}} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}.$$

The Green–Lagrange strain tensor in Cartesian coordinates is

$$e_{IJ} = \frac{1}{2}\left(F_{KI}F_{KJ} - \delta_{IJ}\right),$$

so the only non-zero components are the diagonal entries and we have that

$$e_{IJ} = \frac{\alpha^2 - 1}{2}\,\delta_{IJ}.$$

b.) As on Example Sheet I, let $\xi^1 = r$, $\xi^2 = \theta$, $\xi^3 = \phi$ and because $\boldsymbol{R} = \alpha\boldsymbol{r}$, we have that $\boldsymbol{G}_i = \alpha \boldsymbol{g}_i$. Hence, $G_{ij} = \alpha^2 g_{ij}$ and

$$\gamma_{ij} = \frac{1}{2}\left(G_{ij} - g_{ij}\right) = \frac{\alpha^2 - 1}{2} g_{ij}.$$

From Example Sheet I, we know that g_{ij} is diagonal with the entries $g_{11} = 1$, $g_{22} = r^2$, $g_{33} = r^2 \sin^2\theta$, so

$$\gamma_{11} = \frac{\alpha^2 - 1}{2}, \quad \gamma_{22} = \frac{\alpha^2 - 1}{2} r^2, \quad \gamma_{33} = \frac{\alpha^2 - 1}{2} r^2 \sin^2\theta.$$

If you transform the undeformed coordinates back to Cartesian coordinates then g_{ij} becomes δ_{IJ} and we obtain the same result as in part (a).

7.) The equation

$$G_{ij}n^j - \mu g_{ij}n^j = 0,$$

can be written in the form

$$\frac{\partial X_K}{\partial \xi^i}\frac{\partial X_K}{\partial \xi^j}n^j = \mu\frac{\partial x_K}{\partial \xi^i}\frac{\partial x_K}{\partial \xi^j}n^j \quad \Rightarrow \frac{\partial X_K}{\partial x_I}\frac{\partial x_I}{\partial \xi^i}\frac{\partial X_K}{\partial x_J}\frac{\partial x_J}{\partial \xi^j}n^j = \mu\frac{\partial x_K}{\partial x_I}\frac{\partial x_I}{\partial \xi^i}\frac{\partial x_K}{\partial x_J}\frac{\partial x_J}{\partial \xi^j}n^j.$$

Now the transformation of contravariant components of the normal vector to from Cartesians is

$$n^j = \frac{\partial \xi^j}{\partial x_L}n_L,$$

which means that the equation becomes

$$\frac{\partial X_K}{\partial x_I}\frac{\partial x_I}{\partial \xi^i}\frac{\partial X_K}{\partial x_J}\frac{\partial x_J}{\partial \xi^j}\frac{\partial \xi^j}{\partial x_L}n_L = \mu\frac{\partial x_K}{\partial x_I}\frac{\partial x_I}{\partial \xi^i}\frac{\partial x_K}{\partial x_J}\frac{\partial x_J}{\partial \xi^j}\frac{\partial \xi^j}{\partial x_L}n_L,$$

and so

$$\frac{\partial X_K}{\partial x_I}\frac{\partial X_K}{\partial x_J}\frac{\partial x_I}{\partial \xi^i}\delta_{JL}n_L = \mu\frac{\partial x_K}{\partial x_I}\frac{\partial x_K}{\partial x_J}\frac{\partial x_I}{\partial \xi^i}\delta_{JL}n_L$$

$$\Rightarrow \quad \left(\frac{\partial X_K}{\partial x_I}\frac{\partial X_K}{\partial x_J} - \mu\frac{\partial x_K}{\partial x_I}\frac{\partial x_K}{\partial x_J}\right)n_J\frac{\partial x_I}{\partial \xi^i} = 0.$$

The transformation $\partial x_I/\partial \xi^i$ is assumed to be non-singular and the coordinates x_I are independent, which means that the equation becomes

$$\left(\frac{\partial X_K}{\partial x_I}\frac{\partial X_K}{\partial x_J} - \mu\delta_{KI}\delta_{KJ}\right)n_J = 0, \quad \Rightarrow \quad c_{IJ}n_J - \mu\delta_{IJ}n_J = 0 \quad \Rightarrow \quad c_{IJ}n_J - \mu n_I = 0,$$

where $c_{IJ} = (\partial X_K/\partial x_I)(\partial X_K/\partial x_J)$.

The principal stretches are therefore the square roots of the eigenvalues of the matrix c_{IJ}, which for the deformation given in question 6. is

$$\mathsf{c} = \mathsf{F}^T\mathsf{F} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix}.$$

Hence the eigenvalues are all α^2 with eigenvectors given by $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. The principal stretches are then all equal to α, as we should expect from the nature of the deformation.

