MATH45061: SOLUTION SHEET[1] I

1.) The values a^I are the components of the vector $\boldsymbol{a}$ in the Cartesian coordinate system. We can write $a^I = \boldsymbol{a} \cdot \boldsymbol{e}_I$ from which we deduce that $a^1 = 1$, $a^2 = 2$ and $a^3 = 3$. Similarly, we find that $b_1 = 4$, $b_2 = 2$ and $b_3 = 1$. The scalar product is given by

$$a^I b_I = \sum_{I=1}^{3} a^I b_I = a^1 b_1 + a^2 b_2 + a^3 b_3 = 1 \times 4 + 2 \times 2 + 3 \times 1 = 4 + 4 + 3 = 11.$$

2.) If $\boldsymbol{r} = r^i \boldsymbol{g}_i$, then taking the dot product with the contravariant base vector gives

$$\boldsymbol{r} \cdot \boldsymbol{g}^j = r^i \boldsymbol{g}_i \cdot \boldsymbol{g}^j = r^i \delta_i^j = r^j,$$

and so $r^i = \boldsymbol{r} \cdot \boldsymbol{g}^i$, after changing the free index from j to i. We can write an explicit expression by using the Cartesian representations of the position vector $\boldsymbol{r} = x_K \boldsymbol{e}_K$ and contravariant base vector $\boldsymbol{g}^i = \partial \xi^i / \partial x_K \boldsymbol{e}_K$:

$$r^i = x_K \frac{\partial \xi^i}{\partial x_K}.$$

Thus, $r^i \neq \xi^i$ in general.

In a Cartesian coordinate system, $\xi^i = x^i$ and then

$$r^i = x_K \frac{\partial x^i}{\partial x_K} = x_K \delta_K^i = x^i,$$

so the i-th component of the position vector is equal to the i-th coordinate.

3.) The position vector is given by $\boldsymbol{r} = \xi^2 \xi^1 \boldsymbol{e}_1 + \xi^1 \boldsymbol{e}_2$, which can be written in component form in the global Cartesian basis as

$$\boldsymbol{r} = \begin{pmatrix} \xi^2 \xi^1 \\ \xi^1 \end{pmatrix}.$$

Thus covariant base vectors are therefore given by

$$\boldsymbol{g}_1 = \boldsymbol{r}_{,1} = \begin{pmatrix} \xi^2 \\ 1 \end{pmatrix}, \qquad \boldsymbol{g}_2 = \boldsymbol{r}_{,2} = \begin{pmatrix} \xi^1 \\ 0 \end{pmatrix}.$$

The contravariant base vectors are most easily found by using the inverse transpose of the covariant transformation matrix M, where

$$\begin{pmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \end{pmatrix} = \mathsf{M} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}.$$

In this case,

$$\mathsf{M} = \begin{pmatrix} \xi^2 & 1 \\ \xi^1 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathsf{M}^{-1} = -\frac{1}{\xi^1} \begin{pmatrix} 0 & -1 \\ -\xi^1 & \xi^2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\xi^1} \\ 1 & -\frac{\xi^2}{\xi^1} \end{pmatrix},$$

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$$\Rightarrow \mathsf{M}^{-T} = \begin{pmatrix} 0 & 1 \\ \frac{1}{\xi^1} & -\frac{\xi^2}{\xi^1} \end{pmatrix}.$$

The contravariant base vectors are given by

$$\begin{pmatrix} \boldsymbol{g}^1 \\ \boldsymbol{g}^2 \end{pmatrix} = \mathsf{M}^{-T} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{g}^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{g}^2 = \begin{pmatrix} \frac{1}{\xi^1} \\ -\frac{\xi^2}{\xi^1} \end{pmatrix}.$$

We can check our calculation by confirming that the equation $\boldsymbol{g}_i \cdot \boldsymbol{g}^j = \delta_i^j$ is satisfied (which it is). The coordinate system is not orthogonal because $\boldsymbol{g}_1 \cdot \boldsymbol{g}_2 \neq 0$.

