Chapter 6

Elasticity

6.1 (Perfect) Thermoelastic Materials

A solid body that undergoes reversible deformations is said to be *perfectly elastic*. The key feature of a solid, as opposed to a fluid or a gas, is that a solid body has a natural or rest state in which it is unstrained (any strain measure is zero), but if thermal and/or mechanical loads are applied then the body will deform. For perfectly elastic behaviour, the body must return to its natural state when all the loads are removed. Imagine pulling a spring: under the applied load the spring will deform (extend), but it will return to its rest state once you let go. Similarly, heating a block of rubber will cause it to expand, but once it cools down it will return to its original shape. Perfectly elastic behaviour is an idealisation that is reasonable for many materials over a restricted range of deformations. If you stretch any material too much then it can exhibit plastic behaviour (undergo permanent deformation) or even fracture (break).

The reversibility of perfectly elastic deformation means that there cannot be any internal dissipation of energy. In other words, it must be possible to recover all the energy that goes into the deformation (strain energy). Furthermore, we do not expect the material behaviour to depend on its previous history because then a difference between the loading and unloading times could lead to irreversible deformation. Hence, the material behaviour is expected to depend on the current state of deformation, represented by the deformation gradient tensor F, and current temperature. We propose the following constitutive relationships for the Helmholtz free energy, Cauchy stress, entropy and heat flux

$$\Psi(\mathsf{F}, \Theta, \boldsymbol{\nabla}_{\!R}\Theta), \quad \mathsf{T}(\mathsf{F}, \Theta, \boldsymbol{\nabla}_{\!R}\Theta), \quad \eta(\mathsf{F}, \Theta, \boldsymbol{\nabla}_{\!R}\Theta), \quad \boldsymbol{Q}(\mathsf{F}, \Theta, \boldsymbol{\nabla}_{\!R}\Theta). \tag{6.1}$$

The Clausius–Duhem inequality in free-energy form (4.29) implies that

$$-\rho\dot{\Psi} - \rho\eta\dot{\Theta} - \frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!R}\Theta + \mathsf{T} : \mathsf{D} \geq 0.$$

From the proposed constitutive laws (6.1) and the chain rule the material derivative of the free energy is

$$\dot{\Psi} = \frac{\partial\Psi}{\partial F_{IJ}}\dot{F}_{IJ} + \frac{\partial\Psi}{\partial\Theta}\dot{\Theta} + \frac{\partial\Psi}{\partial\Theta_{,I}}\dot{\Theta}_{,I} = \frac{\partial\Psi}{\partial F_{IJ}}L_{IK}F_{KJ} + \frac{\partial\Psi}{\partial\Theta}\dot{\Theta} + \frac{\partial\Psi}{\partial\Theta_{,I}}\dot{\Theta}_{,I},$$

after using equation (2.51). where for simplicity we have chosen to represent quantities in Cartesian coordinates. Hence, in dyadic form

$$\dot{\Psi} = \frac{\partial\Psi}{\partial\mathsf{F}}\mathsf{F}^T : \mathsf{L} + \frac{\partial\Psi}{\partial\Theta}\dot{\Theta} + \frac{\partial\Psi}{\partial\boldsymbol{\nabla}_{\!R}\Theta}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!R}\dot{\Theta}. \tag{6.2}$$

Using equation (6.2) in the Clausius–Duhem inequality yields

$$-\rho\left[\frac{\partial\Psi}{\partial\mathsf{F}}\mathsf{F}^T:\mathsf{L}+\frac{\partial\Psi}{\partial\Theta}\dot{\Theta}+\frac{\partial\Psi}{\partial\boldsymbol{\nabla}_{\!R}\Theta}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!R}\dot{\Theta}\right]-\rho\eta\dot{\Theta}-\frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!R}\Theta+\mathsf{T}:\mathsf{D}\geq 0.$$

From the derivation in section 4.4.1 or by using the symmetry of the Cauchy stress tensor we know that

$$\mathsf{T}:\mathsf{L}=T_{IJ}L_{IJ}=\frac{1}{2}\left(T_{IJ}L_{IJ}+T_{JI}L_{JI}\right)=T_{IJ}\frac{1}{2}\left(L_{IJ}+L_{JI}\right)=\mathsf{T}:\mathsf{D}.$$

Using the above expression for the stress power and gathering terms in the Clausius–Duhem inequality gives

$$\left[-\rho\frac{\partial\Psi}{\partial\mathsf{F}}\mathsf{F}^T+\mathsf{T}\right]:\mathsf{L}+\left(-\rho\frac{\partial\Psi}{\partial\Theta}-\rho\eta\right)\dot{\Theta}-\rho\frac{\partial\Psi}{\partial\boldsymbol{\nabla}_{\!R}\Theta}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!R}\dot{\Theta}-\frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!R}\Theta\geq 0,$$

which has to be true for all valid thermomechanical processes. One such process is the instantaneous application of a temperature gradient to an initial state with uniform temperature and no deformation. In that case, the inequality becomes

$$\rho\frac{\partial\Psi}{\partial\boldsymbol{\nabla}_{\!R}\Theta}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!R}\dot{\Theta}\leq 0;$$

and because the temperature gradient can be positive or negative, and Ψ does not depend on $\dot{\Theta}$, the only way that the inequality can be satisfied is for

$$\frac{\partial\Psi}{\partial\boldsymbol{\nabla}_{\!R}\Theta}=0. \tag{6.3}$$

Thus, the free energy cannot depend on temperature gradients.

In addition, considering an isothermal process ($\dot{\Theta}=0$, $\boldsymbol{\nabla}_{\!R}\Theta=\mathbf{0}$), we deduce that

$$\mathsf{T}=\rho\frac{\partial\Psi}{\partial\mathsf{F}}\mathsf{F}^T, \tag{6.4a}$$

in order for the inequality to be true for any possible rate of deformation. Similarly, for a process that does not involve any change in deformation, but simply a change in the uniform temperature, the inequality is only satisfied if

$$\eta=-\frac{\partial\Psi}{\partial\Theta}. \tag{6.4b}$$

From the constraint (6.3) the Helmholtz free energy cannot depend on temperature gradients and equations (6.4a,b) demonstrate that the Cauchy stress and the entropy cannot depend on temperature gradients either.

Once the constraints (6.3) and (6.4a,b) are applied the dissipation is zero, as expected, and Clausius–Duhem inequality reduces to

$$-\frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!R}\Theta\geq 0,$$

and because $\Theta>0$, it follows that

$$\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!R}\Theta\leq 0, \tag{6.5}$$

which cannot be reduced further because $\boldsymbol{Q}$ is not necessarily independent of $\boldsymbol{\nabla}_{\!R}\Theta$.

It follows that a general thermoelastic material can be described entirely by the Helmholtz free energy and a heat flux vector

$$\Psi(\mathsf{F}, \Theta), \quad \boldsymbol{Q}(\mathsf{F}, \Theta, \boldsymbol{\nabla}_{\!\boldsymbol{R}}\Theta).$$

For this reason much of the constitutive modelling of elastic bodies is concentrated on modelling the free energy.

In order that the material behaviour is objective then the free energy and heat flux must remain invariant under change in observer[1] $\boldsymbol{R}^* = \mathsf{Q}(t)\boldsymbol{R}$. A little thought shows that the measure of deformation chosen must be based on the Lagrangian representation in order to remain invariant under change in Eulerian observer[2]. The argument is essentially that the free energy is a scalar function of the deformation measure and temperature, so $\Psi^*(\mathsf{a}^*, \Theta^*) = \Psi(\mathsf{a}^*, \Theta^*) = \Psi(\mathsf{a}, \Theta)$, where a is our deformation measure. Hence, the deformation measure must remain invariant under observer transformation $\mathsf{a}^* = \mathsf{a}$ in order that the free energy also remains invariant.

A more long-winded approach is to use the transformation rules, remembering that we do not want the constitutive laws to change form under observer transform,

$$\Psi^*(\mathsf{F}^*, \Theta^*) = \Psi(\mathsf{F}^*, \Theta^*) = \Psi(\mathsf{QF}, \Theta) = \Psi(\mathsf{F}, \Theta), \tag{6.6a}$$

$$\boldsymbol{Q}^*(\mathsf{F}^*, \Theta^*, \boldsymbol{\nabla}_{\!\boldsymbol{R}^*}\Theta^*) = \boldsymbol{Q}(\mathsf{QF}, \Theta, \mathsf{Q}\boldsymbol{\nabla}_{\!\boldsymbol{R}}\Theta) = \mathsf{Q}\boldsymbol{Q}(\mathsf{F}, \Theta, \boldsymbol{\nabla}_{\!\boldsymbol{R}}\Theta). \tag{6.6b}$$

The transformation of the gradient follows from the chain rule and fact that Q is orthogonal:

$$\left[\boldsymbol{\nabla}_{\!\boldsymbol{R}^*}\Theta\right]_I = \frac{\partial \Theta}{\partial X_I^*} = \frac{\partial \Theta}{\partial X_J}\frac{\partial X_J}{\partial X_I^*};$$

but,

$$X_I^* = Q_{IJ}X_J \quad \Rightarrow \quad Q^T_{KI}X_I^* = Q^T_{KI}Q_{IJ}X_J = \delta_{KJ}X_J = X_K \quad \Rightarrow \quad \frac{\partial X_K}{\partial X_I^*} = Q^T_{KI};$$

and so

$$\left[\boldsymbol{\nabla}_{\!\boldsymbol{R}^*}\Theta\right]_I = \frac{\partial \Theta}{\partial X_J}Q^T_{JI} = Q_{IJ}\frac{\partial \Theta}{\partial X_J} \quad \Rightarrow \quad \boldsymbol{\nabla}_{\!\boldsymbol{R}^*}\Theta = \mathsf{Q}\boldsymbol{\nabla}_{\!\boldsymbol{R}}\Theta.$$

Equations (6.6a,b) must be satisfied for every possible choice of Q. In particular they must be true for the orthogonal matrix R^T in the polar decomposition $\mathsf{F} = \mathsf{RU}$. Hence,

$$\Psi(\mathsf{R}^T\mathsf{F}, \Theta) = \Psi(\mathsf{U}, \Theta) = \Psi(\mathsf{F}, \Theta) \text{ and } \mathsf{R}\boldsymbol{Q}(\mathsf{R}^T\mathsf{F}, \Theta, \mathsf{R}^T\boldsymbol{\nabla}_{\!\boldsymbol{R}}\Theta) = \mathsf{R}\boldsymbol{Q}(\mathsf{U}, \Theta, \mathsf{R}^T\boldsymbol{\nabla}_{\!\boldsymbol{R}}\Theta) = \boldsymbol{Q}(\mathsf{F}, \Theta, \boldsymbol{\nabla}_{\!\boldsymbol{R}}\Theta).$$

Hence, the constitutive relations must depend only on the right stretch tensor or, equivalently, the right Cauchy–Green deformation tensor, c, because $\mathsf{U}^2 = \mathsf{c} = \mathsf{F}^T\mathsf{F}$. The right Cauchy–Green deformation tensor is invariant under change in Eulerian observer, as expected.