8.) The deformation is given by

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

a.) The deformation gradient tensor is given by

$$\mathsf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix},$$

and so the Green–Lagrange strain tensor is

$$\mathsf{e} = \frac{1}{2}(\mathsf{F}^T\mathsf{F} - \mathsf{I}) = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \gamma & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right].$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma \\ 0 & \gamma & \gamma^2 \end{pmatrix}.$$

b.) The extreme values of the stretches are given by the principal stretches, which are the square-roots of the eigenvalues of the tensor

$$\mathsf{c} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \gamma \\ 0 & \gamma & 1+\gamma^2 \end{pmatrix}.$$

The eigenvalues are given by the roots of the characteristic equation

$$(1-\mu)[(1-\mu)(1+\gamma^2-\mu)-\gamma^2] = (1-\mu)[1+\gamma^2-\mu-\gamma^2-\mu-\mu\gamma^2+\mu^2]$$
$$= (1-\mu)[1-\mu(2+\gamma^2)+\mu^2],$$

which has roots

$$\mu = 1, \mu = \frac{(2+\gamma^2) \pm \sqrt{(2+\gamma^2)^2 - 4}}{2} = \frac{(2+\gamma^2) \pm \gamma\sqrt{4+\gamma^2}}{2},$$

so

$$\mu = 1, 1 + \frac{1}{2}\gamma^2 \pm \gamma\sqrt{1 + \frac{1}{4}\gamma^2}.$$

The associated directions (eigenvectors) are given by the solutions of the linear equations

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \gamma \\ 0 & \gamma & 1+\gamma^2 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \mu\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}.$$

Thus, the eigenvalue associated with $\mu = 1$ is simply $\boldsymbol{e}_1$,

$$\mu = 1, \quad \boldsymbol{n} = (1, 0, 0)^T.$$

Using the third linear equation for the other eigenvalues we have that

$$\gamma n_2 + (1+\gamma^2)n_3 = \left(1 + \frac{1}{2}\gamma^2 \pm \gamma\sqrt{1 + \frac{1}{4}\gamma^2}\right)n_3 \quad \Rightarrow \quad \gamma n_2 = \left(-\frac{1}{2}\gamma^2 \pm \gamma\sqrt{1 + \frac{1}{4}\gamma^2}\right)n_3;$$

which means that if we choose $n_3 = 1$, then the associated eigenvalues are

$$\boldsymbol{n} = \left(0, -\frac{1}{2}\gamma \pm \sqrt{1 + \frac{1}{4}\gamma^2}, 1\right)^T.$$

The maximum stretch is given by the positive root and the minimum stretch by the negative root.

c.) In general coordinates the angle between the deformed line elements is given by

$$\cos\Theta = \frac{n^i G_{ij} m^j}{\sqrt{n^i G_{ij} n^j}\sqrt{m^i G_{ij} m^j}};$$

from the lecture notes. Transforming to Cartesian coordinates gives

$$\cos\Theta = \frac{n_I c_{IJ} m_J}{\sqrt{n_I c_{IJ} n_J}\sqrt{m_I c_{IJ} c_J}},$$

and we choose $\boldsymbol{n} = (0, 1, 0)$ and $\boldsymbol{m} = (0, 0, 1)$ in the Cartesian coordinates, so

$$\cos\Theta = \frac{c_{23}}{\sqrt{c_{22}c_{33}}} = \frac{\gamma}{\sqrt{1+\gamma^2}}.$$

Hence the change in angle or shear between the two line elements is

$$\frac{\pi}{2} - \cos^{-1}\frac{\gamma}{\sqrt{1+\gamma^2}}.$$

As $\gamma \to 0$, $\cos\Theta \to 0$ and so there is no change in angle between the two line elements. As $\gamma \to \infty$, $\cos\Theta \to 1$ and the change in angle tends to $\pi/2$.

9.) a.) The entire bar is stretched by the amount λ_1 in the x direction so the initial deformation is given by $\tilde{X} = \lambda_1 x$. The second deformation is given by $X = \lambda_2 \tilde{X}$ and so the entire deformation is given by

$$X = \lambda_2 \lambda_1 x.$$

b.) The Green–Lagrange strain tensor associated with the first deformation is simply

$$e = \frac{1}{2}\left(\lambda_1^2 - 1\right);$$

the (incremental) Green–Lagrange strain tensor associated with the second deformation is

$$\tilde{e} = \frac{1}{2}\left(\lambda_2^2 - 1\right),$$

because we assume that the Lagrangian coordinate is $\lambda_1 x$. The Green–Lagrange strain tensor associated with the entire deformation is

$$\tilde{e}_{\text{Total}} = \frac{1}{2}\left(\lambda_1^2 \lambda_2^2 - 1\right).$$