4.) This is actually discussed in the online notes. In an orthonormal coordinate system the covariant and contravariant base vectors must coincide, so

$$\boldsymbol{g}_i = \boldsymbol{g}^i \quad \Rightarrow \quad \frac{\partial x^K}{\partial \xi^i} \boldsymbol{e}_K = \frac{\partial \xi^i}{\partial x^K} \boldsymbol{e}_K.$$

The global Cartesian base vectors $\boldsymbol{e}_K$ are orthogonal, which means that if coordinate system ξ^i is orthogonal then

$$\frac{\partial x^K}{\partial \xi^i} = \frac{\partial \xi^i}{\partial x^K}. \tag{1}$$

Multiplying equation (1) by $\partial \xi^i / \partial x^L$ gives

$$\frac{\partial x^K}{\partial \xi^i} \frac{\partial \xi^i}{\partial x^L} = \frac{\partial \xi^i}{\partial x^K} \frac{\partial \xi^i}{\partial x^L} \quad \Rightarrow \quad \frac{\partial x^K}{\partial x^L} = \frac{\partial \xi^i}{\partial x^K} \frac{\partial \xi^i}{\partial x^L},$$

after using the chain rule. The global Cartesian coordinates are independent and so $\partial x^K / \partial x^L = \delta_L^K$, so

$$\frac{\partial \xi^i}{\partial x^K} \frac{\partial \xi^i}{\partial x^L} = \delta_L^K. \tag{2}$$

If, as usual, we write the components of the contravariant transform as a matrix

$$\mathsf{Q} = \begin{pmatrix} \frac{\partial \xi^1}{\partial x^1} & \frac{\partial \xi^1}{\partial x^2} & \frac{\partial \xi^1}{\partial x^3} \\ \frac{\partial \xi^2}{\partial x^1} & \frac{\partial \xi^2}{\partial x^2} & \frac{\partial \xi^2}{\partial x^3} \\ \frac{\partial \xi^3}{\partial x^1} & \frac{\partial \xi^3}{\partial x^2} & \frac{\partial \xi^3}{\partial x^3} \end{pmatrix},$$

then equation (2) can be written in the form

$$\mathsf{Q}^T \mathsf{Q} = \mathsf{I},$$

where I is the identity matrix. Hence the contravariant transformation from the Cartesian base vectors is orthogonal and because it is identical to the covaraint transform that is also orthogonal. Hence, any orthonormal basis can only be obtained by orthogonal transformation from Cartesian base vectors.

5.) The easiest way to proceed is to consider what happens under a change in coordinates from ξ^i to $\chi^{\bar{i}}$. The components of the vectors transform contravariantly so that

$$a^{\bar{i}} = \frac{\partial \chi^{\bar{i}}}{\partial \xi^j} a^j \quad \Rightarrow \quad a^i = \frac{\partial \xi^i}{\partial \chi^{\bar{j}}} a^{\bar{j}},$$

which means that the equation $a^i = T(i,j)\, b^j$ becomes

$$\frac{\partial \xi^i}{\partial \chi^{\overline{k}}}\, a^{\overline{k}} = T(i,j)\, \frac{\partial \xi^j}{\partial \chi^{\overline{l}}}\, b^{\overline{l}} \quad \Rightarrow \quad a^{\overline{k}} = \frac{\partial \chi^{\overline{k}}}{\partial \xi^i}\, T(i,j)\, \frac{\partial \xi^j}{\partial \chi^{\overline{l}}}\, b^{\overline{l}}. \tag{3}$$

We write $T(\overline{i},\overline{j})$ for the coefficients in the transformed coordinates so that we can write

$$a^{\overline{k}} = T(\overline{k},\overline{l}) b^{\overline{l}},$$

and subtracting equation (3) gives

$$\left[T(\overline{k},\overline{l}) - \frac{\partial \chi^{\overline{k}}}{\partial \xi^i}\, T(i,j)\, \frac{\partial \xi^j}{\partial \chi^{\overline{l}}} \right] b^{\overline{l}} = 0.$$

Provided that the coefficients $T(i,j)$ are independent of $\boldsymbol{b}$, then

$$T(\overline{k},\overline{l}) = \frac{\partial \chi^{\overline{k}}}{\partial \xi^i}\, T(i,j)\, \frac{\partial \xi^j}{\partial \chi^{\overline{l}}};$$

and the coefficients $T(i,j)$ transform contravariantly in the first index and covariantly in the second. Hence, $T(i,j)$ can be written as T^i_j and are the components of a tensor in the basis $\boldsymbol{g}_i \otimes \boldsymbol{g}^j$.