For the heat flux, one way to ensure that the entropy constraint (equation 6.5) is satisfied is to define

$$\boldsymbol{Q}(\mathsf{c}, \Theta, \boldsymbol{\nabla}_{\!\boldsymbol{R}}\Theta) = -\mathsf{K}(\mathsf{c}, \Theta)\boldsymbol{\nabla}_{\!\boldsymbol{R}}\Theta.$$

In fact, it is a reasonable assumption on physical grounds that the heat flux depends on the temperature gradient and is often known as Fourier's Law. Under this assumption,

$$Q_I = -K_{IJ}\Theta_{,J} \tag{6.7}$$

[1]Here we reach another unfortunate clash in standard notation. The heat flux vector $\boldsymbol{Q}$ and the orthogonal matrix, Q, are both represented by a capital letter Q. Hopefully the choice of fonts will keep the distinction clear; a vector is not the same entity as a second-order tensor after all.

[2]Recall from section 5.2.2 that the Lagrangian measures are such that $\mathsf{c}^* = \mathsf{c}$, $\mathsf{e}^* = \mathsf{e}$.

and therefore

$$\boldsymbol{Q}\cdot\boldsymbol{\nabla}_{\!R}\Theta = Q_I\Theta_{,I} = -K_{IJ}\Theta_{,J}\Theta_{,I} \leq 0,$$

which will be true provided that K is positive semi-definite (all the eigenvalues are positive or zero). Moreover, objectivity is guaranteed provided that

$$-\mathsf{QKQ}^T\boldsymbol{\nabla}_{\!R}\Theta = -\mathsf{K}\boldsymbol{\nabla}_{\!R}\Theta,$$

and because this must be true for all possible temperature gradients, it follows that $\mathsf{K} = \kappa(\mathsf{c}, \Theta)\mathsf{I}$ is a second-order isotropic tensor, known as the heat conductivity tensor. Anisotropy can only be included if we change our constitutive assumptions to include dependence on particular directions.

6.2 Hyperelastic materials

A material is said to be hyperelastic (or Green elastic) if there exists a strain energy function $\mathcal{W}$ such that the stress power per unit undeformed volume is the material derivative of the strain energy.

We have already established from thermodynamic constraints that a thermoelastic material can be described by a Helmholtz free energy and heat flux. The components of the Cauchy stress are given by

$$T_{IJ} = \rho\frac{\partial\Psi}{\partial F_{IK}}F_{JK},$$

which means that the stress power per unit deformed volume is

$$T_{IJ}L_{IJ} = \rho\frac{\partial\Psi}{\partial F_{IK}}F_{JK}L_{IJ} = \rho\frac{\partial\Psi}{\partial F_{IK}}\dot{F}_{IK} = \rho\dot{\Psi}.$$

The last equality is only true if the temperature during the deformation remains constant. Equivalently, we can use the constraint (6.4b), to write $\Psi(\mathsf{F}, \eta)$ and then the equality is valid if the entropy is constant during the deformation.

The stress power per unit undeformed volume is therefore

$$\rho\dot{\Psi}J = \rho\dot{\Psi}\frac{\rho_0}{\rho} = \rho_0\dot{\Psi} = \rho_0\dot{\psi}.$$

The above equation follows from the relationship between the two volume elements, equation (2.41a), which states that $\mathrm{d}\mathcal{V}_t = J\,\mathrm{d}\mathcal{V}_0$. The density ρ_0 is constant and if we now write the stress power per unit undeformed volume as the material derivative of the strain energy

$$\dot{\mathcal{W}} = \rho_0\dot{\psi},$$

then the expression can be integrated directly to demonstrate the existence of the strain energy function

$$\mathcal{W} = \rho_0\psi. \tag{6.8}$$

Hence, for isothermal or isentropic deformations the strain energy function exists and is simply the density of the undeformed body multiplied by the free energy. The mechanical behaviour of the material is then determined entirely by its deformation, in which case $\mathcal{W}(\mathsf{c})$.

If a strain energy function exists then in Lagrangian variables from equation (4.12)

$$\dot{\mathcal{W}} = s^{ij}\dot{\gamma}_{ij},$$

where s^{ij} are components of the second Piola–Kirchhoff stress tensor and γ_{ij} are components of the Green–Lagrange strain tensor. The Green–Lagrange strain tensor is invariant under Eulerian observer transformation and is directly related to the right Cauchy–Green deformation tensor ($\mathsf{e} = \mathsf{c} - \mathsf{I}$), which means that we can write $\mathcal{W}(\gamma_{ij})$. Hence,

$$\dot{\mathcal{W}} = \frac{\partial \mathcal{W}}{\partial \gamma_{ij}} \dot{\gamma}_{ij} = s^{ij} \dot{\gamma}_{ij},$$

and because the rate of deformation is arbitrary it follows that

$$s^{ij} = \frac{\partial \mathcal{W}}{\partial \gamma_{ij}}.$$

Alternative stress measures can be derived from the strain energy function by using kinematic relationships and the chain rule. Another simple relationship is given in terms of the first Piola–Kirchhoff stress tensor because

$$\dot{\mathcal{W}} = \mathsf{p} : \dot{\mathsf{F}} = p_{IJ} \dot{F}_{IJ}, \tag{6.9}$$

see example sheet 4. From the chain rule

$$\dot{\mathcal{W}} = \frac{\partial \mathcal{W}}{\partial F_{IJ}} \dot{F}_{IJ}, \tag{6.10}$$

and by comparing equations (6.9) and (6.10) we see that

$$p_{IJ} = \frac{\partial \mathcal{W}}{\partial F_{IJ}}, \tag{6.11}$$

which may be generalised to arbitrary curvilinear coordinates

$$p_{\bar{i}}{}^{j} = \frac{\partial \xi^j}{\partial x_J} \frac{\partial X_I}{\partial \chi^{\bar{i}}} p_{IJ} = \frac{\partial \mathcal{W}}{\partial(\partial X_I / \partial x_J)} \frac{\partial \xi^j}{\partial x_J} \frac{\partial X_I}{\partial \chi^{\bar{i}}} = \frac{\partial \mathcal{W}}{\partial(\partial \chi^{\bar{i}} / \partial \xi^j)} = \frac{\partial \mathcal{W}}{\partial \chi^{\bar{i}}_{,j}}.$$

A wide variety of material behaviours can be described by using different strain energy functions. We shall concentrate on the simplest: homogeneous, isotropic materials.

6.2.1 Homogeneous, Isotropic Materials

The simplest elastic materials are homogeneous (independent of material coordinate) and isotropic (with no preferred direction). In this case the strain energy must depend only on the invariants of the strain tensor because the result must be the same for any local change in material coordinates and cannot depend explicitly on the material coordinates. In other words,

$$\mathcal{W}(I_1, I_2, I_3),$$

where I_1, I_2 and I_3 are the strain invariants defined in section 2.4.3. Hence,

$$s^{ij} = \frac{\partial \mathcal{W}}{\partial \gamma_{ij}} = \frac{\partial \mathcal{W}}{\partial I_1} \frac{\partial I_1}{\partial \gamma_{ij}} + \frac{\partial \mathcal{W}}{\partial I_2} \frac{\partial I_2}{\partial \gamma_{ij}} + \frac{\partial \mathcal{W}}{\partial I_3} \frac{\partial I_3}{\partial \gamma_{ij}}; \tag{6.12}$$

and from the definitions of the strain invariants

$$I_1 = G^i_i = g^{ij} G_{ji} = g^{ij} G_{ij} = 2 g^{ij} \gamma_{ij} + g^{ij} g_{ij} = 2 g^{ij} \gamma_{ij} + \delta^i_i = 3 + 2 g^{ij} \gamma_{ij},$$

$$I_2 = \frac{1}{2}\left[g^{ik}G_{ik}g^{jl}G_{jl} - g^{ik}G_{jk}g^{jl}G_{il}\right] = \frac{1}{2}\left[g^{ik}g^{jl}\left(2\gamma_{ik}+g_{ik}\right)\left(2\gamma_{jl}+g_{jl}\right) - g^{ik}g^{jl}\left(2\gamma_{jk}+g_{jk}\right)\left(2\gamma_{il}+g_{il}\right)\right].$$

$$I_3 = G/g.$$

Thus,

$$\frac{\partial I_1}{\partial \gamma_{ij}} = 2g^{ij}, \tag{6.13a}$$

$$\frac{\partial I_2}{\partial \gamma_{ij}} = \frac{1}{2}\left[g^{ij}g^{kl}2G_{kl} + g^{lk}g^{ji}2G_{lk} - g^{ik}g^{jl}2G_{kl} - g^{ik}g^{jl}2G_{lk}\right] = 2\left(g^{ij}g^{kl} - g^{ik}g^{jl}\right)G_{kl}, \tag{6.13b}$$

$$\frac{\partial I_3}{\partial \gamma_{ij}} = \frac{1}{g}\frac{\partial G}{\partial \gamma_{ij}} = \frac{1}{g}\frac{\partial G}{\partial G_{kl}}\frac{\partial G_{kl}}{\partial \gamma_{ij}} = \frac{G}{g}G^{kl}\,2\,\delta^i_k\,\delta^j_l = 2\,I_3\,G^{ij}, \tag{6.13c}$$

where we have used the equation (2.53), $\partial G/\partial G_{kl} = GG^{kl}$; the fact that the undeformed metric tensor is fixed; and that

$$G_{kl} = 2\gamma_{kl} + g_{kl} = 2\gamma_{ij}\,\delta^i_k\,\delta^j_l + g_{kl} \quad \Rightarrow \quad \frac{\partial G_{kl}}{\partial \gamma_{ij}} = 2\,\delta^i_k\,\delta^j_l.$$

Using the expressions (6.13a–c) in the expression (6.12) gives

$$s^{ij} = 2\frac{\partial \mathcal{W}}{\partial I_1}g^{ij} + \frac{\partial \mathcal{W}}{\partial I_2}2\left(g^{ij}g^{kl} - g^{ik}g^{jl}\right)G_{kl} + \frac{\partial \mathcal{W}}{\partial I_3}2I_3G^{ij},$$

which can be rewritten in the form

$$s^{ij} = a\,g^{ij} + b\;B^{ij} + p\,G^{ij}, \tag{6.14}$$

where

$$a = 2\frac{\partial \mathcal{W}}{\partial I_1}, \quad b = 2\frac{\partial \mathcal{W}}{\partial I_2}, \quad p = 2I_3\frac{\partial \mathcal{W}}{\partial I_3},$$

and

$$B^{ij} = \left(g^{ij}g^{kl} - g^{ik}g^{jl}\right)G_{kl} = I_1g^{ij} - g^{ik}g^{jl}G_{kl}.$$

Note that in general a, b and p will be functions of position and time because the strain invariants can vary with position and time.

