PC. Right.

M. 11am lecture.

W. 9am lecture
10 am lecture/tutorial

Assessment

2 × coursework 10%, 15%
1 × 3 hour exam 75%

Office hours 12:00pm Monday
11:00am Wednesday

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∇f independent of coordinate system, compact, but not explicit. (concise)

MATH45061: Continuum Mechanics

Study of matter on length scales where it can be regarded as continuous, i.e. represented by continuous fields.

Key concept: behaviour should not depend on coordinate system.

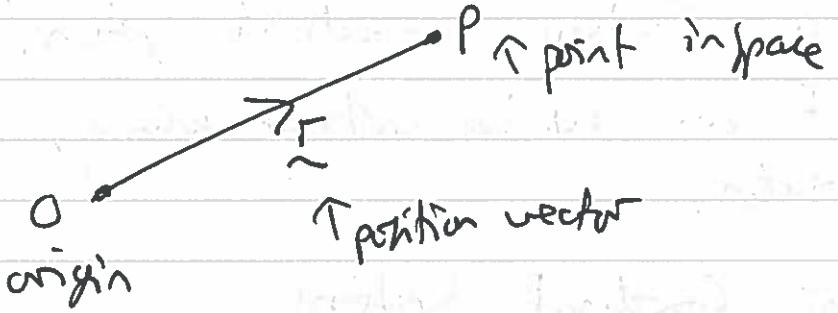
We can use continuum mechanics to study fluids, solids and all "in between".

We build on conservation of mass, energy and balance of linear and angular momentum and a thermodynamic inequality.

Chapter 1 Describing the world

Vectors

Space is 3D & Euclidean (flat)

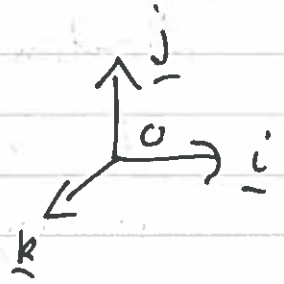


The vector is independent of coordinate system, to describe it we do have to pick a coordinate system and determine the representation of the vector in it.

Cartesian

In Euclidean space, we can always choose a reference Cartesian coordinate system with fixed, orthonormal base vector:

$$\underset{\sim}{e}_1 = \underset{\sim}{i}, \quad \underset{\sim}{e}_2 = \underset{\sim}{j}, \quad \underset{\sim}{e}_3 = \underset{\sim}{k}$$



We can write any vector

$$\underset{\sim}{a} = a^1 \underset{\sim}{e}_1 + a^2 \underset{\sim}{e}_2 + a^3 \underset{\sim}{e}_3 = \sum_{I=1}^{3} a^I \underset{\sim}{e}_I$$

components (scalars)

base vectors

Use summation convention : "drop the Σ"

So $\underset{\sim}{a} = a^I \underset{\sim}{e}_I = a^J \underset{\sim}{e}_J = a^K \underset{\sim}{e}_K$

Sum over a repeated index; one raised, one lowered

For clarity, a capital Roman index always means a Cartesian coordinate system!

Note a^I are useless without knowing the base vectors.

Curvilinear Coordinate Systems

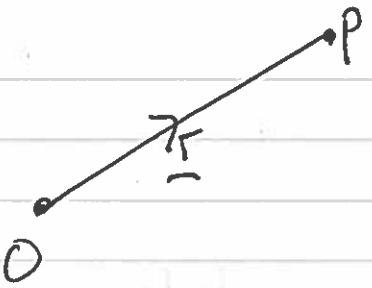
Position vector $\underset{\sim}{r} = r^K \underset{\sim}{e}_K$ or $x^K \underset{\sim}{e}_K = \underset{\sim}{x} = \underset{\sim}{r}$

($\underset{\sim}{r}$ and $\underset{\sim}{x}$ are interchangeable)

So $$\frac{\partial \underset{\sim}{r}}{\partial x^K} = \underset{\sim}{e}_K$$

i.e. $\underset{\sim}{x} = (x, y, z)$ $\quad \frac{\partial \underset{\sim}{r}}{\partial x} = (1, 0, 0)$

Derivative of position vector in coordinate direction gives a vector tangent to the coordinate direction — this is true in any coordinate system.



$$\frac{\partial \underline{r}}{\partial x^k} = \underset{\sim}{e_k}$$

In a general coordinate system ξ^i

$$\underline{r}(\xi^1, \xi^2, \xi^3) = x^k(\xi^i)\, \underset{\sim}{e_k}$$

e.g. cylindrical polars $\xi^1 = r$, $\xi^2 = \theta$, $\xi^3 = z$

$$x^1 = x = r\cos\theta = \xi^1 \cos\xi^2, \quad x^2 = y = r\sin\theta = \xi^1 \sin\xi^2$$

$$x^3 = z = \xi^3$$

A target vector to the ξ^i direction is given by

$$\underset{\sim}{g_i} = \frac{\partial \underline{r}}{\partial \xi^i}$$

e.g. plane polars $\underline{r} = \begin{pmatrix} \xi^1 \cos\xi^2 \\ \xi^1 \sin\xi^2 \\ \xi^3 \end{pmatrix}$

$$\underset{\sim}{g_1} = \frac{\partial \underline{r}}{\partial \xi^1} = \begin{pmatrix} \cos\xi^2 \\ \sin\xi^2 \\ 0 \end{pmatrix}, \quad \underset{\sim}{g_2} = \frac{\partial \underline{r}}{\partial \xi^2} = \begin{pmatrix} -\xi^1 \sin\xi^2 \\ \xi^1 \cos\xi^2 \\ 0 \end{pmatrix}$$

$$\underset{\sim}{g_3} = \frac{\partial \underline{r}}{\partial \xi^3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Systematic — no guarantee of orthonormality

$$\underset{\sim}{g}_i = \frac{\partial \underset{\sim}{r}}{\partial \xi^i} = \frac{\partial}{\partial \xi^i}\left(x^k \underset{\sim}{e}_k\right)$$

$$= \frac{\partial x^k}{\partial \xi^i} \underset{\sim}{e}_k$$

This is a local linear transform between Cartesian base vectors and the tangent vectors.

$$\begin{pmatrix} \underset{\sim}{g}_1 \\ \underset{\sim}{g}_2 \\ \underset{\sim}{g}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial x^1}{\partial \xi^1} & \frac{\partial x^2}{\partial \xi^1} & \frac{\partial x^3}{\partial \xi^1} \\ \frac{\partial x^1}{\partial \xi^2} & \cdots & \\ \cdots & & \end{pmatrix}}_{M} \begin{pmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{pmatrix}$$

If $\det M \neq 0$ $\underset{\sim}{g}_i$ do form a basis of the Euclidean space.

The $\underset{\sim}{g}_i$ are called covariant base vectors.

1.1.4. Contravariant base vectors

In cartesians $\underset{\sim}{a} \cdot \underset{\sim}{e}_I = a^k \underset{\sim}{e}_k \cdot \underset{\sim}{e}_I = a^I$ (Orthonormal)

In covariant basis

$$\underset{\sim}{a} \cdot \underset{\sim}{g}_i = (a^k \underset{\sim}{g}_k) \cdot \underset{\sim}{g}_i = a^k \underbrace{\underset{\sim}{g}_k \cdot \underset{\sim}{g}_i}$$

Cannot simplify.