This result (and its obvious generalisations to as many different indices as you like) is sometimes called the quotient rule and it can be used to establish whether not not an object is a tensor.

6.) The covariant metric tensor is $g_{ij} = \boldsymbol{g}^i \boldsymbol{\cdot} \boldsymbol{g}^j$, so can be written in matrix form as

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{g}_1 \boldsymbol{\cdot} \boldsymbol{g}_1 & \boldsymbol{g}_1 \boldsymbol{\cdot} \boldsymbol{g}_2 \\ \boldsymbol{g}_2 \boldsymbol{\cdot} \boldsymbol{g}_1 & \boldsymbol{g}_2 \boldsymbol{\cdot} \boldsymbol{g}_2 \end{pmatrix} = \begin{pmatrix} 1 + \xi^2\, \xi^2 & \xi^1\, \xi^2 \\ \xi^1\, \xi^2 & \xi^1\, \xi^1 \end{pmatrix}.$$

Its determinant is given by

$$g = \xi^1\, \xi^1 \left(1 + \xi^2\, \xi^2\right) - \xi^1\, \xi^1\, \xi^2\, \xi^2 = \xi^1\, \xi^1.$$

The contravariant metric tensor g^{ij} can be found by taking the inverse of the covariant metric tensor g_{ij} or from the relationships $g^{ij} = \boldsymbol{g}^i \boldsymbol{\cdot} \boldsymbol{g}^j$, either way it is

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\xi^2}{\xi^1} \\ -\frac{\xi^2}{\xi^1} & \frac{1+\xi^2\, \xi^2}{\xi^1\, \xi^1} \end{pmatrix}.$$

Its determinant is given by

$$\frac{1 + \xi^2\, \xi^2}{\xi^1\, \xi^1} - \frac{\xi^2\, \xi^2}{\xi^1\, \xi^1} = \frac{1}{\xi^1\, \xi^1} = \frac{1}{g}.$$

For the verification of $\boldsymbol{g}^i = g^{ij} \boldsymbol{g}_j$, we shall consider the two cases $i = 1$ and $i = 2$ separately:

$$\boldsymbol{g}^1 = g^{11} \boldsymbol{g}_1 + g^{12} \boldsymbol{g}_2 = 1(\xi^2\, \boldsymbol{e}_1 + \boldsymbol{e}_2) - \frac{\xi^2}{\xi^1}(\xi^1\, \boldsymbol{e}_1) = (\xi^2 - \xi^2)\boldsymbol{e}_1 + \boldsymbol{e}_2 = \boldsymbol{e}_2,$$

$$\boldsymbol{g}^2 = g^{21} \boldsymbol{g}_1 + g^{22} \boldsymbol{g}_2 = -\frac{\xi^2}{\xi^1}(\xi^2\, \boldsymbol{e}_1 + \boldsymbol{e}_2) - \frac{1 + \xi^2\, \xi^2}{\xi^1\, \xi^1}(\xi^1\, \boldsymbol{e}_1) = -\frac{1}{\xi^1}\boldsymbol{e}_1 - \frac{\xi^2}{\xi^1}\boldsymbol{e}_2,$$

in agreement with question 3.

7.) The easiest approach here is to establish the identity in Cartesian coordinates and then because it is a tensor identity is will be true in any coordinate system. In Cartesian coordinates, we have

$$e^{IJK}e_{KLM} = \delta^I_L\delta^J_M - \delta^I_M\delta^J_L.$$

We first observe that swapping I and J or L and M will change the sign of LHS and also changes the sign of the RHS, as it should. If I and J and L and M are swapped simultaneously then both LHS and RHS are unchanged. If $I = J$ and/or $L = M$, then both LHS and RHS are zero. We are therefore left with the nine cases $IJ, LM = 12, 13, 23$.