Once we have an expression for one of the stresses then any of the others can be found from the standard transformations. For example, the body stress (or Cauchy stress in convected coordinates, the basis in the deformed configuration associated with the Lagrangian coordinates), is given by

$$JT^{ij} = \sqrt{I_3}T^{ij} = s^{ij},$$

which means that

$$T^{ij} = Ag^{ij} + BB^{ij} + PG^{ij}, \tag{6.15}$$

where

$$A = \frac{a}{\sqrt{I_3}} = \frac{2}{\sqrt{I_3}}\frac{\partial \mathcal{W}}{\partial I_1}, \quad B = \frac{b}{\sqrt{I_3}} = \frac{2}{\sqrt{I_3}}\frac{\partial \mathcal{W}}{\partial I_2}, \quad P = \frac{p}{\sqrt{I_3}} = 2\sqrt{I_3}\frac{\partial \mathcal{W}}{\partial I_3}.$$

6.2.2 Incompressibility constraints

If the body is incompressible then its volume cannot change and so $I_3 = 1$, see section 4.1. Hence, the strain energy function does not vary with I_3 because I_3 is fixed. In general, if a continuum is subject to a scalar internal constraint on the deformation, it can be written in the objective form

$$C(\gamma_{ij}) = 0. \tag{6.16}$$

For example, the incompressibility constraint is that

$$\sqrt{G/g} - 1 = 0. \tag{6.17}$$

Taking the material derivative of equation (6.16) yields

$$\dot{C} = \frac{\partial C}{\partial \gamma_{ij}} \dot{\gamma}_{ij} = 0.$$

and so we can add an arbitrary multiple of $\partial C/\partial \gamma_{ij}$ to the second Piola–Kirchhoff stress tensor without altering the stress power, $\mathsf{s} : \dot{\mathsf{e}}$:

$$s^{ij} = \frac{\partial \mathcal{W}}{\partial \gamma_{ij}} + q \frac{\partial C}{\partial \gamma_{ij}}.$$

The field $q(\boldsymbol{x}, t)$ is an unknown (Lagrange) multiplier that is the part of the stress responsible for enforcing the constraint.

For the incompressibility constraint (6.17)

$$\frac{\partial C}{\partial \gamma_{ij}} = \frac{\partial \sqrt{G/g}}{\partial \gamma_{ij}} = \frac{1}{2\sqrt{Gg}} \frac{\partial G}{\partial G_{kl}} \frac{\partial G_{kl}}{\partial \gamma_{ij}} = \frac{1}{2\sqrt{Gg}} G G^{kl} 2 \delta^i_k \delta^j_l = \sqrt{G/g} G^{ij} = G^{ij},$$

where we have used equation (2.53); the facts that $G_{ij} = 2\gamma_{ij} + g_{ij}$ and g_{ij} does not vary with γ_{ij} (or anything in fact); and that $\sqrt{G/g} = \sqrt{I_3} = 1$.

Thus, for incompressible materials,

$$s^{ij} = \frac{\partial \mathcal{W}(I_1, I_2)}{\partial \gamma_{ij}} + q G^{ij},$$

and on comparison with equation (6.14) we see that $q = p$ and once again

$$s^{ij} = a\, g^{ij} + b\, B^{ij} + p\, G^{ij}, \tag{6.18}$$

where, as before,

$$a = 2\frac{\partial \mathcal{W}}{\partial I_1}, \quad b = 2\frac{\partial \mathcal{W}}{\partial I_2},$$

but $p(\boldsymbol{r}, t)$ is now an independent (pressure) field that is responsible for enforcing the incompressibility constraint. Note that the choice of sign for p is different from that of the ideal gas.

6.2.3 The governing equations for hyperelastic materials

In a hyperelastic material, conservation of mass and the entropy inequality have already been used in the formulation of the strain energy function and balance of angular momentum is built into the symmetry properties of the stress tensor. The system is isothermal (or isentropic), which means that there can be no external sources of heat or entropy, so the energy equation becomes

$$\rho \frac{D\Phi}{Dt} = \mathsf{T} : \mathsf{D} = \frac{\rho}{\rho_0}\frac{D\mathcal{W}}{Dt} \quad \Rightarrow \quad \Phi = \frac{1}{\rho_0}\mathcal{W} + \text{constant}.$$

In other words the strain energy is the internal energy of the solid, apart from an arbitrary reference value, which means that from equation (4.15) the material derivative of the total kinetic and strain energy is equal to the power supplied by the external loads

$$\frac{D}{Dt}(U + K) = \int_{\partial\Omega_t} \boldsymbol{T} \cdot \boldsymbol{V} \, \mathrm{d}\mathcal{S}_t + \int_{\Omega_t} \rho \boldsymbol{F} \cdot \boldsymbol{V} \, \mathrm{d}\mathcal{V}_t, \tag{6.19}$$

where the total internal energy is given by $U = \int_{\Omega_t} \frac{\rho}{\rho_0}\mathcal{W}\,\mathrm{d}\mathcal{V}_t$. If there are no external forces then the total energy is conserved and simply converted between the strain energy and kinetic energy. Thus, the strain energy is equivalent to a potential energy in classic particle dynamics; and for this reason the strain energy is sometimes termed elastic potential energy.

We can use the equation (6.19) to solve problems, but it is not independent of the linear momentum equation. In fact, it follows directly from the linear momentum equation in integral form after taking the dot product with $\boldsymbol{V}$ and integrating with respect to time.

Thus, the only remaining equation is the linear momentum equation or Cauchy's equation, which from equation (4.8) is given in components in the covariant basis associated with the Eulerian coordinates $\chi^{\bar{i}}$ in the deformed configuration, $\boldsymbol{G}_{\bar{i}} = \frac{\partial \boldsymbol{R}}{\partial \chi^{\bar{i}}}$ by

$$\rho\left[\frac{\partial V^{\bar{i}}}{\partial t} + V^{\bar{j}} V^{\bar{i}}\|_{\bar{j}}\right] = T^{\bar{i}\bar{j}}\|_{\bar{j}} + \rho F^{\bar{i}}, \tag{6.20}$$

where the symmetry of the stress tensor has been used. The symbol $\|_{\bar{j}}$ refers to covariant differentiation with respect to the base vectors $\boldsymbol{G}_{\bar{j}}$, so that

$$V^{\bar{i}}\|_{\bar{j}} = \frac{\partial V^{\bar{i}}}{\partial \chi^{\bar{j}}} + \overline{\Gamma}^{\bar{i}}_{\bar{k}\bar{j}} V^{\bar{k}}, \quad T^{\bar{i}\bar{j}}\|_{\bar{j}} = \frac{\partial T^{\bar{i}\bar{j}}}{\partial \chi^{\bar{j}}} + \overline{\Gamma}^{\bar{j}}_{\bar{j}\bar{k}} T^{\bar{i}\bar{k}} + \overline{\Gamma}^{\bar{i}}_{\bar{k}\bar{j}} T^{\bar{k}\bar{j}} \quad \text{and} \quad \overline{\Gamma}^{\bar{i}}_{\bar{k}\bar{j}} = \boldsymbol{G}^{\bar{i}} \cdot \frac{\partial \boldsymbol{G}_{\bar{j}}}{\partial \chi^{\bar{k}}}.$$

In fact, it is more convenient to work in the convected (Lagrangian) coordinates, ξ^i, so that we can use the expression for the body stress from the strain energy function (6.15) directly. The equation (6.20) was written in tensorial form, so we need simply remove the overbars to convert to the new coordinate system[3]

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[3]You might wonder why the acceleration term is not simply $\partial V^i/\partial t$ because we are now working with Lagrangian coordinates ξ^i. The issue is that we are working with the convected Lagrangian coordinates in the deformed configuration and so the derivative of the entire velocity vector is

$$\frac{D\boldsymbol{V}}{Dt} = \frac{D(V^i \boldsymbol{G}_i)}{Dt} = \frac{DV^i}{Dt}\boldsymbol{G}_i + V^i \frac{D\boldsymbol{G}_i}{Dt} = \frac{\partial V^i}{\partial t}\boldsymbol{G}^i + V^i V^k\|_i \boldsymbol{G}_k,$$

after using equation (2.46) to evaluate $D\boldsymbol{G}_i/Dt$. Thus, the component of the acceleration in the direction $\boldsymbol{G}_i$ is

$$\frac{\partial V^i}{\partial t} + V^j V^i\|_j,$$

and we can see that the second (advective) term arises from the variation of the Lagrangian basis vectors as the body deforms.

$$\rho\left[\frac{\partial V^i}{\partial t} + V^j V^i||_j\right] = T^{ij}||_j + \rho F^i. \tag{6.21}$$

Here, the symbol $||_j$ refers to covariant differentiation with respect to the base vectors $\boldsymbol{G}_j$, so that

$$V^i||_j = \frac{\partial V^i}{\partial \xi^j} + \overline{\Gamma}^i_{kj} V^k, \quad T^{ij}||_j = \frac{\partial T^{ij}}{\partial \xi^j} + \overline{\Gamma}^j_{jk} T^{ik} + \overline{\Gamma}^i_{kj} T^{kj} \quad \text{and} \quad \overline{\Gamma}^i_{kj} = \boldsymbol{G}^i \boldsymbol{\cdot} \frac{\partial \boldsymbol{G}_j}{\partial \xi^k}.$$

Naturally, the governing equations can also be expressed in Lagrangian form in components in the basis $\boldsymbol{g}_i$

$$\rho_0 \frac{\partial v^i}{\partial t} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^k} \left(\sqrt{g} p^{\bar{j}k} \boldsymbol{G}_{\bar{j}} \right) \boldsymbol{\cdot} \boldsymbol{g}^i + \rho_0 f^i,$$

which is, of course, much simpler in Cartesian coordinates

$$\rho_0 \frac{\partial v_I}{\partial t} = \frac{\partial p_{JI}}{\partial x_J} + \rho_0 f_I.$$

In fact, the equation can be expressed entirely in terms of the displacement $\boldsymbol{u}(\boldsymbol{r}, t)$ because $v_I = \partial u_I / \partial t$ and

$$\frac{\partial p_{JI}}{\partial x_J} = \frac{\partial p_{JI}}{\partial F_{KL}} \frac{\partial F_{KL}}{\partial x_J} = \frac{\partial p_{JI}}{\partial F_{KL}} \frac{\partial^2 X_K}{\partial x_J \partial x_L} = \frac{\partial^2 \mathcal{W}}{\partial F_{KL} \partial F_{JI}} \frac{\partial^2 (x_K + u_K)}{\partial x_J \partial x_L} = \frac{\partial^2 \mathcal{W}}{\partial F_{KL} \partial F_{JI}} \frac{\partial^2 u_K}{\partial x_J \partial x_L},$$

which means that the governing equation becomes

$$\rho_0 \frac{\partial^2 u_I}{\partial^2 t} = \frac{\partial^2 \mathcal{W}}{\partial F_{KL} \partial F_{JI}} \frac{\partial^2 u_K}{\partial x_J \partial x_L} + \rho_0 f_I.$$

Most analytic treatments of the equations favour working in the deformed configuration because it is then easier to apply the physical boundary conditions. For numerical work, it is easier to work in the Lagrangian formulation so that the domain remains fixed.

6.2.4 Common strain energy functions

A number of strain energy functions have been proposed over the years. It is common to write the functions in terms of the invariants, but it is also possible to write the strain energy in terms of the principal stretches, λ_1, λ_2 and λ_3. By considering the deformed metric tensor in the eigenbasis, equation (2.39), the strain invariants are seen to be

$$I_1 = \text{trace}(\mathcal{G}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{6.22a}$$

$$I_2 = \frac{1}{2}\left[(\text{trace}(\mathcal{G}))^2 - \text{trace}(\mathcal{G}^2)\right] = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \tag{6.22b}$$

$$I_3 = \det(\mathcal{G}) = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{6.22c}$$

Some of the most popular strain energy functions are shown below

- **Neo-Hookean**

$$\mathcal{W} = C_1(I_1 - 3) + cI_3 - d\log(\sqrt{I}_3), \tag{6.23}$$

where C_1, c and d are constants. The original Neo-Hookean model was incompressible and here the choice of dependence of I_3 ensures that the strain energy becomes unbounded under extreme extension $I_3 \to \infty$ or compression $I_3 \to 0$.