We define another set of base vectors $\underset{\sim}{g}^i$ such that $\underset{\sim}{g}^i \cdot \underset{\sim}{g}_j = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ otherwise} \end{cases} \equiv \delta^i_j$

δ^i_j is the Kronecker delta.

$$a^i = \delta^i_j a^j$$ (Prove by writing out components)

In orthonormal systems $\underset{\sim}{g}^i = \underset{\sim}{g}_i$ e.g. $\underset{\sim}{e}_I = \underset{\sim}{e}^I$

It can be shown (online notes) that

$$\underset{\sim}{g}^i = \underbrace{\frac{\partial \xi^i}{\partial x_K}}_{M^{-T}} \underset{\sim}{e}_K$$

$\underset{\sim}{g}^i$ are called contravariant base vectors.

e.g. polars

$$\underset{\sim}{g}_1 = \begin{pmatrix} \cos\xi^2 \\ \sin\xi^2 \\ 0 \end{pmatrix}, \quad \underset{\sim}{g}_2 = \begin{pmatrix} -\xi^1 \sin\xi^2 \\ \xi^1 \cos\xi^2 \\ 0 \end{pmatrix}$$

$$\underset{\sim}{g}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

What are $\underset{\sim}{g}^i$? $\underset{\sim}{g}_1 \cdot \underset{\sim}{g}_2 = 0$

System is orthogonal $\underset{\sim}{g}_2 \cdot \underset{\sim}{g}_3 = 0$, $\underset{\sim}{g}_1 \cdot \underset{\sim}{g}_3 = 0$

and $|\underset{\sim}{g}_1|^2 = 1$, $|\underset{\sim}{g}_3|^2 = 1$, $|\underset{\sim}{g}_2|^2 = \underset{\sim}{g}_2 \cdot \underset{\sim}{g}_2 = (\xi^1)^2$

Hence $\underset{\sim}{g}_1 = \underset{\sim}{g}^1$, $\underset{\sim}{g}_3 = \underset{\sim}{g}^3$ and $\underset{\sim}{g}^2 = \dfrac{\underset{\sim}{g}_2}{(\xi^1)^2}$

Then $\underset{\sim}{g}_i \cdot \underset{\sim}{g}^j = \delta_i^{\ j}$ as required.

1.1.5 Components in co/contravariant bases

We can write $\underset{\sim}{a} = a^k \underset{\sim}{g}_k = a_j \underset{\sim}{g}^j$

$$\underset{\sim}{a} \cdot \underset{\sim}{g}^i = (a^k \underset{\sim}{g}_k) \cdot \underset{\sim}{g}^i = a^k \delta_k{}^i = a^i$$

$$\underset{\sim}{a} \cdot \underset{\sim}{g}_i = a_i$$

1.1.6. Significance of raised and lowered indices

An object with a lowered (raised) index transforms covariantly (contravariantly).

(on line notes) $a_i = \underbrace{\frac{\partial x^k}{\partial \xi^i}}_{M} a_k$ is covariant

$$a^i = \frac{\partial \xi^i}{\partial x^k} a_k \quad \text{is contravariant.}$$

These properties follow directly from the invariance of vectors:

$$\underset{\sim}{a} = a_i \underset{\sim}{g}^i = a^i \underset{\sim}{g}_i$$

must be invariant under coordinate transformations

We change from ξ^i to $x^{\bar{j}}$

$$\underset{\sim}{g}_{\bar{i}} = \frac{\partial x^k}{\partial x^{\bar{i}}} \underset{\sim}{e}_k = \underbrace{\frac{\partial \xi^j}{\partial x^{\bar{i}}} \boxed{\frac{\partial x^k}{\partial \xi^j}} \underset{\sim}{e}_k}_{\text{chain rule}}$$

$$\underset{\sim}{g}_{\bar{i}} := \frac{\partial \xi^j}{\partial x^{\bar{i}}} \underset{\sim}{g}_j$$

↑ new coordinate is down so this is covariant

Thus,

$$\underset{\sim}{a} = a^i \underset{\sim}{g}_i = a^{\bar{i}} \underset{\sim}{g}_{\bar{i}} = a^{\bar{i}} \frac{\partial \xi^j}{\partial x^{\bar{i}}} \underset{\sim}{g}_j$$

$$\Rightarrow a^i \delta^j_i \underset{\sim}{g}_j = a^{\bar{i}} \frac{\partial \xi^j}{\partial x^{\bar{i}}} \underset{\sim}{g}_j$$

$$\Rightarrow a^i \delta^j_i = a^{\bar{i}} \frac{\partial \xi^j}{\partial x^{\bar{i}}} = a^j$$

Hence,

$$a^{\bar{i}} = \overset{\curvearrowright}{\frac{\partial x^{\bar{i}}}{\partial \xi^j}} a^j$$

↑ contravariant transform

new coordinate is up

The sum convention ensures summed quantities remain invariant.

1.1.7 Orthonormal Coordinates

The covariant and contravariant bases and transforms are the <u>same</u>. All indices can be down. This simplification is adopted in <u>many</u> textbooks. We can do this for capital indices $a^{\Pi} = a_{\Pi}$.

1.2 Tensors

A tensor is a geometric object that does not depend on coordinate system. Scalars and vectors are examples of tensors, but the concept is more general! They are linear in nature.

Consider a linear map between vectors

$$\underset{\sim}{a} = M(\underset{\sim}{b})$$

$\underset{\sim}{a}$ and $\underset{\sim}{b}$ are invariant under coordinate transform $\Rightarrow$ map is invariant as well

It is called a second-order (or second-rank) tensor.

In order to describe M precisely we need two coordinate systems: one for $\underset{\sim}{a}$, one for $\underset{\sim}{b}$

In Cartesian: $a_I \underset{\sim}{e}_I = M(b_J \underset{\sim}{e}_J)$

$= b_J M(\underset{\sim}{e}_J)$ by linearity

Here,

$$a_I \underbrace{\underset{\sim}{e}_I \cdot \underset{\sim}{e}_K}_{\delta_{IK}} = b_J M(\underset{\sim}{e}_J) \cdot \underset{\sim}{e}_K$$

$$\Rightarrow a_K = \underbrace{\underset{\sim}{e}_K \cdot M(\underset{\sim}{e}_J)}_{M_{KJ}} b_J \Rightarrow a_K = M_{KJ} b_J$$

M_{KJ} are components of the map in the Cartesian basis.

Need to know both coordinate systems for this to make sense!

