Chapter 4

Physical Conservation and Balance Laws & Thermodynamics

The laws of classical physics are, for the most part, expressions of conservation or balances of certain quantities: *e.g.* mass, momentum, angular momentum, energy. These are fundamental postulates or axioms that follow from experimental work. In this chapter we shall develop these balance laws in forms that can be applied to continua in both the Eulerian and Lagrangian viewpoints. We have already seen the expressions for balance of momenta (both linear and angular) in chapter 3 and how they can be used to derive governing equations of equilibrium in the Eulerian viewpoint and to demonstrate that the Cauchy stress tensor is symmetric.

4.1 Conservation of mass

The axiom of conservation of mass states that the total mass of a material region remains constant under any motion, provided that there are no sources or sinks of mass. Thus, assuming an initial mass density $\rho_0(\boldsymbol{r})$ in the undeformed region,

$$\int_{\Omega_0} \rho_0(\boldsymbol{r}) \mathrm{d}\mathcal{V}_0 = \int_{\Omega_t} \rho(\boldsymbol{R}, t) \mathrm{d}\mathcal{V}_t, \tag{4.1}$$

where ρ is the mass density in the deformed region, as defined in equation (3.5). We saw in chapter 2, that the relationship between material volume elements is given by equation (2.41a),

$$\mathrm{d}\mathcal{V}_t = \sqrt{G/g}\, \mathrm{d}\mathcal{V}_0 = J\, \mathrm{d}\mathcal{V}_0.$$

Thus,

$$\int_{\Omega_0} \rho_0 \, \mathrm{d}\mathcal{V}_0 = \int_{\Omega_0} \rho \sqrt{G/g}\, \mathrm{d}\mathcal{V}_0 \quad \Rightarrow \quad \int_{\Omega_0} (\rho_0 - \rho J)\, \mathrm{d}\mathcal{V}_0 = 0,$$

and because this relationship must be true for all volumes, we have that

$$\rho_0(\boldsymbol{r}) = \rho(\boldsymbol{R}(\boldsymbol{r}, t), t) J, \tag{4.2}$$

which is the Lagrangian expression of conservation of mass.

Alternatively, we can express the relationship in Eulerian form by insisting that the material derivative of the total mass is zero,

$$\frac{D}{Dt} \int_{\Omega_0} \rho_0(\boldsymbol{r}) \mathrm{d}\mathcal{V}_0 = \frac{D}{Dt} \int_{\Omega_t} \rho(\boldsymbol{R}, t) \mathrm{d}\mathcal{V}_t = 0,$$

from equation (4.1). Using the Reynolds transport theorem we can write

$$\frac{D}{Dt}\int_{\Omega_t} \rho \, \mathrm{d}\mathcal{V}_t = \int_{\Omega_t} \left[\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} (\rho \boldsymbol{V})\right] \mathrm{d}\mathcal{V}_t = 0.$$

The equation must be satisfied for all possible volumes, which means that the integrand is zero and we have the Eulerian expression for conservation of mass, also known as the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} (\rho \boldsymbol{V}) = 0. \tag{4.3}$$

Expanding out the second term gives the alternative expression

$$\frac{\partial \rho}{\partial t} + \boldsymbol{V} \boldsymbol{\cdot} \boldsymbol{\nabla}_{\boldsymbol{R}} \rho + \rho \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V} = \frac{D\rho}{Dt} + \rho \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V} = 0, \tag{4.4}$$

which is the Eulerian form of the conservation of mass equation.

If the material is incompressible then its volume cannot change

$$\sqrt{G/g} = J = I_3^{\frac{1}{2}} = 1,$$

and so from equation (4.2) the density does not change $\rho(\boldsymbol{R}(\boldsymbol{r}), t) = \rho_0(\boldsymbol{r})$ provided that mass is conserved (which is obvious from basic physical reasoning). Thus, $D\rho/Dt = 0$ and the continuity equation becomes

$$\rho \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V} = 0 \quad \Rightarrow \quad \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V} = \mathsf{D} : \mathsf{I} = 0.$$

From equation (2.55), it follows that the incompressibility constraint obtained in this way is consistent with the condition that the dilation (relative volume change) is zero, as one might hope.