There is no **simple** relationship between the three tensors.

c.) The Hencky strain tensor associated with the first deformation is

$$e_H = \ln\lambda_1.$$

The (incremental) Hencky strain tensor associated with the second deformation is

$$\tilde{e}_H = \ln\lambda_2.$$

The Hencky strain tensor associated with the total deformation is

$$e_{H\text{Total}} = \ln(\lambda_1 \lambda_2).$$

There is a simple relationship between the three tensors:

$$e_{H\text{Total}} = \ln(\lambda_1 \lambda_2) = \ln \lambda_1 + \ln \lambda_2 = e_H + \tilde{e}_H.$$

Thus, the sum of incremental Hencky strain tensors gives the total strain. For this reason it is sometimes called the "true" strain tensor and is often used in plasticity theory.

10.) a.) The deformation is given by

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1+\alpha t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Thus the Lagrangian velocity is given by

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -\omega\sin(\omega t) & \omega\cos(\omega t) & 0 \\ -\omega\cos(\omega t) & -\omega\sin(\omega t) & 0 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The Eulerian velocity is found by making use of the inverse deformation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & (1+\alpha t)^{-1} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

to write

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} -\omega\sin(\omega t) & \omega\cos(\omega t) & 0 \\ -\omega\cos(\omega t) & -\omega\sin(\omega t) & 0 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & (1+\alpha t)^{-1} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & \alpha/(1+\alpha t) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

Hence, the Eulerian velocity gradient tensor is given by

$$V_{I,J} = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & \alpha/(1+\alpha t) \end{pmatrix}.$$

b.) The rate of deformation tensor is the symmetric part of $V_{I,J}$ and the spin tensor is the antisymmetric part so

$$D_{IJ} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha/(1+\alpha t) \end{pmatrix}, \quad W_{IJ} = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The deformation consists of a rotation in the x_1-x_2 plane with angular velocity ω and a stretch in the x_3 direction that increases with time. The rotation is associated only with the spin tensor and the stretch only with the rate of deformation tensor.

11.) The Eulerian rate of strain tensor is related to the Eulerian rate of deformation tensor by the formula

$$\dot{E}_{IJ} = D_{IJ} - E_{IK}V_{K,J} - E_{JK}V_{K,I},$$

see the lecture notes. In one dimension we can only have

$$\dot{E}_{11} = D_{11} - E_{11}V_{1,1} - E_{11}V_{1,1}$$

and $D_{11} = V_{1,1}$ which means that

$$\dot{E}_{11} = V_{1,1}(1 - 2E_{11}) = D_{11}(1 - 2E_{11}).$$

Thus, if the rate of deformation tensor is zero, then the rate of strain tensor must also be zero. If the rate of deformation tensor is not zero then the rate of strain tensor is proportional to the rate of deformation, but will never equal it unless the strain is instantaneously zero.

12.) **a.)** The deformation is a stretch of α in the x_1 direction followed by rigid-body rotation about the origin of the Cartesian coordinate system.

b.) The Eulerian rate of deformation tensor can be found by using the same techniques as in q. 9. The Eulerian velocity will be given by

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} -\alpha\omega\sin(\omega t) & \omega\cos(\omega t) & 0 \\ -\alpha\omega\cos(\omega t) & -\omega\sin(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\omega t)/\alpha & -\sin(\omega t)/\alpha & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix},$$

which means that

$$V_{I,J} = W_{IJ} = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $D_{IJ} = 0$ and there is no rate of deformation because the body is simply rotating rigidly after the initial (instantaneous) stretch.

The spatial deformation gradient tensor is

$$H_{IJ} = \begin{pmatrix} \cos(\omega t)/\alpha & -\sin(\omega t)/\alpha & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which means that the Eulerian strain tensor is then

$$E_{IJ} = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \cos(\omega t)/\alpha & \sin(\omega t) & 0 \\ -\sin(\omega t)/\alpha & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\omega t)/\alpha & -\sin(\omega t)/\alpha & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$= \begin{pmatrix} \cos^2(\omega t)(1 - \alpha^{-2}) & \cos(\omega t)\sin(\omega t)(\alpha^{-2} - 1) & 0 \\ \cos(\omega t)\sin(\omega t)(\alpha^{-2} - 1) & \sin^2(\omega t)(1 - \alpha^{-2}) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that the strain tensor is a function of t because the strained body is rotating rigidly about the origin. When $t = n\pi/\omega$, the only non-zero component is $E_{11} = (1-\alpha^{-2})$, and when $t = n\pi/(2\omega)$, the only non-zero component is $E_{22} = (1-\alpha^{-2})$.