If we now work in general coordinates,

$$a^i \underset{\sim}{g}_i = M(b_k \underset{\sim}{g}^k)$$

$$\Rightarrow a^i \underset{\sim}{g}_i \cdot \underset{\sim}{g}^j = b_k \underset{\sim}{g}^j \cdot M(\underset{\sim}{g}^k)$$

$$\Rightarrow a^j = b_k M^{jk}$$

Now $M^{jk} = \underset{\sim}{g}^j \cdot M(\underset{\sim}{g}^k)$

$$= \frac{\partial \xi^j}{\partial x_\pi} \underset{\sim}{e}^\pi \cdot M\left(\frac{\partial \Sigma^k}{\partial x^\lambda} \underset{\sim}{e}^\lambda\right)$$

$$= \frac{\partial \xi^j}{\partial x_\pi} \frac{\partial \Sigma^k}{\partial x_\lambda} \underbrace{\left[\underset{\sim}{e}^\pi \cdot M(\underset{\sim}{e}^\lambda)\right]}_{M^{\pi\lambda}}$$

↑ ↑ two contravariant transformations

a tensor is contravariant if both indices obey a contravariant transform.

- The order of a tensor is number of indices (coordinate systems needed to describe it)

Tensor of order 0: scalar. a

1: vector $a^i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $a_i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

2: $a^{ij} \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $b_{ij} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ $a^i{}_j \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

↑ contra ↑ co ↑ mixed

1.2.1 Cartesian tensors

Here if we change coordinates the rule is

$x_{\bar{I}}$ & x_K are two Cartesian coordinate systems

$$a_{\bar{I}} = \frac{\partial x_K}{\partial x^{\bar{I}}} a_K = \frac{\partial x^{\bar{I}}}{\partial x^K} a_K$$

$$= Q_{\bar{I}K} a_K$$

↑ orthogonal matrix

then $M_{\bar{I}\bar{J}} = Q_{\bar{I}K} M_{KL} Q_{\bar{J}L}$

or $\bar{M} = QMQ^T$ in matrix notation

In many textbooks this transformation defines a (Cartesian) tensor.

$$\underset{\sim}{g}_i = \frac{\partial \underset{\sim}{r}}{\partial \xi^i} \qquad \underset{\sim}{g}_i \cdot \underset{\sim}{g}^j = \delta_i^{\ j} = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$$

Two types of transformation: covariant $\frac{\partial x_I}{\partial \xi^i}$

contravariant $\frac{\partial \xi^i}{\partial x_I}$

1.4 The metric tensor

In Cartesians

$$\underset{\sim}{e}_I \cdot \underset{\sim}{e}_J = \delta_{IJ} = \delta^{IJ} = \delta^I_{\ J} = \begin{cases} 1, & I=J \\ 0, & \text{otherwise} \end{cases}$$

and transforming to general coordinates gives the metric tensor:

$$g_{ij} = \frac{\partial x^I}{\partial \xi^i}\frac{\partial x^J}{\partial \xi^j}\delta_{IJ} = \frac{\partial x^I}{\partial \xi^i}\frac{\partial x^I}{\partial \xi^j} = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_j$$

and $g^{ij} = \underset{\sim}{g}^i \cdot \underset{\sim}{g}^j$ and $g^i_{\ j} = \underset{\sim}{g}^i \cdot \underset{\sim}{g}_j = \delta^i_{\ j}$ ✓

invariant in all coordinate systems.

The tensors are symmetric $g_{ij} = g_{ji}$

and $g^{ij} g_{jk} = \delta^i_{\ k} \Rightarrow g^{ij}$ is the inverse of g_{jk}

Index raising and lowering

Now, $a^i = g^{ij} a_j$ and $a_i = g_{ij} a^j$

Metric tensor "converts" between co and contravariant bc. coordinates.

The determinant of g_{ij} is denoted by g

$$g = |g_{ij}| = \left| \frac{\partial x^i}{\partial \xi^j} \right|^2 > 0$$

and $|g^{ij}| = 1/g$

Measuring space

Line Elements

$\underline{dr}$, $|\underline{dr}| << 1$

$$\underline{dr} = \frac{\partial \underline{r}}{\partial \xi^i} d\xi^i = \underline{g}_j \, d\xi^i$$

It's length squared is

$$ds^2 = \underline{dr} \cdot \underline{dr} = \underline{g}_i \cdot d\xi^i \cdot \underline{g}_j \, d\xi^j = g_{ij} \, d\xi^i d\xi^j$$

i.e. g_{ij} gives the scale, or metric, of the line element / unit coordinate.

Surface Elements

An element in the surface ξ^i = constant is given by

$$dA_i = \sqrt{g \, g^{ii}} \; d\xi^j d\xi^k, \quad i \text{ is not summed}$$

c.f. Cartesians $dA_x = dy\,dz$ $\quad j \neq k \neq i$

Volume Elements

$$dV = \sqrt{g} \, d\xi^1 d\xi^2 d\xi^3$$

c.f. standard change of coordinates in integrals.

Calculus & Differentiation

Differentiating vector

$$\frac{\partial \underset{\sim}{a}}{\partial \xi^j} = \underset{\sim}{a}_{,j} = (a^i \underset{\sim}{g}_i)_{,j} = a^i{}_{,j}\,\underset{\sim}{g}_i + a^i \underset{\sim}{g}_{i,j}$$

$a^i{}_{,j}$: variation of component; $\underset{\sim}{g}_{i,j}$: variation of base vector

$$= a^i{}_{,j}\,\underset{\sim}{g}_i + a^i \Gamma_{ijk}\, \underset{\sim}{g}^k$$

$\Gamma_{ijk} = \underset{\sim}{g}_{i,j} \cdot \underset{\sim}{g}_k$

Then,

$$\underset{\sim}{a}_{,j} = a^i{}_{,j}\,\underset{\sim}{g}_i + a^i \Gamma_{ijk} g^{kl}\,\underset{\sim}{g}_l$$

$$= \left[a^i{}_{,j} + a^i \Gamma_{ijk} g^{kl}\right] \underset{\sim}{g}_i$$

$$= \left[a^i{}_{,j} + a^i \Gamma^l{}_{ij}\right] \underset{\sim}{g}_l$$

$a^i{}_{,j}$: coordinate derivative; $\Gamma^l{}_{ij}$: base vector derivative

Then, $\underset{\sim}{a}_{,j} = a^k{}_{|j}\,\underset{\sim}{g}_k$, where

$$a^k{}_{|j} = a^k{}_{,j} + a^i \Gamma^k{}_{ij}$$

$a^k{}_{|j}$: covariant derivative of a^k. it is a mixed tensor. $a_{k|j}$ is not a tensor.

We can show that $a_{i}|_{j} = a_{i,j} - \Gamma^{k}_{ji}\, a_k$

In Cartesians $\Gamma_{IJK} = 0 \quad \forall I, J, K$ so

$a_I|_K = a^I|_K = a_{I,K}$ and covariant derivative = partial derivative.