If $IJ = LM$ ($I \neq J$), then the LHS is 1 and so is the RHS. If we swap IJ and LM, then the RHS is unchanged by symmetry of the Kronecker delta. Hence, we are left with three cases to check

$$IJ = 12, LM = 13, 23$$

and $IJ = 13, LM = 23$. In the first two cases, the LHS must be zero, because from e^{12K}, the only non-zero value occurs when $K = 3$, but then $e_{KLM} = 0$; and the RHS is also zero. In the final case, $K = 2$, and so $e_{KLM} = 0$ and the LHS is zero, as it the RHS. Thus, we have established the identity in Cartesian coordinates and the general result follow from coordinate transform.

The contravariant components of the double vector product can be written in index notation as

$$\epsilon^{ijk}a_j\epsilon_{klm}b^lc^m = \epsilon^{ijk}\epsilon_{klm}a_jb^lc^m = \left(g^i_l\, g^j_m - g^i_m\, g^j_l\right) a_jb^lc^m = b^ia_mc^m - c^ia_lb^l,$$

which gives

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}.$$

8.) The Christoffel symbol is defined by

$$\Gamma_{ijk} = \frac{\partial^2 \boldsymbol{r}}{\partial\xi^i\partial\xi^j}\boldsymbol{\cdot}\frac{\partial \boldsymbol{r}}{\partial\xi^k}.$$

Under a change in coordinates from ξ^i to $\chi^{\bar{i}}$,

$$\Gamma_{\bar{i}\bar{j}\bar{k}} = \frac{\partial^2 \boldsymbol{r}}{\partial\chi^{\bar{i}}\partial\chi^{\bar{j}}}\boldsymbol{\cdot}\frac{\partial \boldsymbol{r}}{\partial\chi^{\bar{k}}} = \frac{\partial}{\partial\chi^{\bar{i}}}\left(\frac{\partial x^K}{\partial\chi^{\bar{j}}}\right)\frac{\partial x^K}{\partial\chi^{\bar{k}}}.$$

Now using the chain rule, we have

$$\begin{aligned}
\Gamma_{\bar{i}\bar{j}\bar{k}} &= \frac{\partial\xi^l}{\partial\chi^{\bar{i}}}\frac{\partial}{\partial\xi^l}\left(\frac{\partial\xi^m}{\partial\chi^{\bar{j}}}\frac{\partial x^K}{\partial\xi^m}\right)\frac{\partial x^K}{\partial\xi^n}\frac{\partial\xi^n}{\partial\chi^{\bar{k}}} = \frac{\partial\xi^l}{\partial\chi^{\bar{i}}}\frac{\partial\xi^n}{\partial\chi^{\bar{k}}}\frac{\partial}{\partial\xi^l}\left(\frac{\partial\xi^m}{\partial\chi^{\bar{j}}}\frac{\partial x^K}{\partial\xi^m}\right)\frac{\partial x^K}{\partial\xi^n}, \\
&= \frac{\partial\xi^l}{\partial\chi^{\bar{i}}}\frac{\partial\xi^n}{\partial\chi^{\bar{k}}}\left(\frac{\partial}{\partial\xi^l}\left(\frac{\partial\xi^m}{\partial\chi^{\bar{j}}}\right)\frac{\partial x^K}{\partial\xi^m} + \frac{\partial\xi^m}{\partial\chi^{\bar{j}}}\frac{\partial^2 x^K}{\partial\xi^l\partial\xi^m}\right)\frac{\partial x^K}{\partial\xi^n}, \\
&= \left(\frac{\partial\xi^l}{\partial\chi^{\bar{i}}}\frac{\partial\xi^m}{\partial\chi^{\bar{j}}}\frac{\partial\xi^n}{\partial\chi^{\bar{k}}}\right)\frac{\partial^2 x^K}{\partial\xi^l\partial\xi^m}\frac{\partial x^K}{\partial\xi^n} + \frac{\partial}{\partial\chi^{\bar{i}}}\left(\frac{\partial\xi^m}{\partial\chi^{\bar{j}}}\right)\frac{\partial\xi^n}{\partial\chi^{\bar{k}}}\frac{\partial x^K}{\partial\xi^m}\frac{\partial x^K}{\partial\xi^n},
\end{aligned}$$

$$= \left(\frac{\partial \xi^l}{\partial \chi^{\bar{i}}}\frac{\partial \xi^m}{\partial \chi^{\bar{j}}}\frac{\partial \xi^n}{\partial \chi^{\bar{k}}}\right)\Gamma_{lmn} + \frac{\partial^2 \xi^m}{\partial \chi^{\bar{i}}\partial \chi^{\bar{j}}}\frac{\partial \xi^n}{\partial \chi^{\bar{k}}}g_{mn}.$$

The first term corresponds to the tensor transformation rule, but the presence of the second term means that the Christoffel symbol does not in general transform as a tensor. Therefore, the Christoffel symbol is not a tensor.