- **Mooney–Rivlin**

$$\mathcal{W} = C_1(I_1 - 3) + C_2(I_2 - 3) + cI_3 - d\log(\sqrt{I}_3), \tag{6.24}$$

where C_1, C_2, c and d are constants. The original Mooney–Rivlin model is also incompressible and the dependence on I_3 is chosen to be the same as in the Neo-Hookean model.

- **General Ogden Models**
Ogden considered very general forms of the strain energy function given in terms of the invariants. He based the form on the fact that for an unstrained body the strain energy should be zero.

$$\mathcal{W} = \sum_{p,q,r} c_{pqr}(I_1 - 3)^p(I_2 - 3)^q(I_3 - 1)^r, \tag{6.25}$$

where c_{pqr} are constants. Note that the incompressible Neo–Hookean and Mooney–Rivlin models are special cases of this model.

An alternative form in terms of the principal stretches is given by

$$\mathcal{W} = \sum_{p,q,r} a_{pqr}\left[\left(\lambda_1^p(\lambda_2^q + \lambda_3^q) + \lambda_2^p(\lambda_1^q + \lambda_3^q) + \lambda_3^p(\lambda_1^q + \lambda_2^q)\right)(\lambda_1\lambda_2\lambda_3)^r - 6\right], \tag{6.26}$$

which satisfies the property that it is unchanged under permutation of the principal stretches and is zero when the body is unstrained; as you might expect, a_{pqr} are constants.

- **Specific Ogden**
In the case when the body is incompressible and $q = 0$ the second Ogden model reduces to

$$\mathcal{W} = \sum_{p} 2a_p\left(\lambda_1^p + \lambda_2^p + \lambda_3^p - 3\right), \tag{6.27}$$

which is equivalent to the Mooney–Rivlin model if the only non zero terms are $p = \pm 2$. The incompressibility constraint means that $\lambda_1^{-2} = \lambda_2^2\lambda_3^2$, etc. Ogden noticed that p does not have to be restricted to integer powers in equation (6.27).

- **St. Venant –Kirchhoff**

$$\mathcal{W} = \frac{\lambda}{2}\gamma_i^i\gamma_j^j + \mu(\gamma_j^i\gamma_i^j) = \frac{\lambda}{8}(I_1 - 3)^2 + \frac{\mu}{4}\left[(I_1 - 3)^2 + 4(I_1 - 3) - 2(I_2 - 3)\right], \tag{6.28}$$

where $\gamma_j^i = g^{ik}\gamma_{kj}$ and λ and μ are constants. This model fits within the framework of the general Odgen model, but it has been criticised because although it exhibits unbounded strain energy under extreme tension ($\lambda_{(I)} \to \infty$), it does not under extreme compression ($\lambda_{(I)} \to 0$).

6.2.5 Example: Pure Torsion of a circular cylinder

We wish to find the state of internal stress and strains associated with the purely torsional (twisting) deformation of a circular cylinder of undeformed length l and radius a. The deformation is such that the angle of twist increases linearly with distance along the cylinder, but that the cross-sections of the cylinder do not move out of plane. We shall assume that the cylinder is made of an incompressible, homogeneous and isotropic hyperelastic material so that after deformation the length is l and radius is a.

It is natural to treat the Lagrangian coordinates as a cylindrical polar coordinate system in the undeformed configuration, but our governing equation (6.21) is formulated in the Lagrangian coordinates associated with the deformed position. For this reason, so that we don't have to work with any more complicated Christoffel symbols than necessary, we shall consider a cylindrical polar coordinate system aligned with the cylinder, $\xi^1 = r$, $\xi^2 = \theta$ and $\xi^3 = z$, in which the deformed position is given by

$$X_1 = r\cos\theta, \quad X_2 = r\sin\theta, \quad X_3 = z, \quad \text{where } 0 \le r \le a, \quad 0 \le z \le l.$$

Before the torsional deformation

$$x_1 = r\cos(\theta - Cz), \quad x_2 = r\sin(\theta - Cz), \quad x_3 = z,$$

where C is a constant that expresses the magnitude of the deformation.

The deformed covariant base vectors in the global Cartesian coordinates are

$$\boldsymbol{G}_1 = (\cos\theta, \sin\theta, 0)^T, \quad \boldsymbol{G}_2 = (-r\sin\theta, r\cos\theta, 0)^T, \quad \boldsymbol{G}_3 = (0,0,1)^T,$$

so the deformed metric tensors are therefore given by

$$G_{ij} = \boldsymbol{G}_i \cdot \boldsymbol{G}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The undeformed covariant base vectors are

$$\boldsymbol{g}_1 = (\cos(\theta - Cz), \sin(\theta - Cz), 0)^T, \quad \boldsymbol{g}_2 = (-r\sin(\theta - Cz), r\cos(\theta - Cz), 0)^T,$$

$$\boldsymbol{g}_3 = (Cr\sin(\theta - Cz), -Cr\cos(\theta - Cz), 1)^T,$$

which leads to the undeformed metric tensors

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & -Cr^2 \\ 0 & -Cr^2 & 1 + C^2r^2 \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & C^2 + \frac{1}{r^2} & C \\ 0 & C & 1 \end{pmatrix}.$$

Calculating the determinants gives $g = G = r^2$, so $I_3 = G/g = 1$ and the deformation is indeed isochoric (volume preserving). The other two strain invariants are given by

$$I_1 = g^{ij}G_{ij} \equiv G^i_i = 3 + C^2r^2,$$

$$I_2 = \frac{1}{2}\left[I_1^2 - G^i_j G^j_i\right] = 3 + C^2r^2 = I_1.$$

Thus, the strain energy is a function only of r, which means that the functions A and B in the body stress tensor (6.15) are also functions only of r.

In addition,

$$B^{ij} = (3 + C^2r^2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & C^2 + \frac{1}{r^2} & C \\ 0 & C & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & C^2 + \frac{1}{r^2} & C \\ 0 & C & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & C^2 + \frac{1}{r^2} & C \\ 0 & C & 1 \end{pmatrix},$$

$$= \begin{pmatrix} 2 + C^2r^2 & 0 & 0 \\ 0 & C^2 + \frac{2}{r^2} & C \\ 0 & C & 2 \end{pmatrix}.$$

Hence, from equation (6.15),

$$T^{ij} = A\begin{pmatrix} 1 & 0 & 0 \\ 0 & C^2+\frac{1}{r^2} & C \\ 0 & C & 1 \end{pmatrix} + B\begin{pmatrix} 2+C^2r^2 & 0 & 0 \\ 0 & C^2+\frac{2}{r^2} & C \\ 0 & C & 2 \end{pmatrix} + P\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In other words, the only non-zero components of the stress tensor are

$$\begin{aligned} T^{11} =& A + (2 + C^2r^2)B + P, \\ T^{22} =& A(C^2 + 1/r^2) + B(C^2 + 2/r^2) + P/r^2, \\ T^{33} =& A + 2B + P, \\ T^{23} = T^{32} =& C(A + B), \end{aligned}$$

If the cylinder is in equilibrium and there are no body forces, then Cauchy's equations (6.21) become

$$T^{ij}||_j = T^{ij}_{,j} + \overline{\Gamma}^j_{jk}T^{ik} + \overline{\Gamma}^i_{kj}T^{kj} = 0.$$

In the deformed metric, cylindrical polars, the only non-zero Christoffel symbols are

$$\overline{\Gamma}^1_{22} = -r, \quad \overline{\Gamma}^2_{12} = \overline{\Gamma}^2_{21} = \frac{1}{r}.$$

Thus, the non-zero terms in the governing equations are

$$T^{11}_{,1} + \frac{1}{r}T^{11} - rT^{22} = 0, \tag{6.29a}$$
$$T^{22}_{,2} + T^{23}_{,3} = 0, \tag{6.29b}$$
$$T^{32}_{,2} + T^{33}_{,3} = 0. \tag{6.29c}$$

Now, C is constant and $A(r)$, $B(r)$, but because the pressure P is an independent field that enforces the incompressibility constraint, in principle $P(r, \theta, z)$. The first equation (6.29a) becomes

$$\frac{\partial}{\partial r}\left(A + (2 + C^2r^2)B + P\right) + A/r + (2+C^2r^2)B/r + P/r - A(rC^2+1/r) - B(C^2r+2/r) - P/r = 0,$$

$$\Rightarrow \frac{\partial}{\partial r}\left(A + (2 + C^2r^2)B + P\right) - ArC^2 = 0;$$

and the equations (6.29b,c) reduce to

$$\frac{\partial P}{\partial \theta} = \frac{\partial P}{\partial z} = 0.$$

Thus, the pressure is a function of r only and is given by

$$P = -A - B(2 + C^2r^2) + C^2\int^r A(s)s\,\mathrm{d}s + \text{constant},$$

after direct integration of equation (6.29a).

The outer unit normal to the curved surface of the cylinder is given by $\boldsymbol{G}^1/\sqrt{G^{11}} = \boldsymbol{G}^1 = \boldsymbol{G}_1$, because $G^{11} = 1$, so in the deformed coordinate system $N_1 = 1$, $N_2 = N_3 = 0$. Thus, if we assume that the curved surface at $r = a$ has a constant normal load of magnitude M then the boundary traction $\boldsymbol{T} = -M\boldsymbol{G}^1$, which means that

$$T^{ij}N_i = T^j \quad \Rightarrow \quad T^{11} = -N, \quad T^{12} = T^{13} = 0, \quad \text{at } r = a.$$

We already know that $T^{12} = T^{13} = 0$, and the only non-trivial condition is

$$T^{11}\Big|_{r=a} = -M \quad \Rightarrow \quad P(a) = -M - A(a) - (2 + C^2a^2)B(a),$$

which means that

$$P = -M - A - B(2 + C^2r^2) + C^2 \int_a^r A(s)s\,\mathrm{d}s.$$

The internal pressure therefore depends on the details of the constitutive model, but has the explicit form given above,

At the top ($z = l$) /bottom ($z = 0$) surfaces, the outer unit normal is given by $\pm\boldsymbol{G}^3 = \pm\boldsymbol{G}_3$, so $N_1 = N_2 = 0$ and $N_3 = \pm 1$. Thus, the traction at these surfaces is

$$T^j = \pm T^{3j},$$

and so

$$T^1 = 0,$$
$$T^2 = \pm C(A + B),$$
$$T^3 = \pm\left[A + 2B - M - A - 2B - BC^2r^2 + C^2\int_a^r A(s)s\,\mathrm{d}s\right] = \pm\left[-M + C^2\left(\int_a^r A(s)s\,\mathrm{d}s - Br^2\right)\right].$$

Thus, in order to maintain the deformation equal and opposite non-zero azimuthal and axial tractions must be applied at the ends of the cylinder. Integrating the tractions over the surface shows that there is no resultant azimuthal force, but there is a resultant moment (or couple). The resultant couple is to balance the twist within the cylinder and the resultant axial force is required to ensure that the volume of the cylinder does not change.