Vector Differential operators

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x^I} \underset{\sim}{e}_I$$

$$= \underset{\sim}{g}^i \frac{\partial f}{\partial \xi^i}$$

so $\underset{\sim}{\nabla} = \underset{\sim}{g}^i \frac{\partial}{\partial \xi^i}$ ← invariant under coordinate transform:

$$\text{div } \underset{\sim}{F} = \underset{\sim}{\nabla} \cdot \underset{\sim}{F} = \underset{\sim}{g}^i \cdot \underset{\sim}{F}_{,i} = F^i|_i$$

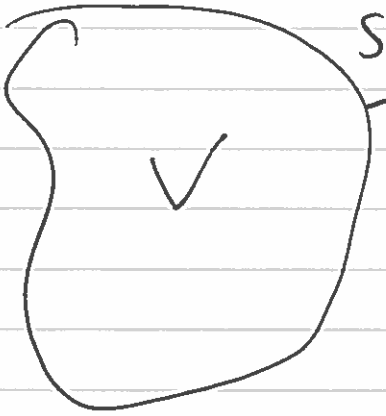
$$\text{curl } \underset{\sim}{F} = \underset{\sim}{\nabla} \times \underset{\sim}{F} = \underset{\sim}{g}^i \times \frac{\partial}{\partial \xi^i}\left(F_j \underset{\sim}{g}^j\right)$$

$$= \varepsilon^{ijk} F_j|_i \, \underset{\sim}{g}_k$$

$$\text{Laplacian } \nabla^2 f = \underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} f = g^{ij} f|_{ij}$$

$$\text{or } \frac{1}{\sqrt{g}} \left(\sqrt{g}\, g^{ij} f_{,i}\right)_{,j} \quad \text{(not obvious)}$$

The divergence theorem



$$\iint_S \underset{\sim}{F} \cdot \underset{\sim}{n}\, dS = \int_V \operatorname{div} \underset{\sim}{F}\, dV$$

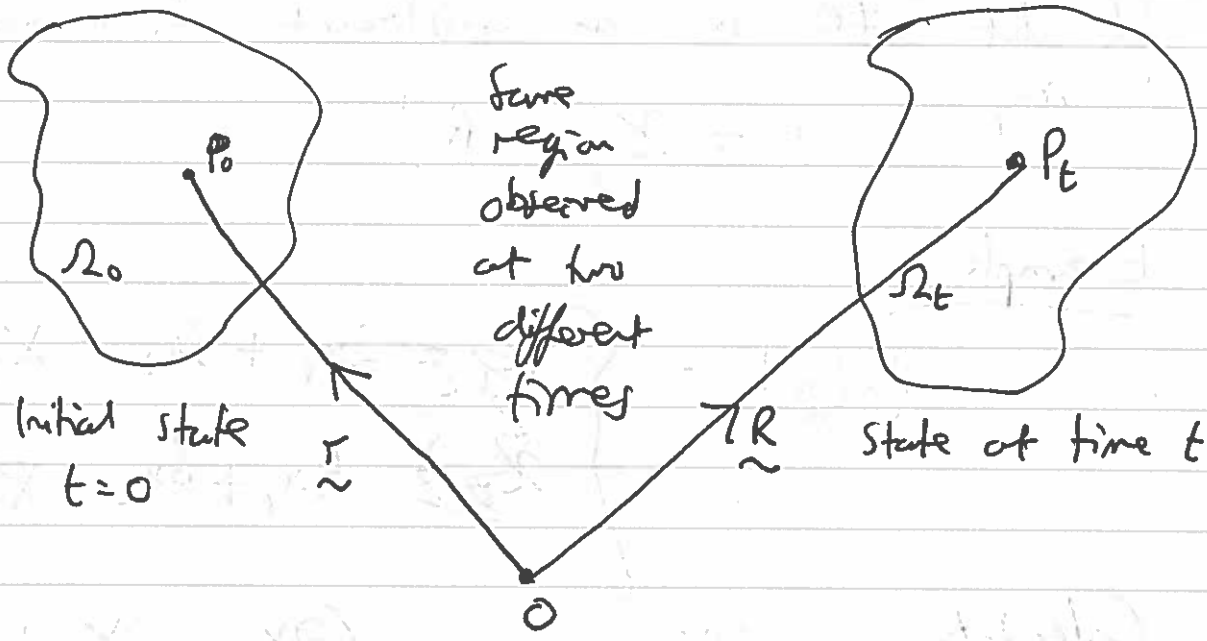
In general tensor notation:

$$\int_S F^i n_i\, dS = \int_V F^i{}_{|i}\, dV$$

$$= \int_V \frac{1}{\sqrt{g}} \left(F^i \sqrt{g}\right)_{,i}\, dV$$

Chapter 2: Kinematics; Deformation & Flow

Matter can move and deform and we need to be able to quantify and distinguish both effects.



The position of a point at $t=0$ is $\underset{\sim}{r} = x_K \underset{\sim}{e}_K$ in global Cartesian coordinates.

The position of the same point at t is $\underset{\sim}{R} = X_K \underset{\sim}{e}_K$

Change in configuration described by a deformation map $\chi_t : \Omega_0 \to \Omega_t$

$$\underset{\sim}{R} = \underset{\sim}{\chi}_t(\underset{\sim}{r})$$

We expect: (a) matter cannot be created or destroyed
(b) matter does not interpenetrate.

This means that mapping is 1-1 and

$J = \det F$, where $F_{IJ} = \dfrac{\partial X_I}{\partial x_J}$

is such that $J > 0$.

F is the tensor representation of the mapping.

If $\det F \neq 0$ we can construct the inverse map

$$\underset{\sim}{r} = \chi_t^{-1}(\underset{\sim}{R}) .$$

Example

$$\underset{\sim}{\chi}(\underset{\sim}{x}) = \begin{cases} 2x_1 + 3x_2 = X_1 \\ 4x_1 + 6x_2 = X_2 \end{cases}$$

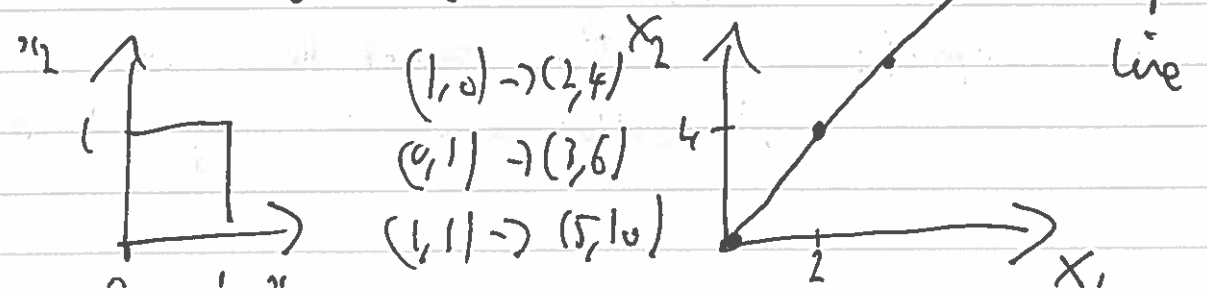
Calculate $F_{IJ} = \dfrac{\partial X_I}{\partial x_J}$: $F = \begin{pmatrix} \dfrac{\partial X_1}{\partial x_1} & \dfrac{\partial X_1}{\partial x_2} \\ \dfrac{\partial X_2}{\partial x_1} & \dfrac{\partial X_2}{\partial x_2} \end{pmatrix}$

$\Rightarrow F = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$: $\det F = 0$

Mapping is not physical.

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

collapse to line



2.2 Lagrangian (Material) and Eulerian (Lab) Descriptions

(i) Lagrangian: Treat everything as a function of initial position, $\underset{\sim}{r}$

$\underset{\sim}{R}(\underset{\sim}{r}, t)$

common in solid mechanics.