4.2 Balance of linear momentum

In chapter 3, we derived the governing equations of equilibrium

$$\boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \mathsf{T} + \rho \boldsymbol{F} = \boldsymbol{0},$$

in the Eulerian viewpoint, from the conservation of linear momentum in the absence of acceleration terms. In order to include the effects of acceleration we must return to the integral form of the balance law law (3.6)

$$\frac{D\boldsymbol{P}}{Dt} = \frac{D}{Dt}\int_{\Omega_t} \rho \boldsymbol{V} \, \mathrm{d}\mathcal{V}_t = \int_{\partial\Omega_t} \boldsymbol{T} \, \mathrm{d}\mathcal{S}_t + \int_{\Omega_t} \rho \boldsymbol{F} \, \mathrm{d}\mathcal{V}_t. \tag{4.5}$$

Using equation (3.16) in place of the force integrals we have that

$$\frac{D}{Dt}\int_{\Omega_t} \rho \boldsymbol{V} \, \mathrm{d}\mathcal{V}_t = \int_{\Omega_t} \frac{1}{\sqrt{G}} \frac{\partial \boldsymbol{T}^{\bar{i}}}{\partial \chi^{\bar{i}}} + \rho \boldsymbol{F} \mathrm{d}\mathcal{V}_t = \boldsymbol{0}.$$

Applying the Reynolds transport theorem to the momentum integral[1]

[1]You should be slightly concerned here because we only derived the Reynolds transport theorem for the transport of a scalar, ϕ. Fortunately, the argument easily generalises to vectors (and tensors) because the product rule always applies and the product of a scalar and tensor remains a tensor of the same type

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \boldsymbol{V} \, \mathrm{d}\mathcal{V}_t = \int_{\Omega_0} \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{V}\sqrt{G/g}\right) \mathrm{d}\mathcal{V}_0 = \int_{\Omega_0} \frac{D\boldsymbol{V}}{Dt}\sqrt{G/g} + \boldsymbol{V}\frac{D\sqrt{G/g}}{Dt} \, \mathrm{d}\mathcal{V}_0$$
$$= \int_{\Omega_0} \left[\frac{D\boldsymbol{V}}{Dt} + \boldsymbol{V} \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V}\right] \sqrt{G/g} \, \mathrm{d}\mathcal{V}_0 = \int_{\Omega_t} \frac{D\boldsymbol{V}}{Dt} + \boldsymbol{V} \, \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V} \, \mathrm{d}\mathcal{V}_t.$$

gives

$$\frac{D}{Dt}\int_{\Omega_t} \rho \boldsymbol{V}\,\mathrm{d}\mathcal{V}_t = \int_{\Omega_t}\left[\frac{D(\rho\boldsymbol{V})}{Dt} + \rho\boldsymbol{V}\,\boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V}\right]\mathrm{d}\mathcal{V}_t,$$

$$= \int_{\Omega_t}\left[\frac{D\rho}{Dt}\boldsymbol{V} + \rho\frac{D\boldsymbol{V}}{Dt} + \rho\boldsymbol{V}\,\boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V}\right]\mathrm{d}\mathcal{V}_t = \int_{\Omega_t}\left[\rho\frac{D\boldsymbol{V}}{Dt} + \boldsymbol{V}\left(\frac{D\rho}{Dt} + \rho\boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V}\right)\right]\mathrm{d}\mathcal{V}_t.$$

The expression in round brackets is zero by conservation of mass (4.4), and so the expression for balance of linear momentum (4.5) becomes

$$\int_{\Omega_t}\left[\rho\frac{D\boldsymbol{V}}{Dt} - \frac{1}{\sqrt{G}}\frac{\partial \boldsymbol{T}^{\bar{i}}}{\partial \chi^{\bar{i}}} - \rho\boldsymbol{F}\right]\mathrm{d}\mathcal{V}_t = \boldsymbol{0}. \tag{4.6}$$

The integral must be zero for all possible volumes, which means that the integrand is zero and so

$$\rho\frac{D\boldsymbol{V}}{Dt} = \frac{1}{\sqrt{G}}\frac{\partial \boldsymbol{T}^{\bar{i}}}{\partial \chi^{\bar{i}}} + \rho\boldsymbol{F}, \tag{4.7}$$

which can be written as

$$\rho\frac{D\boldsymbol{V}}{Dt} = \boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\mathsf{T} + \rho\boldsymbol{F},$$

or in component form in the basis $\boldsymbol{G}_{\bar{i}}$

$$\rho\left[\frac{\partial V^{\bar{i}}}{\partial t} + V^{\bar{j}}V^{\bar{i}}\|_{\bar{j}}\right] = T^{\bar{j}\,\bar{i}}\|_{\bar{j}} + \rho F^{\bar{i}}. \tag{4.8}$$

Irrespective of how we write them, these are Cauchy's equations of motion for a continuum in the Eulerian representation.