9.) The covariant derivative of the scalar ϕ coincides with the partial derivative, so $\phi|_i = \phi_{,i}$. Now the combination

$$a_j|_i b^j + a_j b^j|_i = (a_{j,i} - \Gamma^k_{ij}a_k)b^j + a_j(b^j_{,i} + \Gamma^j_{ki}b^k) = a_{j,i}b^j + a_j b^j_{,i} = (a_j b^j)_{,i},$$

because the partial derivative does obey the product rule and the terms involving Christoffel symbols are the same (one dummy indices are interchanged). Thus,

$$a_j|_i b^j + a_j b^j|_i = (a_j b^j)_{,i} = \phi_{,i} = \phi|_i = (a_j b^j)|_i,$$

and the covariant derivative satisfies the product rule.

10.) The quantity $\alpha = A_{ij}b^i c^j$ is a scalar invariant, so $\alpha_{,r}$ is covariant. Then

$$\alpha_{,r} = A_{ij,r}b^i c^j + A_{ij}b^i_{,r}c^j + A_{ij}b^i c^j_{,r}.$$

From the definition of the covariant derivative for vectors

$$b^i_{,r} = b^i|_r - \Gamma^i_{kr}b^k \quad \text{and} \quad c^j_{,r} = c^j|_r - \Gamma^j_{kj}c^k,$$

so

$$\begin{aligned}\alpha_{,r} &= A_{ij,r}b^i c^j + A_{ij}c^j\left(b^i|_r - \Gamma^i_{kr}b^k\right) + A_{ij}b^i\left(c^j|_r - \Gamma^j_{lr}c^l\right).\\ &= A_{ij}c^j b^i|_r + A_{ij}b^i c^j|_r + \left(A_{kl,r} - A_{il}\Gamma^i_{kr} - A_{kj}\Gamma^j_{lr}\right)b^k c^l\end{aligned}$$

All three terms must be covariant tensors of first order and we can define the third-order covariant tensor

$$A_{ij}|_r = A_{ij,r} - A_{kj}\Gamma^k_{ir} - A_{ik}\Gamma^k_{jr},$$

so that

$$(A_{ij}b^i c^j)|_r = A_{ij}|_r b^i c^j + A_{ij}bi|_r c^j + A_{ij}b^i c^j|_r,$$

and the product rule of covariant differentiation is satisfied.

11.) This is a worthwhile exercise, but it's a bit tedious (sorry). Let's start with cylindrical polars:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Let us define $\xi^1 = r$, $\xi^2 = \theta$, $\xi^3 = z$ and then we have that the covariant base vectors in components in the global Cartesian basis are

$$\boldsymbol{g}_1 = \begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix}, \quad \boldsymbol{g}_2 = \begin{pmatrix}-r\sin\theta\\ r\cos\theta\\ 0\end{pmatrix}, \quad \boldsymbol{g}_3 = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}.$$

The covariant metric tensor is given by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{with determinant} \quad g = r^2,$$

and we note that the basis is orthogonal, but not orthonormal because the distance associated with a change in angle varies as a function of distance from the origin. The contravariant metric tensor is then simply

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the contravariant base vectors are

$$\boldsymbol{g}^1 = \boldsymbol{g}_1, \quad \boldsymbol{g}^2 = \frac{1}{r^2}\boldsymbol{g}_2, \quad \boldsymbol{g}^3 = \boldsymbol{g}_3.$$

Hence, the gradient of a scalar field f is

$$\nabla f = \boldsymbol{g}^i f_{,i} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \frac{\partial f}{\partial r} + \begin{pmatrix} -\frac{\sin\theta}{r} \\ \frac{\cos\theta}{r} \\ 0 \end{pmatrix} \frac{\partial f}{\partial \theta} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \frac{\partial f}{\partial z}.$$