(ii) Eulerian: Treat everything as a function of current position $\underset{\sim}{R}$

$\underset{\sim}{r}(\underset{\sim}{R}, t)$

- observe at a fixed location in space

common in fluid mechanics.

2.3 Displacement, Velocity & Acceleration

Newton's laws are Lagrangian...

2.3.1 Displacement

Displacement = change in position of a material point.

displacement $\underset{\sim}{u} = \underset{\sim}{R} - \underset{\sim}{r}$

either $\underset{\sim}{u}(\underset{\sim}{r}) = \underset{\sim}{R}(\underset{\sim}{r}) - \underset{\sim}{r}$

or $\underset{\sim}{U}(\underset{\sim}{R}) = \underset{\sim}{R} - \underset{\sim}{r}(\underset{\sim}{R})$

Example

$$\underset{\sim}{R}(\underset{\sim}{r}, t) = \underset{\sim}{r}(1+t)$$

Find displacement in Eulerian and Lagrangian viewpoints....

Solution

Lagrangian: $\underset{\sim}{u} = \underset{\sim}{R} - \underset{\sim}{r} = \underset{\sim}{r}(1+t) - \underset{\sim}{r} = t\underset{\sim}{r}$

Eulerian: Find $\underset{\sim}{r}(\underset{\sim}{R})$: $\underset{\sim}{R} \to \underset{\sim}{r} = \underset{\sim}{R}/(1+t)$

so $\underset{\sim}{u}(\underset{\sim}{R}) = \dfrac{\underset{\sim}{R}\,t}{(1+t)}$ only valid if $\underset{\sim}{R}$ is within the body at time t.

Velocity

Velocity is rate of change of displacement of material pt. w.r.t. time.

Lagrangian: $\underset{\sim}{v}(\underset{\sim}{r}, t) = \dfrac{\partial \underset{\sim}{u}(\underset{\sim}{r}, t)}{\partial t} = \dfrac{\partial \underset{\sim}{R}(\underset{\sim}{r}, t)}{\partial t}$

Velocity is calculated at fixed material point

Eulerian: $\underset{\sim}{V}(\underset{\sim}{R}, t) = \left.\dfrac{\partial \underset{\sim}{U}(\underset{\sim}{R}, t)}{\partial t}\right|_{\underset{\sim}{r}\ \text{fixed}} = \underset{\sim}{v}(\underset{\sim}{r}(\underset{\sim}{R}, t), t)$

Acceleration

- **Lagrangian:** $\underset{\sim}{a}(\underset{\sim}{r},t) = \left.\frac{\partial^2 \underset{\sim}{R}}{\partial t^2}\right|_{\underset{\sim}{r}} = \left.\frac{\partial \underset{\sim}{v}}{\partial t}\right|_{\underset{\sim}{r}}$

- **Eulerian:**

$$\underset{\sim}{A}(\underset{\sim}{R},t) = \left.\frac{\partial \underset{\sim}{V}(\underset{\sim}{R},t)}{\partial t}\right|_{\underset{\sim}{r}}$$

$$= \left.\frac{\partial \underset{\sim}{V}(\underset{\sim}{R}(\underset{\sim}{r},t),t)}{\partial t}\right|_{\underset{\sim}{r}}$$

$$= \left.\frac{\partial \underset{\sim}{V}}{\partial t}\right|_{\underset{\sim}{R}} + \left.\frac{\partial \underset{\sim}{V}}{\partial \underset{\sim}{R}}\right|_{t} \cdot \left.\frac{\partial \underset{\sim}{R}}{\partial t}\right|_{\underset{\sim}{r}}$$

$$= \left.\frac{\partial \underset{\sim}{V}}{\partial t}\right|_{\underset{\sim}{R}} + \underset{\sim}{V} \cdot \left.\frac{\partial \underset{\sim}{V}}{\partial \underset{\sim}{R}}\right|_{t}$$

$$= \frac{\partial \underset{\sim}{V}}{\partial t} + \underset{\sim}{V} \cdot \nabla_{\underset{\sim}{R}} \underset{\sim}{V}$$

In components $A_I = \frac{\partial V_I}{\partial t} + V_J \frac{\partial V_I}{\partial X_J}$

↑ material velocity change

↑ material point passes fixed point in space.

Material Derivative

Define

Eulerian $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underset{\sim}{V} \cdot \nabla_{\underset{\sim}{R}}$

Lagrangian $\frac{D}{Dt} = \frac{\partial}{\partial t}$

In curvilinear coordinates $x^{\bar{i}}$ (Eulerian)

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x^{\bar{i}}} V^{\bar{i}}$$

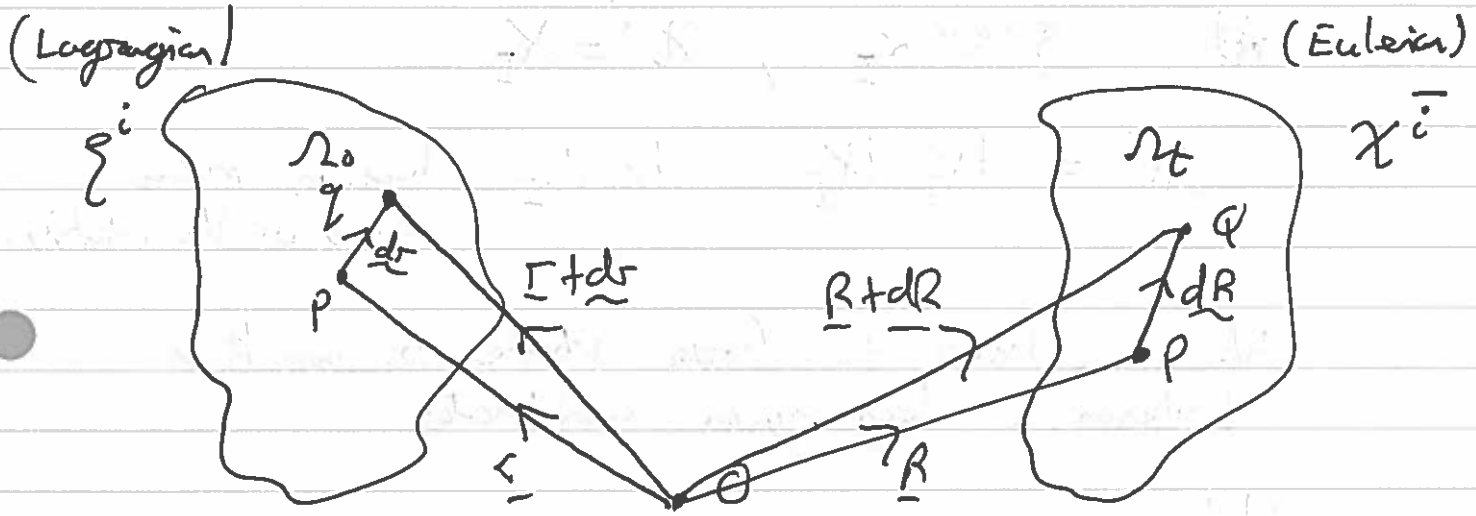
where we have the bases $\underset{\sim}{G}_{\bar{i}} = \frac{\partial \underset{\sim}{R}}{\partial x^{\bar{i}}}$

$$\frac{D\underset{\sim}{V}}{Dt} = \left(\frac{\partial V^{\bar{i}}}{\partial t} + V^{\bar{j}} V^{\bar{i}} \|_{\bar{j}}^{\bar{i}} \right) G_{\bar{i}}$$

$$= \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \underline{\nabla}_{\underline{R}} \underline{V}$$