6.2.6 Principles of Virtual Work and Displacements

We can express the governing equations of hyperelasticity as a variational principle. Consider a deformable body that is loaded by a surface traction $\boldsymbol{T}$ and a body force $\boldsymbol{F}$. The linear momentum of the body is in balance and it is then subject to an instantaneous, infinitesimal virtual displacement field $\delta\boldsymbol{R}$ that is consistent with all boundary conditions[4].

[4] If you are unfamiliar with variational calculus then the important point is that $\delta\boldsymbol{R}$ represents an infinitesimal virtual change in the function $\boldsymbol{R}(\boldsymbol{r}, t)$, such that the new function $\boldsymbol{R}^*(\boldsymbol{r}, t) = \boldsymbol{R}(\boldsymbol{r}, t) + \delta\boldsymbol{R}(\boldsymbol{r})$. Here, the variation $\delta\boldsymbol{R} = \epsilon\boldsymbol{f}(\boldsymbol{r}) = \boldsymbol{R}^* - \boldsymbol{R}$, where $\epsilon \ll 1$ and $\boldsymbol{f}(\boldsymbol{r})$ can be any function that satisfies the appropriate (homogeneous) boundary conditions. The term virtual is used because we are not making the choice of any specific displacement, but considering all possible infinitesimal displacements, which is not the same as a specific infinitesimal change $\mathrm{d}\boldsymbol{R}$ caused by a known change in Lagrangian coordinate, $\mathrm{d}\xi^i$, for example.

The first variation commutes with derivatives and definite integrals because they are both linear operators:

$$\delta\frac{\partial\boldsymbol{R}}{\partial x} = \frac{\partial\boldsymbol{R}^*}{\partial x} - \frac{\partial\boldsymbol{R}}{\partial x} = \frac{\partial}{\partial x}(\boldsymbol{R}^* - \boldsymbol{R}) = \frac{\partial\delta\boldsymbol{R}}{\partial x},$$

$$\delta\int \boldsymbol{R}\,\mathrm{d}\mathcal{V}_0 = \int \boldsymbol{R}^*\,\mathrm{d}\mathcal{V}_0 - \int \boldsymbol{R}\,\mathrm{d}\mathcal{V}_0 = \int \boldsymbol{R}^* - \boldsymbol{R}\,\mathrm{d}\mathcal{V}_0 = \int \delta\boldsymbol{R}\,\mathrm{d}\mathcal{V}_0.$$

The principle of virtual work states that the net external work done by the virtual displacements is equal to the internal work done by the virtual strains consistent with the virtual displacements.

The net external virtual work (assuming only stress boundary conditions, so that $\delta\boldsymbol{R} \neq \boldsymbol{0}$ on the boundary) is given by

$$\delta W = \int_{\Omega_t} \left(\rho \boldsymbol{F} - \rho \frac{D\boldsymbol{V}}{Dt}\right) \boldsymbol{\cdot} \delta\boldsymbol{R} \, \mathrm{d}\mathcal{V}_t + \int_{\partial\Omega_t} \boldsymbol{T} \boldsymbol{\cdot} \delta\boldsymbol{R} \, \mathrm{d}\mathcal{S}_t, \tag{6.30}$$

and from the divergence theorem and the definition of the Cauchy stress tensor, the surface integral can be expressed in the form

$$\int_{\partial\Omega_t} \left(\delta\boldsymbol{R} \boldsymbol{\cdot} \frac{\boldsymbol{T}^i}{\sqrt{G}}\right) N_i \, \mathrm{d}\mathcal{S}_t = \int_{\Omega_t} \frac{1}{\sqrt{G}} \frac{\partial}{\partial \xi^i} \left(\delta\boldsymbol{R} \boldsymbol{\cdot} \boldsymbol{T}^i\right) \mathrm{d}\mathcal{V}_t = \int_{\Omega_t} \frac{1}{\sqrt{G}} \frac{\partial}{\partial \xi^i} \left(\sqrt{G} T^{ij} \boldsymbol{G}_j \boldsymbol{\cdot} \delta\boldsymbol{R}\right) \mathrm{d}\mathcal{V}_t,$$

$$= \int_{\Omega_t} T^{ij}||_i \, \delta R_j + T^{ij} \boldsymbol{G}_j \boldsymbol{\cdot} (\delta\boldsymbol{R})_{,i} \, \mathrm{d}\mathcal{V}_t.$$

Hence, equation (6.30) becomes

$$\delta W = \int_{\Omega_t} \left[\left(\rho F^j - \rho \frac{DV^j}{Dt} + T^{ij}||_i\right) \delta R_j + T^{ij} \boldsymbol{G}_j \boldsymbol{\cdot} (\delta\boldsymbol{R})_{,i}\right] \mathrm{d}\mathcal{V}_t = \int_{\Omega_t} T^{ij} \boldsymbol{G}_j \boldsymbol{\cdot} (\delta\boldsymbol{R})_{,i} \, \mathrm{d}\mathcal{V}_t, \tag{6.31}$$

after using the balance of linear momentum, equation (4.8). The equation (6.31) represents the internal work due to the virtual strains, which can be seen by using the symmetry property of the Cauchy stress tensor to write

$$\delta W = \int_{\Omega_t} T^{ij} \frac{1}{2} \left[\boldsymbol{G}_i \boldsymbol{\cdot} (\delta\boldsymbol{R})_{,j} + \boldsymbol{G}_j \boldsymbol{\cdot} (\delta\boldsymbol{R})_{,i}\right] \mathrm{d}\mathcal{V}_t = \int_{\Omega_t} T^{ij} \delta\gamma_{ij} \, \mathrm{d}\mathcal{V}_t, \tag{6.32}$$

because the first variation of the strain is given by

$$\delta\gamma_{ij} = \delta \frac{1}{2} \left(G_{ij} - g_{ij}\right) = \frac{1}{2} \delta \left(\boldsymbol{G}_i \boldsymbol{\cdot} \boldsymbol{G}_j\right) = \frac{1}{2} \left(\boldsymbol{G}_i \boldsymbol{\cdot} \delta\boldsymbol{R}_{,j} + \delta\boldsymbol{R}_{,i} \boldsymbol{\cdot} \boldsymbol{G}_j\right).$$

In deriving this equation, we have used the fact that the product rule applies to the variation symbol δ and the variation symbol and the partial derivative commute — properties that follow directly from the definition of δ.

Combining equations (6.30) and (6.32) we obtain a formal statement of the principle of virtual work:

$$\int_{\Omega_t} \left(\rho \boldsymbol{F} - \rho \frac{D\boldsymbol{V}}{Dt}\right) \boldsymbol{\cdot} \delta\boldsymbol{R} \, \mathrm{d}\mathcal{V}_t + \int_{\partial\Omega_t} \boldsymbol{T} \boldsymbol{\cdot} \delta\boldsymbol{R} \, \mathrm{d}\mathcal{S}_t = \int_{\Omega_t} T^{ij} \delta\gamma_{ij} \, \mathrm{d}\mathcal{V}_t, \tag{6.33}$$

which can be put into the Lagrangian representation by transforming the volume integral to the undeformed domain

$$\int_{\Omega_0} \rho_0 \left[\boldsymbol{f} - \frac{\partial \boldsymbol{v}}{\partial t}\right] \boldsymbol{\cdot} \delta\boldsymbol{R} - s^{ij} \delta\gamma_{ij} \, \mathrm{d}\mathcal{V}_0 + \int_{\partial\Omega_t} \boldsymbol{T} \boldsymbol{\cdot} \delta\boldsymbol{R} \, \mathrm{d}\mathcal{S}_t = 0. \tag{6.34}$$

The first variation of a general functional $J(\boldsymbol{R})$ is the directional derivative (Gâteaux derivative) in the direction $\delta\boldsymbol{R}$,

$$\delta J = \frac{\mathrm{d}}{\mathrm{d}\alpha} \left\{J(\boldsymbol{R} + \alpha \delta\boldsymbol{R})\right\}\big|_{\alpha=0}.$$

where α is a scalar.

The principle of virtual displacements is simply the principle of virtual work restated in terms only of variations in the displacements

$$\int_{\Omega_0} \rho_0 \left[\boldsymbol{f} - \frac{\partial \boldsymbol{v}}{\partial t} \right] \cdot \delta \boldsymbol{R} - s^{ij} \boldsymbol{R}_{,i} \cdot \delta \boldsymbol{R}_{,j} \, \mathrm{d}\mathcal{V}_0 + \int_{\partial\Omega_t} \boldsymbol{T} \cdot \delta \boldsymbol{R} \, \mathrm{d}\mathcal{S}_t = 0. \tag{6.35}$$

Strictly speaking this is not a variational principle because it cannot necessarily be written as the variation of a single functional. If the material is hyperelastic, however, there exists a strain energy function $\mathcal{W}$, and virtual internal work can be written as the variation of a single functional because the first variation of the total internal energy is then

$$\delta W = \delta \int_{\Omega_0} \rho_0 \psi \, \mathrm{d}\mathcal{V}_0 = \delta \int_{\Omega_0} \mathcal{W} \, \mathrm{d}\mathcal{V}_0 = \int_{\Omega_0} \delta \mathcal{W} \, \mathrm{d}\mathcal{V}_0 = \int_{\Omega_0} \frac{\partial \mathcal{W}}{\partial \gamma_{ij}} \delta \gamma_{ij} \, \mathrm{d}\mathcal{V}_0 = \int_{\Omega_0} s^{ij} \delta \gamma_{ij} \, \mathrm{d}\mathcal{V}_0.$$

If the body force and surface loads are conservative, so that they can be written as the gradients of a potential energy, G, then the variational principle can be formulated as the minimum of the total energy of the system. The fact that it is a minimum follows from an assumption that the strain energy is positive definite. Note that the existence of a variational principle provides formal justification for using Lagrange multipliers to enforce kinematic constraints.

A very brief introduction to finite element methods

Equation (6.35) is the basis of finite element methods for the numerical solution of the equations of nonlinear elasticity. We derived the governing variational principle (6.35) using general coordinates in which the undeformed position is given by

$$\boldsymbol{r}(\xi^i) = r^k(\xi^i) \boldsymbol{e}_k.$$

For computational purposes, unless there are special symmetries in the initial domain, it is easiest to choose the Lagrangian coordinates to be the global Cartesian coordinates. In that case the components of undeformed position are $r^j(\xi^i) = \xi^j$; the tangent vectors are Cartesian base vectors

$$\boldsymbol{g}_i = \boldsymbol{e}_i;$$

and the undeformed metric tensor is the Kronecker delta, $g_{ij} = \delta_{ij}$.