If we convert the vectors to unit vectors in the coordinate directions, we obtain the standard expression

$$\nabla f = \boldsymbol{e}_r \frac{\partial f}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \boldsymbol{e}_z \frac{\partial f}{\partial z}.$$

The Laplacian of a scalar field is given by

$$\nabla^2 f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^j}\left(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial \xi^i}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(r\frac{1}{r^2}\frac{\partial f}{\partial \theta}\right) + \frac{1}{r}\frac{\partial}{\partial z}\left(r\frac{\partial f}{\partial z}\right),$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}.$$

The divergence of a vector $\boldsymbol{F}$ is given by

$$\mathrm{div}\boldsymbol{F} = F^i|_i = F^i_{,i} + \Gamma^i_{ik} F^k,$$

which means that we need to calculate the Christoffel symbols.

$$\Gamma^i_{jk} = \boldsymbol{g}_{j,k} \cdot \boldsymbol{g}^i.$$

We already know the base vectors so we can determine that the only non-zero derivatives are

$$\boldsymbol{g}_{1,2} = \boldsymbol{g}_{2,1} = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{g}_{2,2} = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \\ 0 \end{pmatrix},$$

which means that the only non-zero Christoffel symbols are

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \Gamma^1_{22} = -r.$$

Thus,

$$\mathrm{div}\boldsymbol{F} = F^1_{,1} + F^2_{,2} + F^3_{,3} + \frac{1}{r}F^1,$$

and because

$$\boldsymbol{F} = F^1\boldsymbol{g}_1 + F^2\boldsymbol{g}_2 + F^3\boldsymbol{g}_3 = F^1\boldsymbol{e}_r + F^2 r\boldsymbol{e}_\theta + F^3\boldsymbol{e}_z,$$

so $F^1 = F_r$, $F^2 = \frac{1}{r}F_\theta$ and $F^3 = F_z$, where F_r, F_θ and F_z are the components of the vector in the directions of the standard unit base vectors. Hence,

$$\mathrm{div}\boldsymbol{F} = \frac{\partial F_r}{\partial r} + \frac{1}{r}F_r + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}.$$

Turning to spherical polars we have

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

Let us define $\xi^1 = r$, $\xi^2 = \theta$, $\xi^3 = \phi$ and then the covariant base vectors in components in the global Cartesian basis are

$$\boldsymbol{g}_1 = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \quad \boldsymbol{g}_2 = \begin{pmatrix} r\cos\theta\cos\phi \\ r\cos\theta\sin\phi \\ -r\sin\theta \end{pmatrix}, \quad \boldsymbol{g}_3 = \begin{pmatrix} -r\sin\theta\sin\phi \\ r\sin\theta\cos\phi \\ 0 \end{pmatrix}.$$

The covariant metric tensor is given by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \quad \text{with determinant} \quad g = r^4\sin^2\theta,$$

and we note that the basis is again orthogonal, but not orthonormal. The contravariant metric tensor is then simply

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix},$$

and the contravariant base vectors are

$$\boldsymbol{g}^1 = \boldsymbol{g}_1, \quad \boldsymbol{g}^2 = \frac{1}{r^2}\boldsymbol{g}_2, \quad \boldsymbol{g}^3 = \frac{1}{r^2\sin^2\theta}\boldsymbol{g}_3.$$

Hence, the gradient of a scalar field is

$$\nabla f = \boldsymbol{g}^i f_{,i} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}\frac{\partial f}{\partial r} + \begin{pmatrix} \frac{\cos\theta\cos\phi}{r} \\ \frac{\cos\theta\sin\phi}{r} \\ -\frac{\sin\theta}{r} \end{pmatrix}\frac{\partial f}{\partial \theta} + \begin{pmatrix} -\frac{\sin\phi}{r\sin\theta} \\ \frac{\cos\phi}{r\sin\theta} \\ 0 \end{pmatrix}\frac{\partial f}{\partial \phi}.$$

If we convert the vectors to unit vectors in the coordinate directions, we obtain the standard expression

$$\nabla f = \boldsymbol{e}_r \frac{\partial f}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \boldsymbol{e}_\phi \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}.$$