2.4 Deformation

Deformation occurs when two material points move relative to one another. We must study material lines



We know that:

$$\underset{\sim}{dr}(\xi^i) = \frac{\partial r}{\partial \xi^i} d\xi^i = \underset{\sim}{g_i} d\xi^i$$

so $ds^2 = |dr|^2 = \underset{\sim}{g_i} d\xi^i \cdot \underset{\sim}{g_j} d\xi^j = g_{ij} d\xi^i d\xi^j$

and $\underset{\sim}{dR}(x^{\bar{i}}) = \frac{\partial R}{\partial x^{\bar{i}}} dx^{\bar{i}} = \underset{\sim}{G_{\bar{i}}} dx^{\bar{i}}$

(definition)

Hence $ds^2 = G_{\bar{i}\bar{j}} dx^{\bar{i}} dx^{\bar{j}}$ where $G_{\bar{i}\bar{j}} = \underset{\sim}{G_{\bar{i}}} \cdot \underset{\sim}{G_{\bar{j}}}$

Hence, change in (square) length is

$$\Delta s^2 = ds^2 - ds^2 = G_{\bar{i}\bar{j}} dx^{\bar{i}} dx^{\bar{j}} - g_{ij} d\xi^i d\xi^j$$

If $\Delta s^2 = 0$ $\forall$ possible points, there is no deformation.

There are different representations depending on choice of coordinates:

A) Cartesian

$\xi^i = x_I$, $\chi^{\bar{i}} = X_I$

$$\Delta s^2 = dX_I\, dX_I - dx_I\, dx_I$$ because metric tensors are the identity.

We now have to choose whether to work in Eulerian or Lagrangian coordinates:

i) Lagrangian

$$dX_I = \frac{\partial X_I}{\partial x_J}\, dx_J = F_{IJ}\, dx_J$$ ← Deformation gradient tensor.

So

$$\Delta s^2 = \left(F_{IK}\, dx_K\, F_{IL}\, dx_L - \delta_{KL}\, dx_K\, dx_L \right)$$

$$= \left(F_{IK} F_{IL} - \delta_{KL} \right) dx_L\, dx_K$$

We define the Green-Lagrange strain tensor to be

$$e_{IJ} = \frac{1}{2}\left(C_{IJ} - \delta_{IJ} \right), \text{ where } C_{IJ} = F_{KI} F_{KJ}$$

[$C = F^T F$ – right Cauchy Green deformation tensor

$b = F F^T$ – left " " " (Finger tensor)]

(ii) Eulerian

$$\Delta S^2 = (\delta_{IJ} - C_{IJ})\, dX_I\, dX_J$$

where

$$C_{IJ} = F^{-1}_{kI} F^{-1}_{kJ}, \quad C = (FF^T)^{-1}$$

Cauchy deformation tensor

Then we define

$$E_{IJ} = \frac{1}{2}(\delta_{IJ} - C_{IJ})$$ – the Almansi strain tensor.

Almansi, ALMANSI

Note e_{IJ} and E_{IJ}, represent the same thing, just in different coordinates.

B) Curvilinear

It most common to choose convected coordinates $\chi^i = \xi^i$

then

$$\Delta S^2 = (G_{ij} - g_{ij})\, d\xi^i\, d\xi^j$$

where $\underline{G}_i = \dfrac{\partial \underline{R}}{\partial \xi^i}$

Now Green-Lagrange strain tensor is

deformed metric $\gamma_{ij} = \frac{1}{2}(G_{ij} - g_{ij})$ undeformed metric

γ_{ij}, e_{IJ} and E_{IJ} all represent the same object (strain tensor) and can be interconverted by changing coordinates.

2.4.1 Deformation and Displacement

G.-L. strain tensor is $\gamma_{ij} = \frac{1}{2}(G_{ij} - g_{ij})$

But $\underset{\sim}{R} = \underset{\sim}{r} + \underset{\sim}{u}$ ← displacement

So

$$\underset{\sim}{G}_i \equiv \frac{\partial \underset{\sim}{R}}{\partial \xi^i} = \frac{\partial \underset{\sim}{r}}{\partial \xi^i} + \frac{\partial \underset{\sim}{u}}{\partial \xi^i} = \underset{\sim}{g}_i + \underset{\sim}{u}_{,i}$$

$G_{ij} = \underset{\sim}{G}_i \cdot \underset{\sim}{G}_j$

Hence,

$$\gamma_{ij} = \frac{1}{2}\left[(\underset{\sim}{g}_i + \underset{\sim}{u}_{,i}) \cdot (\underset{\sim}{g}_j + \underset{\sim}{u}_{,j}) - g_{ij} \right]$$

$$= \frac{1}{2}\left[\cancel{\underset{\sim}{g}_i \cdot \underset{\sim}{g}_j} + \underset{\sim}{u}_{,i} \cdot \underset{\sim}{g}_j + \underset{\sim}{g}_i \cdot \underset{\sim}{u}_{,j} + \underset{\sim}{u}_{,i} \cdot \underset{\sim}{u}_{,j} - \cancel{g_{ij}} \right]$$

Let $\underset{\sim}{u} = u_j \underset{\sim}{g}^j \Rightarrow \underset{\sim}{u}_{,i} = u_j|_i \, \underset{\sim}{g}^j$

↑ covariant w.r.t. g_{ij}

So

$$\gamma_{ij} = \frac{1}{2}\left[u_i|_j + u_j|_i + u^k|_i \, u_k|_j \right]$$

and $u_i|_j$ are components of displacement gradient tensor

$$\underset{\sim}{\nabla}_{\underset{\sim}{r}} \otimes \underset{\sim}{u}$$

In Cartesian,

$$e_{IJ} = \frac{1}{2}\left[\underbrace{u_{I,J} + u_{J,I}}_{\text{Linear}} + \underbrace{u_{K,I} \, u_{K,J}}_{\text{Nonlinear}} \right]$$

Interpretation

Consider a unit vector $\underset{\sim}{n}$, $\underset{\sim}{n} \cdot \underset{\sim}{n} = 1$

and we write $d\xi^i = n^i d\Sigma$

Stretch in the direction $\underset{\sim}{n}$ is defined to be

$$\lambda(\underset{\sim}{n}) = \frac{|d\underset{\sim}{R}(\underset{\sim}{n})|}{|d\underset{\sim}{r}(\underset{\sim}{n})|} = \sqrt{n^i G_{ij} n^j}$$

Strain Invariants and Principal Strains

Direction of maximum stretch is given by

$$\max_{\underset{\sim}{n}} \lambda(\underset{\sim}{n}) \quad \text{subject to constraint} \quad |\underset{\sim}{n}| = 1$$

(See online notes for details)

The directions of max stretch are eigenvectors of G_{ij}. Each eigenvector has an associated eigenvalue, μ, and these are the squares of the stretches in the direction of the eigenvectors. These eigenvalues are called principal stretches.