In the finite element approach, we approximate the Lagrangian coordinates by discrete sum over known basis functions, P_l

$$\xi^i = \sum_l \hat{\xi}^i_l \, P_l,$$

$\hat{\xi}^i_l$ is the i-th Lagrangian coordinate at the l-th node. It is common to take an isoparametric approach and use the same basis functions for the unknown deformed positions

$$R^k = \sum_l \hat{R}^k_l \, P_l.$$

The basis functions are fixed and never vary, which means that the variations in the position are given by variations only in the discrete variables

$$\delta \boldsymbol{R} = \sum_l \delta \hat{R}^k_l P_l \boldsymbol{e}_k \quad \text{and} \quad \delta \frac{\partial \boldsymbol{R}}{\partial \xi^j} = \sum_l \delta \hat{R}^k_l \frac{\partial P_l}{\partial \xi^j} \boldsymbol{e}_k.$$

Thus, the principle of virtual displacements (6.35) becomes

$$\sum_l \int_{\Omega_0} \left[s^{ij} \frac{\partial R^k}{\partial \xi^i} \frac{\partial P_l}{\partial \xi^j} - \rho_0 \left(f^k - \frac{\partial^2 R^k}{\partial t^2} \right) P_l \right] \delta \hat{R}_l^k \sqrt{g}\, \mathrm{d}\xi^1\, \mathrm{d}\xi^2\, \mathrm{d}\xi^3 - \int_{\partial \Omega_t} T_k P_l \delta \hat{R}_l^k \, \mathrm{d}\mathcal{S}_t = 0. \tag{6.36}$$

The discrete variations may be taken outside the integrals because they are not functions of space

$$\sum_l \left\{ \int_{\Omega_0} \left[s^{ij} \frac{\partial R^k}{\partial \xi^i} \frac{\partial P_l}{\partial \xi^j} - \rho_0 \left(f^k - \frac{\partial^2 R^k}{\partial t^2} \right) P_l \right] \sqrt{g}\, \mathrm{d}\xi^1\, \mathrm{d}\xi^2\, \mathrm{d}\xi^3 - \int_{\partial \Omega_t} [T_k P_l] \, \mathrm{d}\mathcal{S}_t \right\} \delta\, \hat{R}_l^k = 0. \tag{6.37}$$

The variations of the unknowns are independent, so the only way that equation (6.37) can be satisfied for all possible variations is for each term in braces to be zero, which gives one discrete equation for each unknown

$$\int_{\Omega_0} \left[s^{ij} \frac{\partial R^k}{\partial \xi^i} \frac{\partial P_l}{\partial \xi^j} - \rho_0 \left(f^k - \frac{\partial^2 R^k}{\partial t^2} \right) P_l \right] \sqrt{g}\, \mathrm{d}\xi^1\, \mathrm{d}\xi^2\, \mathrm{d}\xi^3 - \int_{\partial \Omega_t} [T_k P_l] \, \mathrm{d}\mathcal{S}_t = 0.$$

These contributions to the problem can be assembled in an element-by-element manner. We divide the undeformed domain into into elements, which are related to the support of the basis functions, and for each element compute the contribution to the discrete volume residual

$$\mathcal{R}_{kl} = \int_{\Omega_0} \left[s^{ij} \frac{\partial R^k}{\partial \xi^i} \frac{\partial P_l}{\partial \xi^j} - \left(f^k - \rho_0 \frac{\partial^2 R^k}{\partial t^2} \right) P_l \right] \sqrt{g}\, \mathrm{d}\xi^1\, \mathrm{d}\xi^2\, \mathrm{d}\xi^3$$

The integration takes place over the Lagrangian coordinates (undeformed domain), which means that we only need to generate a mesh in the undeformed domain — this can be a considerable advantage.

In order to complete the discrete equations, we must loop over the surfaces to add any tractions

$$\int_{\partial \Omega_t} [T_k P_l] \, \mathrm{d}\mathcal{S}_t.$$

Note that this integral is over the **deformed** surface, so the deformed normal vector must be computed from the local mapping $\boldsymbol{R}(\xi^i)$. Finally, we assemble the contributions into a global nonlinear residuals vector, which is typically solved by a multidimensional Newton method.

6.3 Linear Thermoelasticity

The governing equations for a general thermoelastic body are simply the balance of linear momentum and conservation of energy. In the Eulerian form these are

$$\rho \frac{D\boldsymbol{V}}{Dt} = \boldsymbol{\nabla}_{\!R} \cdot \mathsf{T} + \rho \boldsymbol{F}, \tag{6.38a}$$

$$\rho \frac{D\Phi}{Dt} = \mathsf{T} : \mathsf{D} + \rho B - \boldsymbol{\nabla}_{\!R} \cdot \boldsymbol{Q}. \tag{6.38b}$$

In addition, we have the relations that follow from entropy inequality

$$\mathsf{T} = \rho \frac{\partial \Psi}{\partial \mathsf{F}} \mathsf{F}^T, \quad \eta = -\frac{\partial \Psi}{\partial \Theta},$$

where $\Psi(\mathsf{F}, \Theta)$ is the Helmholtz free energy; and our assumed form for the heat flux

$$\mathbf{Q} = -\mathsf{K} \boldsymbol{\nabla}_{\boldsymbol{R}} \Theta.$$

In general, given specific Ψ, K, $\boldsymbol{F}$ and B, we must solve the coupled system of equations (6.38a) for a given set of boundary conditions. The boundary conditions will consist of specified displacements, tractions, temperatures or heat fluxes on the boundaries of the domain and we must also be given an initial position, initial velocity and initial temperature. The full solution of the nonlinear equations of thermoelasticity is difficult and is usually treated numerically. However, we can reduce the system to a simpler set of equations if we linearise about a known strain-free state.

In general solid mechanics problems, nonlinearity arises from two distinct sources: (i) geometric nonlinearity, which occurs when the displacements are large; and (ii) material nonlinearity, which is a consequence of the constitutive law. It is possible to construct a theory in which the displacements are large, but the strains remain small (such as the bending theory of shells), in which geometric nonlinearity is retained, but material nonlinearity is neglected. We could also construct a theory in which the deformations are small, but the material behaviour has a nonlinear dependence on temperature. In what follows, we shall eliminate all sources of nonlinearity by assuming both that the deformations are infinitesimal and that the constitutive law is linear. In other words, we shall assume that

$$\boldsymbol{R} = \boldsymbol{r} + \boldsymbol{u} = \boldsymbol{r} + \epsilon \tilde{\boldsymbol{u}} \quad \text{and} \quad \Theta = \theta = \theta_0 + \epsilon \tilde{\theta}, \tag{6.39}$$

where $\epsilon \ll 1$ and $\tilde{\boldsymbol{u}}$ and $\tilde{\theta}$ are both $\mathcal{O}(1)$.

6.3.1 Linear strain and stress measures

In the linear approximation, all the strain and stress measures can be reduced to the classical infinitesimal strain and Cauchy stress, respectively.

Strain

The Green–Lagrange strain tensor in general curvilinear coordinates is defined by

$$\gamma_{ij} = \frac{1}{2} \left(\frac{\partial \boldsymbol{R}}{\partial \xi^i} \cdot \frac{\partial \boldsymbol{R}}{\partial \xi^j} - \frac{\partial \boldsymbol{r}}{\partial \xi^i} \cdot \frac{\partial \boldsymbol{r}}{\partial \xi^j} \right),$$

see equation (2.26). Using the expressions (6.39) we have

$$\gamma_{ij} = \frac{1}{2} \left(\frac{\partial \boldsymbol{r} + \epsilon \tilde{\boldsymbol{u}}}{\partial \xi^i} \cdot \frac{\partial \boldsymbol{r} + \epsilon \tilde{\boldsymbol{u}}}{\partial \xi^j} - \frac{\partial \boldsymbol{r}}{\partial \xi^i} \cdot \frac{\partial \boldsymbol{r}}{\partial \xi^j} \right)$$

$$= \frac{1}{2} \left(\frac{\partial \boldsymbol{r}}{\partial \xi^i} \cdot \frac{\partial \boldsymbol{r}}{\partial \xi^j} + \epsilon \frac{\partial \tilde{\boldsymbol{u}}}{\partial \xi^i} \cdot \frac{\partial \boldsymbol{r}}{\partial \xi^j} + \epsilon \frac{\partial \boldsymbol{r}}{\partial \xi^i} \cdot \frac{\partial \tilde{\boldsymbol{u}}}{\partial \xi^j} + \epsilon^2 \frac{\partial \tilde{\boldsymbol{u}}}{\partial \xi^i} \cdot \frac{\partial \tilde{\boldsymbol{u}}}{\partial \xi^j} - \frac{\partial \boldsymbol{r}}{\partial \xi^i} \cdot \frac{\partial \boldsymbol{r}}{\partial \xi^j} \right).$$

Neglecting the term multiplied by ϵ^2 because $\epsilon \ll 1$, we obtain

$$\gamma_{ij} \approx \epsilon \frac{1}{2} \left(\tilde{\boldsymbol{u}}_{,i} \cdot \boldsymbol{g}_j + \tilde{\boldsymbol{u}}_{,j} \cdot \boldsymbol{g}_i \right) = \epsilon \frac{1}{2} \left(\tilde{u}_i|_j + \tilde{u}_j|_i \right),$$

where the line | denotes covariant differentiation with respect to the undeformed basis $\boldsymbol{g}_i$. Thus, the Green–Lagrange strain tensor for small deformations is simply ϵ multiplied by the classic infinitesimal strain tensor, $\gamma_{ij} = \epsilon \tilde{e}_{ij}$, where

$$\tilde{e}_{ij} = \frac{1}{2}\left(\tilde{u}_i|_j + \tilde{u}_j|_i\right).$$

In fact, all our strain measures, by construction, will reduce to the infinitesimal strain tensor $\epsilon\tilde{\mathsf{e}}$ at leading order when the deformation is small. The general strain measures described in section 2.4.4 can all be written in the form

$$\mathcal{F}(\mathcal{U}) = \sum_{\hat{I}=1}^{3} f\left(\lambda_{(\hat{I})}\right) \boldsymbol{v}_{\hat{I}} \otimes \boldsymbol{v}_{\hat{I}}, \tag{6.40}$$

where $\lambda_{(\hat{I})}$ are the principal stretches and $\boldsymbol{v}_{\hat{I}}$ are the corresponding principal directions, or eigenvectors of the deformed metric tensor, see equation (2.36). In the linear approximation, from equation (2.36), the required eigenvalues and eigenvectors of the deformed metric tensor are defined by

$$G_{ij} v^j = \left\{g_{ij} + \epsilon\left(\tilde{u}_i|_j + \tilde{u}_j|_i\right)\right\} v^j = \left(g_{ij} + \epsilon 2\tilde{e}_{ij}\right) v^j = \mu g_{ij} v^j. \tag{6.41}$$

If we expand the eigenvalue $\mu = \mu_0 + \epsilon 2\tilde{\mu}$, then collect together terms at for each power of ϵ, equation (6.41) becomes

$$g_{ij} v^j = \mu_0 g_{ij} v^j, \quad \text{at } \mathcal{O}(1),$$
$$2\tilde{e}_{ij} v^j = 2\tilde{\mu} g_{ij} v^j, \quad \text{at } \mathcal{O}(\epsilon).$$

Thus $\mu = 1 + \epsilon 2\tilde{\mu}$, where $\tilde{\mu}$ are the eigenvalues of the infinitesimal strain tensor. Moreover, from equation (2.40), the stretch in the direction of the eigenvectors is given by

$$\lambda(\boldsymbol{v}_{\hat{I}}) = \sqrt{\mu_{(\hat{I})}} = \sqrt{1 + 2\epsilon\tilde{\mu}_{(\hat{I})}} \approx 1 + \epsilon\tilde{\mu}_{(\hat{I})},$$

from the binomial theorem. Thus, the general strain measures (6.40) become

$$\mathcal{F}(\mathcal{U}) = \sum_{\hat{I}=1}^{3} \left[f(1) + \epsilon f'(1)\tilde{\mu}_{(\hat{I})}\right] \boldsymbol{v}_{\hat{I}} \otimes \boldsymbol{v}_{\hat{I}}.$$

The constraints on f were that $f(1) = 0$ and $f'(1) = 1$, which gives

$$\mathcal{F}(\mathcal{U}) = \epsilon \sum_{\hat{I}=1}^{3} \tilde{\mu}_{(\hat{I})} \boldsymbol{v}_{\hat{I}} \otimes \boldsymbol{v}_{\hat{I}} = \epsilon\tilde{\mathsf{e}},$$

because the left-hand side is simply diagonalised form of the infinitesimal strain tensor. Hence, we conclude that under the assumption of small deformations, all strain measures are equivalent to the infinitesimal strain measure.