The Laplacian of a scalar field is given by

$$\nabla^2 f = \frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^j}\left(\sqrt{g}\, g^{ij}\frac{\partial f}{\partial \xi^i}\right) = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(r^2\sin\theta\frac{1}{r^2}\frac{\partial f}{\partial \theta}\right)$$

$$+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \phi}\left(r^2\sin\theta\frac{1}{r^2\sin^2\theta}\frac{\partial f}{\partial \phi}\right),$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}.$$

For the Christoffel symbols, the only non-zero derivatives are

$$\boldsymbol{g}_{1,2} = \boldsymbol{g}_{2,1} = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \quad \boldsymbol{g}_{1,3} = \boldsymbol{g}_{3,1} = \begin{pmatrix} -\sin\theta\sin\phi \\ \sin\theta\cos\phi \\ 0 \end{pmatrix}$$

$$\boldsymbol{g}_{2,3} = \boldsymbol{g}_{3,2} = \begin{pmatrix} -r\cos\theta\sin\phi \\ r\cos\theta\cos\phi \\ 0 \end{pmatrix}, \quad \boldsymbol{g}_{2,2} = \begin{pmatrix} -r\sin\theta\cos\phi \\ -r\sin\theta\sin\phi \\ -r\cos\theta \end{pmatrix}$$

$$\boldsymbol{g}_{3,3} = \begin{pmatrix} -r\sin\theta\cos\phi \\ -r\sin\theta\sin\phi \\ 0 \end{pmatrix},$$

which means that the only non-zero Christoffel symbols are

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \frac{\cos\theta}{\sin\theta},$$

$$\Gamma^1_{22} = -r, \quad \Gamma^1_{33} = -r\sin^2\theta, \quad \Gamma^2_{33} = -\cos\theta\sin\theta.$$

Thus,

$$\mathrm{div}\boldsymbol{F} = F^1_{,1} + F^2_{,2} + F^3_{,3} + \frac{2}{r}F^1 + \frac{\cos\theta}{\sin\theta}F^2,$$

and because

$$\boldsymbol{F} = F^1\boldsymbol{g}_1 + F^2\boldsymbol{g}_2 + F^3\boldsymbol{g}_3 = F^1\boldsymbol{e}_r + F^2 r\boldsymbol{e}_\theta + F^3 r\sin\theta\boldsymbol{e}_\phi,$$

so $F^1 = F_r$, $F^2 = \frac{1}{r}F_\theta$ and $F^3 = \frac{1}{r\sin\theta}F_\phi$, where F_r, F_θ and F_ϕ are the components of the vector in the directions of the standard unit base vectors. Hence,

$$\mathrm{div}\boldsymbol{F} = \frac{\partial F_r}{\partial r} + \frac{2}{r}F_r + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{1}{r}\frac{\cos\theta}{\sin\theta}F_\theta + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta F_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}.$$

12.) The curve is given by $\boldsymbol{r}(t)$ and if a vector field $\boldsymbol{a}(\boldsymbol{r})$ is to remain parallel, then the vector $\boldsymbol{a}(\boldsymbol{r}(t))$ must not change as it moves along the curve. In other words, the vector must not change as t varies, so that its rate of change with respect to t is zero, *i.e.*

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = 0.$$

If we have a general coordinate system ξ^i, then we can write $\boldsymbol{a}(\xi(t))$ and so

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = \frac{\mathrm{d}\xi^i}{\mathrm{d}t}\frac{\partial \boldsymbol{a}}{\partial \xi^i} = \boldsymbol{a}_{,i}\frac{\mathrm{d}\xi^i}{\mathrm{d}t} = a^j|_i \boldsymbol{g}_j \frac{\mathrm{d}\xi^i}{\mathrm{d}t} = 0.$$

Now the vectors $\boldsymbol{g}_i$ are not zero because they form a basis of the space (unless something very degenerate is going on). In addition for any non-trivial curve $\mathrm{d}\xi^i/\mathrm{d}t \neq 0$. Therefore the condition for the vector field to be parallel is that

$$a^j|_i = 0, \quad \text{for all } i,\ j.$$