The eigenvalues are scalar invariants - do not change with coordinate system.
We can take combinations to form 3 independent invariants.

Most common choice is (literature) is to define

$$|g^{ik}G_{kj} - \mu \delta^i_j| = 0 = -\mu^3 + I_1\mu^2 - I_2\mu + I_3$$

"$G - \mu I$"

where

$$I_1 = g^{ij}G_{ji} \equiv G^i_i \quad \text{(absolute of rotation)}$$

$$I_2 = \frac{1}{2}\left[G^i_i G^j_j - G^i_j G^j_i\right]$$

$$I_3 = |G^i_j| = \sqrt{G/g}$$

G_{ij} is real and symmetric so we can diagonalize

$$G_{ij} = \begin{pmatrix} \lambda_1^2 & 0 & \\ & \lambda_2^2 & \\ 0 & & \lambda_3^2 \end{pmatrix} \quad \text{in the eigenbasis.}$$

We can now define

$$U = \underset{\text{matrix}}{\sqrt{G}} = \begin{pmatrix} \lambda_1 & 0 & \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix}$$

$F = RU$ (see online notes)

F: deformation gradient; R: orthogonal

I.e. any deformation is locally a rotation and 3 perpendicular stretches.

Alternative measures of strain

The strain is zero when all principal stretches are 1, so $U = \sqrt{G} = I$ in eigenbasis.

matrices

We can consider a more general strain measure

$$\begin{pmatrix} f(\lambda_1) & 0 & \\ & f(\lambda_2) & \\ 0 & & f(\lambda_3) \end{pmatrix}$$ in eigenbasis

$f(x)$ should be monotonically increasing

$f(1) = 0$

$f'(1) = 1$ ← normalisation.

e.g. $f(x) = \frac{1}{n}(x^n - 1)$ $n \neq 0$, $\ln x$ $(n=0)$

$n=2$ – Green-Lagrange $n=1$ Biot

$n=-2$ – Almansi $n=0$ Hencky

Chapter 3: Forces, momentum & stress

3.2 The Continuum Hypothesis

We can replace discrete particles by continuous distributions of matter:

Mass Here we define a density field

$$\rho(\underline{R}, t) = \lim_{\Delta V \to 0} \frac{M_{\Delta V}}{\Delta V}$$

(ρ: density; $M_{\Delta V}$: total mass in region; ΔV: volume of region)

We assume this limit exists, if so:

$$M(t) = \int_{\Omega_t} \rho(\underline{R}, t)\, dV_t$$

($M(t)$: total mass; Ω_t: deformed volume.)

Momentum

We define $\underline{P}(t) = \int_{\Omega_t} \rho(\underline{R}, t)\, \underline{V}(\underline{R}, t)\, dV_t$

($\underline{P}$: total linear momentum)

$$\underline{\underline{H}}_{\underline{\underline{Z}}}(t) = \int_{\Omega_t} (\underline{R} - \underline{\underline{Z}}) \times [\rho \underline{V}]\, dV_t$$

($\underline{\underline{H}}_{\underline{\underline{Z}}}$: total angular momentum about $\underline{\underline{Z}}$)

Balance of Momentum (Newton 2)

$$\frac{D\underline{P}}{Dt} = \frac{D}{Dt}\int_{\Omega_t} \rho \underline{V}\, dV_t = \underline{F}$$

rate of change of momentum

$\underline{F}$: total applied force

$$\frac{D\underline{H}_z}{Dt} = \frac{D}{Dt}\int_{\Omega_t} (\underline{R} - \underline{z}) \times [\rho \underline{V}]\, dV_t = \underline{L}_z$$

rate of change of angular momentum

$\underline{L}_z$: total applied torque

These are sometimes called Cauchy's equations.

In order to use these, we need to know what forces & torques can be applied:

(i) Body forces

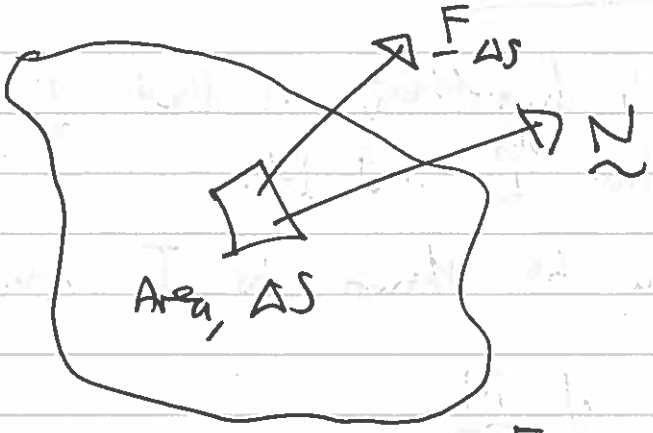
$$\underline{F}^V = \int_{\Omega_t} \rho \underline{F}\, dV_t$$

$\underline{F}^V$: total body force

$\underline{F}$: Force per unit mass

e.g. gravity $\underline{F} = \underline{g}$, $\underline{g}$ is direction of gravity.

(ii) Surface forces



Traction, $$\underset{\sim}{T}(\underline{R}, t) = \lim_{\Delta S \to 0} \frac{\underline{F}\,\Delta S}{\Delta S}$$

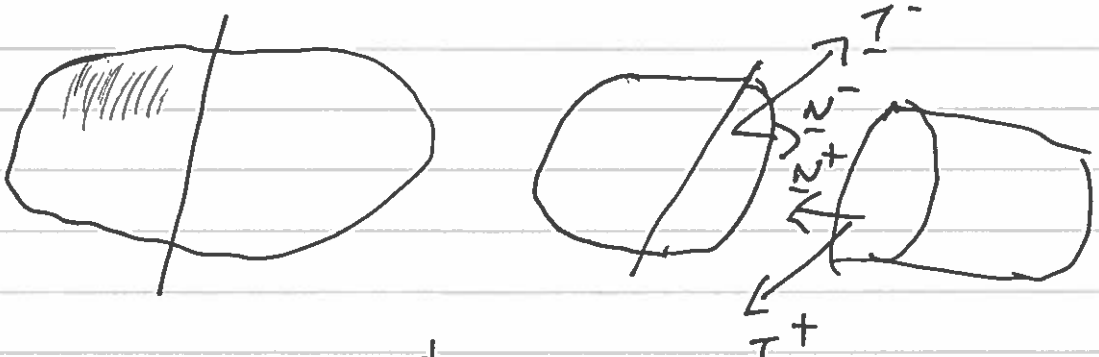
Assume this limit exists, under continuum hypothesis

Total surface force $$\underline{F}_S = \int_{\partial\Omega_t} \underset{\sim}{T}\, dS_t$$

$\partial\Omega_t$ ← surface of Ω_t

(iii) Internal Forces:

These must "cancel" via Newton's 3rd law



$\underline{T}^+ = -\underline{T}^-$ and $\underline{N}^+ = -\underline{N}^-$

Internal tractions are called stresses.