Stress

The only difference between the two different families of stress measures is whether the force is measured per unit deformed or undeformed area. If the deformation remains small, then there is no distinction between the two measures to leading order. The easiest demonstration is from the relationship between the body stress and the second Piola–Kirchhoff stress

$$T^{ij} = J s^{ij}.$$

The jacobian of the mapping from undeformed to deformed configurations is given by

$$J = \sqrt{G/g}, \quad \text{where} \quad G = |g_{ij} + 2\epsilon\tilde{e}_{ij}| \approx g + \mathcal{O}(\epsilon).$$

Hence,

$$J = 1 + \mathcal{O}(\epsilon), \quad \text{and therefore} \quad T^{ij} = s^{ij} + \mathcal{O}(\epsilon),$$

so the two families of stresses are identical to leading order, as claimed. It also follows that from the conservation of mass, $J\rho = \rho_0$, that $\rho = \rho_0 + \mathcal{O}(\epsilon)$ and, in fact, there is little point making a distinction between Lagrangian and Eulerian coordinates, as we shall see in section 6.3.3.

6.3.2 Linear constitutive laws

Stress

We consider the Taylor expansion of the second Piola–Kirchhoff stress $s^{ij}(\gamma_{ij}, \theta)$ about a reference configuration given by a state of zero strain $\gamma_{ij} = 0$ and a constant temperature $\theta_0(\boldsymbol{r})$

$$s^{ij} = s^{ij}\Big|_{\substack{\gamma_{ij}=0,\\ \theta=\theta_0}} + \epsilon\tilde{e}_{kl}\,\frac{\partial s^{ij}}{\partial \gamma_{kl}}\Big|_{\substack{\gamma_{ij}=0,\\ \theta=\theta_0}} + \epsilon\tilde{\theta}\,\frac{\partial s^{ij}}{\partial \theta}\Big|_{\substack{\gamma_{ij}=0,\\ \theta=\theta_0}} + \mathcal{O}(\epsilon^2),$$

which can be written as

$$s^{ij} = s_0^{ij} + E^{ijkl}e_{kl} + \alpha^{ij}(\theta - \theta_0),$$

where s_0^{ij} is a pre-stress; E^{ijkl} is the fourth-order stiffness (or elasticity) tensor; and α^{ij} is the thermal expansion tensor. Here, $e_{kl} = \epsilon\tilde{e}_{kl}$ is (re)defined to be infinitesimal strain tensor

$$e_{ij} = \frac{1}{2}\left(u_i|_j + u_j|_i\right).$$

In the linear approximation, all stress tensors are the same, so we can also write the single stress tensor as

$$\tau^{ij} = \tau_0^{ij} + E^{ijkl}e_{kl} + \alpha^{ij}(\theta - \theta_0), \tag{6.42}$$

the familiar linear thermoelastic constitutive law[5]. The precise form of the tensors E^{ijkl} and α^{ij} will still depend on the material under consideration. The symmetry of the stress and strain tensors means that

$$E^{ijkl} = E^{jikl} = E^{ijlk}, \quad \tau_0^{ij} = \tau_0^{ji}, \quad \alpha^{ij} = \alpha^{ji}.$$

Thus, the number of independent coefficients of the tensor E^{ijkl} is reduced from 81 to 36. If we assume that a strain energy function exists then its linearised form is given by

$$\mathcal{W} = \tau_0^{ij}e_{ij} + \frac{1}{2}E^{ijkl}e_{ij}e_{kl} + \alpha^{ij}e_{ij}\theta + \text{constant},$$

[5] In the absence of thermal terms, this is the generalisation of the classic Hooke's law which states that stress is linearly proportional to strain.

which follows from integration of the relationship $s^{ij} = \partial \mathcal{W}/\partial \gamma_{ij}$. The existence of the strain energy function provides an additional symmetry property, $E^{ijkl} = E^{klij}$, which further reduces the number of independent coefficients to 21.

If the material is isotropic, then its material properties must be invariant under any possible change to the material coordinates, which gives

$$E^{ijkl} = \lambda g^{ij} g^{kl} + \mu \left(g^{ik} g^{jl} + g^{il} g^{jk} \right), \tag{6.43}$$

and

$$\alpha^{ij} = \alpha g^{ij},$$

where λ, μ are the Lamé constants and α is the coefficient of thermal expansion. Thus, the constitutive equation (6.42) for a homogeneous, isotropic, linear thermoelastic material is

$$\tau^{ij} = \tau_0^{ij} + \lambda g^{ij} e_k^k + 2\mu e^{ij} + \alpha g^{ij}(\theta - \theta_0). \tag{6.44}$$

Entropy

We also consider a Taylor expansion of the entropy $\eta(\gamma_{ij}, \theta)$ about the same reference configuration

$$\eta = \eta \,\Big|_{\substack{\gamma_{ij} = 0, \\ \theta = \theta_0}} + \epsilon \tilde{e}_{ij} \left.\frac{\partial \eta}{\partial \gamma_{ij}}\right|_{\substack{\gamma_{ij} = 0, \\ \theta = \theta_0}} + \epsilon \tilde{\theta} \left.\frac{\partial \eta}{\partial \theta}\right|_{\substack{\gamma_{ij} = 0, \\ \theta = \theta_0}} + \mathcal{O}(\epsilon^2). \tag{6.45}$$

From the thermodynamic constraint (6.4a), we have that

$$T_{IJ} = \rho \frac{\partial \Psi}{\partial F_{IK}} F_{JK},$$

which can be converted into the derivative with respect to the nonlinear Green–Lagrange strain tensor by using the chain rule:

$$\begin{aligned} T_{IJ} = \rho \frac{\partial \Psi}{\partial e_{LM}} \frac{\partial e_{LM}}{\partial F_{IK}} F_{JK} &= \rho \frac{\partial \Psi}{\partial e_{LM}} \frac{1}{2} \frac{\partial (F_{PL} F_{PM})}{\partial F_{IK}} F_{JK}, \\ &= \rho \frac{1}{2} \frac{\partial \Psi}{\partial e_{LM}} \left(\delta_{PI} \delta_{LK} F_{PM} + F_{PL} \delta_{PI} \delta_{MK} \right) F_{JK}, \\ &= \rho \frac{\partial \Psi}{\partial e_{LM}} \frac{1}{2} \left(F_{IM} F_{JL} + F_{IL} F_{JM} \right) = \rho F_{IL} \frac{\partial \Psi}{\partial e_{LM}} F_{JM}, \end{aligned}$$

after using the symmetry properties of the Green–Lagrange strain tensor. Thus,

$$\mathsf{T} = \frac{\rho_0}{J} \mathsf{F} \frac{\partial \Psi}{\partial \mathsf{e}} \mathsf{F}^T \quad \Rightarrow \rho_0 \frac{\partial \Psi}{\partial \mathsf{e}} = J \mathsf{F}^{-1} \mathsf{T} \mathsf{F}^{-T} = \mathsf{s},$$

the second Piola–Kirchhoff stress tensor from equation (3.13). We had already established that $s^{ij} = \rho_0 \frac{\partial \Psi}{\partial \gamma_{ij}}$ for a hyperelastic material; and the strain energy function is then $\mathcal{W} = \rho_0 \Psi$. The above argument demonstrates that the result holds for a general thermoelastic material.

The point of the above argument is that we have the two thermodynamic constraints (6.4a,b), or equivalently,

$$s^{ij} = \rho_0 \frac{\partial \Psi}{\partial \gamma_{ij}} \quad \text{and} \quad \eta = -\frac{\partial \Psi}{\partial \theta},$$

which means that

$$\rho_0 \frac{\partial \eta}{\partial \gamma_{ij}} = -\rho_0 \frac{\partial^2 \Psi}{\partial \gamma_{ij} \partial \theta} = -\rho_0 \frac{\partial^2 \Psi}{\partial \theta \partial \gamma_{ij}} = -\frac{\partial s^{ij}}{\partial \theta},$$

and our expansion for η (6.45) becomes

$$\eta = \eta_0 - \epsilon \tilde{e}_{ij} \frac{1}{\rho_0} \alpha^{ij} + \epsilon \tilde{\theta} \beta + \mathcal{O}(\epsilon^2), \tag{6.46}$$

where η_0 is the entropy of the initial state and β is a scalar entropy parameter.

6.3.3 Linearised governing equations

Linear momentum

The balance of momentum equation in component form in the deformed Lagrangian basis is given by a simple transformation of equation from the Eulerian basis ($\boldsymbol{G}_{\bar{i}}$) to the Lagrangian ($\boldsymbol{G}_i$) (4.8)

$$\rho \left[\frac{\partial V^i}{\partial t} + V^j V^i ||_j \right] = T^{ji}||_j + \rho F^i. \tag{6.47}$$

Our assumptions of small deformation (6.39) were that $\boldsymbol{R} = \boldsymbol{r} + \epsilon \tilde{\boldsymbol{u}}$, which means that

$$\boldsymbol{V} = \frac{D\boldsymbol{R}}{Dt} = \frac{D(\boldsymbol{r} + \epsilon \tilde{\boldsymbol{u}})}{Dt} = \epsilon \frac{D\tilde{\boldsymbol{u}}}{Dt},$$

because the undeformed position is fixed. In addition the covariant base vectors in the deformed basis are given by

$$\boldsymbol{G}_i = \boldsymbol{R}_{,i} = (\boldsymbol{r} + \epsilon \tilde{\boldsymbol{u}})_{,i} = \boldsymbol{g}_i + \epsilon \tilde{\boldsymbol{u}}_{,i},$$

and hence the Christoffel symbols associated with $\boldsymbol{G}_i$ are

$$\overline{\Gamma}_{ijk} = \boldsymbol{G}_k \boldsymbol{\cdot} \boldsymbol{G}_{i,j} = (\boldsymbol{g}_k + \epsilon \tilde{\boldsymbol{u}}_{,k}) \boldsymbol{\cdot} (\boldsymbol{g}_{i,j} + \epsilon \tilde{\boldsymbol{u}}_{,ij}) = \Gamma_{ijk} + \mathcal{O}(\epsilon),$$

which means that there is no distinction between the covariant derivatives with respect to the deformed and undeformed bases, *i.e.* we can replace the terms $||_j$ by $|_j$. We denote the stress T^{iJ} by the equivalent infinitesimal stress measure τ^{ij} and use the result

$$\frac{D\boldsymbol{V}}{Dt} = \frac{\partial}{\partial t} \left(\epsilon \frac{D\tilde{\boldsymbol{u}}}{Dt} \right) + \epsilon \frac{D\tilde{\boldsymbol{u}}}{Dt} \boldsymbol{\cdot} \boldsymbol{\nabla} \left(\epsilon \frac{D\tilde{\boldsymbol{u}}}{Dt} \right) = \epsilon \frac{\partial^2 \tilde{\boldsymbol{u}}}{\partial t^2} + \mathcal{O}(\epsilon^2).$$