The rate of strain tensor is given by

$$\dot{E}_{IJ} = D_{IJ} - E_{KI}V_{K,J} - E_{JK}V_{K,I},$$

which gives as non-zero entries

$$\dot{E}_{11} = -2E_{21}V_{2,1} = 2\omega\cos(\omega t)\sin(\omega t)(\alpha^2-1),$$

$$\dot{E}_{12} = \dot{E}_{21} = -E_{11}V_{1,2} - E_{22}V_{2,1} = \omega(E_{22}-E_{11}) = \omega(1-\alpha^{-2})(\sin^2(\omega t) - \cos^2\omega t),$$

$$\dot{E}_{22} = -2E_{12}V_{1,2} = -2\omega\cos(\omega t)\sin(\omega t)(\alpha^2-1)$$

Thus we have

$$\dot{E}_{IJ} = \omega(\alpha^{-2}-1)\begin{pmatrix} \sin(2\omega t) & \cos(2\omega t) & 0 \\ \cos(2\omega t) & -\sin(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is only zero and therefore coincides with D_{IJ} when $\alpha^2 = 1 \Rightarrow \alpha = \pm 1$.

13.) For ease of notation, we drop the overbars in this solution so that in general coordinates ξ^i we can write the i-th component of the cross product as

$$[\boldsymbol{\omega} \times \boldsymbol{v}]_i = \epsilon_{ijk}\omega^j v^k.$$

From the definition of $\boldsymbol{\omega}$, we have that

$$\omega^j = \epsilon^{jmn}W_{nm},$$

and so

$$[\boldsymbol{\omega} \times \boldsymbol{v}]_i = \epsilon_{ijk}\epsilon^{jmn}W_{nm}v^k.$$

Now using the relationship from Example Sheet I, q 5:

$$\epsilon_{ijk}\epsilon^{jmn} = g_k^m g_i^n - g_k^n g_i^m,$$

and then

$$[\boldsymbol{\omega} \times \boldsymbol{v}]_i = (g_k^m g_i^n - g_k^n g_i^m)W_{nm}v^k = W_{ik}v^k - W_{ki}v^k = 2W_{ik}v^k,$$

using the antisymmetry property of the spin tensor. Thus

$$2\mathsf{W}\cdot\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{v}.$$

The Eulerian acceleration is given by

$$\boldsymbol{A} = \frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V} \cdot \boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{V}.$$

For simplicity let's work in Cartesian coordinates in which case we can write

$$A_I = \frac{\partial V_I}{\partial t} + V_J V_{I,J}.$$

Adding and subtracting the term $V_J V_{J,I}$ gives

$$\begin{aligned} A_I &= \frac{\partial V_I}{\partial t} + (V_{I,J} - V_{J,I})V_J + V_{J,I}V_J, \\ &= \frac{\partial V_I}{\partial t} + 2W_{IJ}V_J + \frac{1}{2}(V_J V_J)_{,I}, \end{aligned}$$

which can be written in dyadic notation as

$$\boldsymbol{A} = \frac{\partial \boldsymbol{V}}{\partial t} + 2\mathsf{W}\boldsymbol{\cdot}\boldsymbol{V} + \frac{1}{2}\boldsymbol{\nabla}_{\boldsymbol{R}}(\boldsymbol{V}\boldsymbol{\cdot}\boldsymbol{V});$$

and using the result above we finally obtain

$$\boldsymbol{A} = \frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{V} + \frac{1}{2}\boldsymbol{\nabla}_{\boldsymbol{R}}|\boldsymbol{V}|^2.$$

14.) This is essentially a direct application of the Stokes theorem

$$\Gamma = \int \boldsymbol{V}\boldsymbol{\cdot}\mathbf{d}\boldsymbol{R} = \iint \boldsymbol{\nabla} \times \boldsymbol{V}\boldsymbol{\cdot}\mathbf{d}\ \mathrm{A} = \iint \boldsymbol{\omega}\boldsymbol{\cdot}\mathbf{d}\ \mathrm{A}.$$

If we consider the limit as the area becomes very small then $\boldsymbol{\omega}$ is approximately constant over the area and we can assume that the normal is also constant so that we have

$$\Gamma \approx \boldsymbol{\omega}\boldsymbol{\cdot}\boldsymbol{n}\mathrm{d}A \quad \Rightarrow \quad \boldsymbol{\omega}\boldsymbol{\cdot}\boldsymbol{n} \approx \Gamma/\mathrm{d}A.$$

Thus, the ratio of the circulation around a closed loop to the area enclosed by the loop is equal to the component of vorticity normal to the surface in the limit as the area tends to zero.