3.3 The Stress tensor

Cauchy's hypothesis is that $\underset{\sim}{T}$ depends only on position $\underset{\sim}{R}$ and $\underset{\sim}{N}$.

It can be shown to $\underset{\sim}{T}$ must be a linear function of $\underset{\sim}{N}$.

Hence

$$\underset{\sim}{T} = T^{ij}\, N_i\, \underset{\sim}{G}_j \qquad \text{(Eulerian representation)}$$

$\underset{\sim}{T}$: stress tensor

(Force / unit deformed area)

We can transform to Lagrangian by

$$\underset{\sim}{T} = T^{ij}\, N_i\, \underset{\sim}{G}_j \qquad \text{(everything is a tensor)}$$

This is still force / unit area (deformed)

The stress tensor $\underset{\sim}{T}$ is defined such that the traction $\underset{\sim}{T}$ (force / unit deformed area) is

$$\underset{\sim}{T} = T^{\bar{i}\bar{j}} N_{\bar{i}} \underset{\sim}{G_{\bar{j}}} \quad \text{in Eulerian coordinates } x^{\bar{i}}$$

where $\underset{\sim}{N}$ is the normal to our chosen plane.

We can transform $T^{\bar{i}\bar{j}}$ into whatever coordinates we wish, but it will always represent force / unit deformed area.

Physical components of the stress are recovered by scaling $T^{\bar{i}\bar{j}}$ by the lengths of base vectors:

i.e.

$$\sigma_{(\bar{i}\bar{j})} = \sqrt{G_{\bar{j}\bar{j}} / G^{\bar{i}\bar{i}}}\; T^{\bar{i}\bar{j}}$$

($\bar{j}$ no summation)

e.g.

$$\sigma_{(12)} = \sqrt{G_{22} / G^{11}}\; T^{12}$$

3.3.3 Alternative Form of the Stress Tensor

A different family of stress tensors is formed by considering the force per unit undeformed area.

Consider

$$\underset{\sim}{F} = \underset{\sim}{T}\, dS = T^{\bar{i}\bar{j}} N_{\bar{i}} \underset{\sim}{G_{\bar{j}}}\, dS$$

($\underset{\sim}{T}$: deformed; dS: deformed)

$= T^{\bar{i}\bar{j}} \underset{\sim}{G}_{\bar{j}} \, dA_{\bar{i}}$, where $dA_{\bar{i}} = N_{\bar{i}} \, dS$

Now use Nanson's formula

$$dA_{\bar{i}} = J \frac{\partial \xi^k}{\partial x^{\bar{i}}} da_k \quad \text{do}$$

$$\underset{\sim}{F} = T^{\bar{i}\bar{j}} J \frac{\partial \xi^k}{\partial x^{\bar{i}}} da_k \underset{\sim}{G}_{\bar{j}} \quad , \text{ but } da_k = n_k \, dS$$

$$= T^{\bar{i}\bar{j}} J \frac{\partial \xi^k}{\partial x^{\bar{i}}} \underset{\sim}{G}_{\bar{j}} n_k \, dS \quad \leftarrow \text{undeformed area}$$

$$= \rho^{\bar{j}k} n_k \underset{\sim}{G}_{\bar{j}} \, dS \quad , \text{ where}$$

$$\boxed{\rho^{\bar{j}k} = J T^{\bar{i}\bar{j}} \frac{\partial \xi^k}{\partial x^{\bar{i}}}}$$

ρ is called the first Piola-Kirchhoff stress tensor.

You may see $\rho = J \sigma^T F^{-T}$ in books.

This is a two-point tensor - components of normal are in Lagrangian basis, traction is decomposed into Eulerian basis.

We can write traction in Lagrangian basis

$$\underset{\sim}{t} = S^{kl} n_k \underset{\sim}{G}_l \quad , \text{ where}$$

$$\boxed{S^{kl} = J \frac{\partial \xi^k}{\partial x^{\bar{i}}} T^{\bar{i}\bar{j}} \frac{\partial \xi^l}{\partial x^{\bar{j}}}}$$

S^{kl} is the second Piola-Kirchhoff stress tensor.

3.3.4 Equations of Equilibrium

All accelerations are zero.

Balance of linear momentum:

$$\underset{\sim}{F} = \underset{\sim}{0} \Rightarrow \int_{\Omega_t} \rho \underset{\sim}{F}\, dV_t + \int_{\partial\Omega_t} \underset{\sim}{T}\, dS_t = \underset{\sim}{0}$$

(deformed configuration)

$\int_{\Omega_t} \rho \underset{\sim}{F}\, dV_t$ ↑ total body force

$\int_{\partial\Omega_t} \underset{\sim}{T}\, dS_t$ ↑ total surface force

by principle

$$\int_{\partial\Omega_t} \underset{\sim}{T}\, dS_t = \int_{\partial\Omega_t} \underline{\underline{T}} \cdot \underset{\sim}{N}\, dS_t = \int_{\Omega_t} \underset{\underset{\sim}{R}}{\nabla} \cdot \underline{\underline{T}}\, dV_t$$

by divergence theorem

Then

$$\int_{\Omega_t} \rho \underset{\sim}{F} + \underset{\underset{\sim}{R}}{\nabla} \cdot \underline{\underline{T}}\, dV_t = \underset{\sim}{0} \quad \forall \Omega_t$$

This integral equation being zero for all volumes means that

$$\rho \underset{\sim}{F} + \underset{\underset{\sim}{R}}{\nabla} \cdot \underline{\underline{T}} = \underset{\sim}{0}$$

↑ what does this mean?

We find in components in $\underset{\sim}{G}_j$ that

$$T^{ij}\|_{\bar{j}} + \rho F^{\bar{i}} = 0$$

↑ covariant derivative w.r.t $\underset{\sim}{G}_j$.

In Cartesian,

$$T_{IJ,I} + \rho F_J = 0$$

Angular Momentum about axis $\underline{z}$

$$\int_{\partial\Omega_t} (\underline{R} - \underline{z}) \times \underline{T}\, dS_t + \int_{\Omega_t} \rho (\underline{R} - \underline{z}) \times \underline{F}\, dV_t = \underline{0}$$

↑ total surface torque ↑ total volume torque

$$\Rightarrow \int_{\partial\Omega_t} \underline{R} \times \underline{T}\, dS_t + \int_{\Omega_t} \rho \underline{R} \times \underline{F}\, dV_t$$

$$- \underline{z} \times \left[\int_{\partial\Omega_t} \underline{T}\, dS_t + \int_{\Omega_t} \rho \underline{F}\, dV_t \right] = \underline{0}$$

$\underline{0}$ ← balance of linear momentum.

Apply divergence theorem and manipulate.

$$\varepsilon_{\bar{i}\bar{j}\bar{k}}\, T^{\bar{i}\bar{j}}\, \underline{G}^{\bar{k}} = \underline{0}$$

Hence $\underline{T^{\bar{i}\bar{j}} = T^{\bar{j}\bar{i}}}$ ← stress tensor is symmetric.