Hence, the governing equation (6.47) becomes

$$\epsilon \rho_0 \frac{\partial^2 \tilde{u}^i}{\partial t^2} = \tau^{ji}|_j + \rho_0 F^i + \mathcal{O}(\epsilon^2),$$

and using the linear constitutive law (6.42) gives

$$\epsilon \rho_0 \frac{\partial^2 \tilde{u}^i}{\partial t^2} = \tau_0^{ji}|_j + \epsilon (E^{ijkl} \tilde{e}_{kl})|_j + \epsilon (\alpha^{ij} \tilde{\theta})|_j + \rho_0 F^i + \mathcal{O}(\epsilon^2),$$

so if we decompose the body force into the form, $\boldsymbol{F} = \boldsymbol{F}_0 + \epsilon \tilde{\boldsymbol{f}}$, then we have the $\mathcal{O}(1)$ equation

$$\tau_0^{ij}|_j + \rho_0 F_0^i = 0, \tag{6.48}$$

which determines the steady pre-stress, assuming that the deformation remains small[6]; and the $\mathcal{O}(\epsilon)$ equation which is the linearised balance of linear momentum

$$\rho_0 \frac{\partial^2 u^i}{\partial t^2} = \left(E^{ijkl} e_{kl}\right)|_j + \left(\alpha^{ij}(\theta - \theta_0)\right)|_j + \rho_0 f^i. \tag{6.49}$$

Energy

The conservation of energy equation in component form in the deformed Lagrangian basis is obtained by a simple transformation of equation (4.18)

$$\rho \frac{D\Phi}{Dt} = T^{ij} D_{ij} + \rho B - Q^i||_i. \tag{6.50}$$

The Helmholtz free energy is given by $\Psi = \Phi - \eta\Theta$, so

$$\frac{D\Phi}{Dt} = \frac{D\Psi}{Dt} + \frac{D\eta}{Dt}\Theta + \eta\frac{D\Theta}{Dt}, \tag{6.51}$$

but from the constitutive assumption $\Psi(\mathsf{F}, \Theta)$ and the thermodynamic requirements (6.4a,b)

$$\frac{D\Psi}{Dt} = \frac{\partial\Psi}{\partial\mathsf{F}} : \frac{D\mathsf{F}}{Dt} + \frac{\partial\Psi}{\partial\Theta}\frac{D\Theta}{Dt} = \rho^{-1}\mathsf{T}\mathsf{F}^{-T} : \frac{D\mathsf{F}}{Dt} - \eta\frac{D\Theta}{Dt}. \tag{6.52}$$

Combining equations (6.51) and (6.52) yields

$$\frac{D\Phi}{Dt} = \rho^{-1}\mathsf{T}\mathsf{F}^{-T} : \frac{D\mathsf{F}}{Dt} + \Theta\frac{D\eta}{Dt}. \tag{6.53}$$

Expanding the first term in index notation and using the result (2.51), $D\mathsf{F}/Dt = \mathsf{L}\mathsf{F}$, gives

$$T^{ij} F^{-1}_{kj} L_{il} F_{lk} = T^{ij} L_{il} \delta^l_j = T^{ij} L_{ij} = T^{ij} D_{ij},$$

after using the symmetry properties of the Cauchy stress tensor. Thus, equation (6.53) becomes

$$\frac{D\Phi}{Dt} = \rho^{-1} T^{ij} D_{ij} + \Theta\frac{D\eta}{Dt};$$

and the energy equation (6.50) is simply

$$\rho\Theta\frac{D\eta}{Dt} + Q^i||_i = \rho B.$$

Using our linear expansion for the entropy, equation (6.46), the energy equation becomes

$$\rho_0(\theta_0 + \epsilon\tilde{\theta})\epsilon\left[-\frac{D\tilde{e}_{ij}}{Dt}\frac{1}{\rho_0}\alpha^{ij} + \frac{D\tilde{\theta}}{Dt}\beta\right] + q^i|_i = \rho_0 B.$$

Note that we have used the fact that there is no distinction between $\boldsymbol{Q}$ and $\boldsymbol{q}$ in the linear approximation and that we can replace the covariant derivative $||_i$ by $|_i$. If we use Fourier's law for the heat flux (6.7), then

$$q^i = -k^{ij}(\theta_0 + \epsilon\tilde{\theta})|_j,$$

[6]It is also possible, that the pre-stress has a component of $\mathcal{O}(\epsilon)$ which can then be included at the next order, in equation (6.49).

and the energy balance is then

$$\rho_0\theta_0\beta\epsilon\frac{D\tilde{\theta}}{Dt} = \left(k^{ij}(\theta_0|_j + \epsilon\tilde{\theta}|_j)\right)|_i + \epsilon\theta_0\,\alpha^{ij}\frac{D\tilde{e}_{ij}}{Dt} + \rho_0 B + \mathcal{O}(\epsilon^2).$$

If we now decompose the body heating into the form $B = b_0 + \epsilon\tilde{b}$, then we have the $\mathcal{O}(1)$ equation

$$\left(k^{ij}\theta_0|_j\right)|_i + \rho_0 b_0 = 0,$$

which determines the initial temperature distribution; and the $\mathcal{O}(\epsilon)$ equation which is the required linearised balance of energy

$$\rho_0\theta_0\beta\frac{D\theta}{Dt} = \left(k^{ij}\theta|_j\right)|_i + \theta_0\,\alpha^{ij}\frac{De_{ij}}{Dt} + \rho_0 b.$$

This is simply the heat equation, but with an additional term that represents the heating due to time variations in strain. Note that the material and partial derivatives of a scalar field coincide in the linear approximation, $D\theta/Dt = \partial\theta/\partial t + \mathcal{O}(\epsilon)$.

Thus the linearised equations of thermoelasticity are the coupled equations

$$\rho_0\frac{\partial^2 u^i}{\partial t^2} = \left(E^{ijkl}e_{kl}\right)|_j + \left(\alpha^{ij}(\theta - \theta_0)\right)|_j + \rho_0 f^i, \tag{6.54a}$$

$$\rho_0\theta_0\beta\frac{\partial\theta}{\partial t} = \left(k^{ij}\theta|_j\right)|_i + \theta_0\,\alpha^{ij}\frac{De_{ij}}{Dt} + \rho_0 b. \tag{6.54b}$$

It is common to neglect the elastic heating term in equation (6.54b), in which case the energy equation decouples from the linear momentum equation (6.54a) and the thermal problem can be solved first. This is not the case in the nonlinear system even if the elastic heating is neglected because the deformation of the domain indirectly affects the thermal problem. If $\alpha^{ij} = 0$, then the strain energy does not depend on the temperature and the two equations decouple completely. The transport of heat is unaffected by the deformation of the material and vice versa.

6.3.4 Isotropic materials: Navier–Lamé equations

If we assume that the material is homogeneous then the quantities E^{ijkl}, α^{ij} and k^{ij} do not depend on space, which simplifies the governing equations (6.54a,b)

$$\rho_0\frac{\partial^2 u^i}{\partial t^2} = E^{ijkl}e_{kl}|_j + \alpha^{ij}\theta|_j + \rho_0 f^i,$$
$$\rho_0\theta_0\beta\frac{\partial\theta}{\partial t} = k^{ij}\theta|_{ji} + \theta_0\,\alpha^{ij}\frac{De_{ij}}{Dt} + \rho_0 b.$$

If the material is also isotropic then using the constitutive relation (6.44) yields the equations

$$\rho_0\frac{\partial^2 u^i}{\partial t^2} = \lambda g^{ij}e^k_k|_j + 2\mu e^{ij}|_j + \alpha g^{ij}\theta|_j + \rho_0 f^i,$$
$$\rho_0\theta_0\beta\frac{\partial\theta}{\partial t} = \kappa g^{ij}\theta|_{ji} + \theta_0\,\alpha g^{ij}\frac{De_{ij}}{Dt} + \rho_0 b.$$

If we use the definition of the infinitesimal strain tensor in terms of the displacements $2e_{ij} = u_{i|j} + u_{j|i}$ then we obtain the two equations

$$\rho_0\frac{\partial^2 u^i}{\partial t^2} = \lambda g^{ij}u^k|_{kj} + \mu g^{ik}g^{jl}(u_k|_{lj} + u_l|_{kj}) + \alpha g^{ij}\theta|_j + \rho_0 f^j$$

$$= (\lambda + \mu) g^{ij} u^k|_{kj} + \mu g^{jk} u^i|_{kj} + \alpha g^{ij} \theta|_j + \rho_0 f^i, \tag{6.55a}$$

and the energy equation

$$\rho_0 \theta_0 \beta \frac{\partial \theta}{\partial t} = \kappa g^{ij} \theta|_{ji} + \theta_0\, \alpha \dot{e}^i_i + \rho_0 b = \kappa g^{ij} \theta|_{ij} + \theta_0\, \alpha d^i_i + \rho_0 b = \kappa g^{ij} \theta|_{ij} + \theta_0\, \alpha v^i|_i + \rho_0 b. \tag{6.55b}$$

Note that the material derivative of the infinitesimal strain is the (infinitesimal) rate of deformation tensor[7]

$$\dot{e}_{ij} = d_{ij} + \mathcal{O}(\epsilon^2) = \frac{1}{2}\left(v_i|_j + v_j|_i\right) + \mathcal{O}(\epsilon^2),$$

where $\boldsymbol{v} = \partial \boldsymbol{u}/\partial t$ is the velocity of the material.

In dyadic form equations (6.55a,b) become

$$\rho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda + \mu) \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \mu \boldsymbol{\nabla}^2 \boldsymbol{u} + \alpha \boldsymbol{\nabla} \theta + \rho_0 \boldsymbol{f}, \tag{6.56a}$$

$$\rho_0 \theta_0 \beta \frac{\partial \theta}{\partial t} = \kappa \boldsymbol{\nabla}^2 \theta + \theta_0\, \alpha \boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{u}}{\partial t} + \rho_0 b. \tag{6.56b}$$

If we set $\alpha = 0$, and assume that the system is isothermal then what remains is known as the Navier–Lamé equations

$$\rho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda + \mu) \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \mu \boldsymbol{\nabla}^2 \boldsymbol{u} + \rho_0 \boldsymbol{f}.$$

The Navier–Lamé equations govern isothermal linear elasticity and have been widely studied.

[7] The result follows from equation (2.49), which states that for the finite Eulerian strain tensor

$$\dot{E}_{IJ} = D_{IJ} - E_{IK} V_{K,J} - E_{KJ} V_{K,I} = D_{IJ} - E_{IK}(W_{KJ} + D_{KJ}) - E_{KJ}(W_{KI} + D_{KI}).$$

In the infinitesimal limit $\dot{E}_{IJ} = \dot{e}_{IJ} + \mathcal{O}(\epsilon^2)$ and $D_{IJ} = d_{IJ} + \mathcal{O}(\epsilon^2)$, thus the result follows provided that the spin tensor is also of $\mathcal{O}(\epsilon)$. If the spin tensor contains a component of $\mathcal{O}(1)$ (large rotation) then it's contribution must also be included, corresponding to a limit in which we include some geometric nonlinearities.