MATH45061: EXAMPLE SHEET[1] IV

1.) A region modelled as a continuum is growing, such that mass is produced internally at a rate γ per unit of existing mass. Show that the conservation of mass equation in Eulerian form becomes

$$\frac{D\rho}{Dt} + \rho \boldsymbol{\nabla}_{\boldsymbol{R}} \cdot \boldsymbol{V} = \rho\gamma,$$

and find the equivalent expression in Lagrangian form.

2.) By considering the conservation of angular momentum in Lagrangian form, or otherwise, show that first Piola–Kirchhoff stress tensor is not symmetric and find the relationship between p and p^T.

3.) Consider a body undergoing a rigid motion

$$\boldsymbol{R}(\boldsymbol{r}, t) = \mathsf{A}(t)\boldsymbol{r} + \boldsymbol{C}(t),$$

where A is an orthogonal matrix and $\boldsymbol{C}$ is a vector. Show that the stress power is zero for such a motion.

4.) Show that the stress power per unit undeformed volume can be written as $\mathsf{p} : \dot{\mathsf{F}}$, where the dot denotes the material derivative, p is the first Piola–Kirchhoff stress tensor and F is the (Eulerian) deformation gradient tensor.

5.) If the Cauchy stress in a particular material is given by $\mathsf{T} = -P(\boldsymbol{R}, t)\,\mathsf{I}$, show that the governing conservation laws are given by

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \cdot (\rho \boldsymbol{V}) = 0,$$

$$\frac{\partial \rho \boldsymbol{V}}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \cdot (\rho \boldsymbol{V} \otimes \boldsymbol{V} + P\,\mathsf{I}) = \rho \boldsymbol{F},$$

$$\frac{\partial \rho E}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \cdot ([\rho E + P]\boldsymbol{V}) = \rho B - \boldsymbol{\nabla}_{\boldsymbol{R}} \cdot \boldsymbol{Q} + \rho \boldsymbol{F} \cdot \boldsymbol{V},$$

where $E = \Phi + G(\boldsymbol{V})$, where $G(.)$ is a function to be determined. Note that we need constitutive laws that describe $\boldsymbol{Q}$, P and E in terms of ρ, $\boldsymbol{V}$ and Θ in order to close the system.

6.) Explain why for a material model that has a constant heat flux in a fixed direction, *e.g.* $\boldsymbol{Q}$ is a constant vector, the only thermodynamically admissible temperature gradients in the absence of dissipation are those that decrease in the direction of the heat flux. If we allow dissipation, but insist that the motion is isentropic ($\dot{\eta} = 0$), how can the direction of heat flux be reversed?

7.) A material has the constitutive relations

$$\boldsymbol{q} = -\kappa \boldsymbol{\nabla}_{\boldsymbol{r}} \theta, \quad \phi = \alpha\theta,$$

where κ and α are constants. Consider a body composed of the material that undergoes only rigid body motions.

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a.) Give a physical interpretation of the constitutive equations.

b.) Show that the conservation of energy reduces to the heat equation

$$\frac{\partial \theta}{\partial t} = D\boldsymbol{\nabla}_{\boldsymbol{r}}^2 \theta + S,$$

where the quantities D and S are to be found.

c.) If the Helmholtz free energy and entropy are functions only of the temperature, show that the second law of thermodynamics is only satisfied, in general, if $\kappa \geq 0$ and $\eta_0 = -\frac{\partial \psi}{\partial \theta}$.

8.) The equations of state for an ideal gas are given by

$$P = \rho R \Theta, \quad \Phi = c_v \Theta, \quad \eta = c_v \log \Theta - R \log \rho + \eta_*,$$

when ρ and Θ are the independent variables.

a.) Find the equations of state when ρ and η are treated as the independent variables, *i.e.* find expressions for P, Φ and Θ as functions of ρ and η. You may find it useful to introduce the constant $\gamma = 1 + R/c_v$.

b.) Show that the governing equations of gas dynamics can be written in the form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \cdot (\rho \boldsymbol{V}) = 0,$$

$$\rho \frac{D\boldsymbol{V}}{Dt} + \boldsymbol{\nabla}_{\boldsymbol{R}} P(\rho, \eta) = \rho \boldsymbol{F},$$

$$\frac{\partial \Phi(\rho, \eta)}{\partial \eta} \frac{D\eta}{Dt} = B.$$

c.) When there is no body heating and the entropy at $t = 0$ is constant, show that the entropy remains constant for all time and hence that the equations of gas dynamics reduce to the Euler equations for a perfect fluid:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \cdot (\rho \boldsymbol{V}) = 0,$$

$$\rho \frac{D\boldsymbol{V}}{Dt} + \boldsymbol{\nabla}_{\boldsymbol{R}} P(\rho) = \rho \boldsymbol{F}.$$

9.) **a.)** For an ideal gas at rest, show in the absence of body forces and body heating, a solution is given by

$$\rho(\boldsymbol{R}, t) = \rho^*, \quad \eta(\boldsymbol{R}, t) = \eta^*,$$

where ρ^* and η^* are constants.

b.) The gas is excited by small motions of $\mathcal{O}(\epsilon)$, where $\epsilon \ll 1$, so that deviations from the quiescent state found in part (a.) are $\mathcal{O}(\epsilon)$. By developing a solution as a regular series in powers of ϵ, find the linear equations (those at $\mathcal{O}(\epsilon)$) that describe the motion of the gas.

c.) Show that the linear pressure perturbation $\epsilon P^{(1)} \approx (P - P^*)$ can be the solution of a wave equation

$$\frac{\partial^2 P^{(1)}}{\partial t^2} = c^2 \boldsymbol{\nabla}_{\boldsymbol{R}}^2 P^{(1)},$$

and find the speed of propagation c.