The expression for balance of momentum in the Lagrangian formulation follows by transforming the integrals in equation (4.5) to be over the undeformed, rather than the deformed domain

$$\frac{D}{Dt}\int_{\Omega_0}\rho\boldsymbol{v}\,J\mathrm{d}\mathcal{V}_0 = \int_{\partial\Omega_0}\boldsymbol{t}\,\mathrm{d}\mathcal{S}_0 + \int_{\Omega_0}\rho\boldsymbol{F}\,J\,\mathrm{d}\mathcal{V}_0, \tag{4.9}$$

where $\boldsymbol{t}$ is the traction measured using the area of the undeformed surface. From the definition of the first Piola–Kirchhoff stress tensor we have that

$$\boldsymbol{t} = p^{\bar{j}\,k}n_k\boldsymbol{G}_{\bar{j}},$$

and so we can define a stress vector per unit undeformed area which transforms contravariantly $\boldsymbol{t}^k = \sqrt{g}p^{\bar{j}\,k}\boldsymbol{G}_{\bar{j}}$. Thus, the equation (4.9) becomes

$$\frac{D}{Dt}\int_{\Omega_0}\rho_0\boldsymbol{v}\,\mathrm{d}\mathcal{V}_0 = \int_{\partial\Omega_0}\frac{n_k}{\sqrt{g}}\boldsymbol{t}^k\,\mathrm{d}\mathcal{S}_0 + \int_{\Omega_0}\rho_0\boldsymbol{F}\,\mathrm{d}\mathcal{V}_0,$$

after using the Lagrangian expression for conservation of mass (4.2) in the volume integrals. We can now use the divergence theorem on the surface integral and bring the material derivative inside the fixed integral over the reference domain to obtain

$$\int_{\Omega_0}\left[\rho_0\frac{\partial\boldsymbol{v}}{\partial t} - \frac{1}{\sqrt{g}}\frac{\partial \boldsymbol{t}^i}{\partial \xi^i} - \rho_0\boldsymbol{F}\right]\mathrm{d}\mathcal{V}_0 = 0.$$

Once again, the integrand must vanish so

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} - \frac{1}{\sqrt{g}} \frac{\partial \boldsymbol{t}^i}{\partial \xi^i} - \rho_0 \boldsymbol{F} = 0.$$

The expression for the derivative of the stress vector is actually rather messy when expressed in terms of the first Piola–Kirchhoff stress tensor because we must evaluate $\partial \boldsymbol{G}_{\bar{j}}/\partial \xi^i$. A compact expression can be obtained if we introduce a new representation of the stress tensor that decomposes the vector into components in the undeformed basis $\boldsymbol{t}^k = \sqrt{g}\, t^{jk} \boldsymbol{g}_j$. The governing equation is then

$$\rho_0 \frac{\partial v^j}{\partial t} = t^{ij}|_i + \rho_0 f^j,$$

where $\boldsymbol{F} = f^j \boldsymbol{g}_j$, $\boldsymbol{v} = v^j \boldsymbol{g}_j$ and the line $|$ represents covariant differentiation with respect to the Lagrangian coordinates in the undeformed configuration.

In fact, in Cartesian coordinates the two tensors $p^{\bar{j}k}$ and t^{jk} coincide, which follows from

$$t^{jk} \boldsymbol{g}_j = p^{\bar{j}k} \boldsymbol{G}_{\bar{j}} \quad \Rightarrow \quad t^{jk} \frac{\partial \boldsymbol{r}}{\partial \xi^j} = p^{\bar{j}k} \frac{\partial \boldsymbol{R}}{\partial \chi^{\bar{j}}} \quad \Rightarrow \quad t^{jk} \frac{\partial x_I}{\partial \xi^j} = p^{\bar{j}k} \frac{\partial X_I}{\partial \chi^{\bar{j}}}.$$

The choice of Cartesian coordinates corresponds to choosing $\xi^j = x_j$ and $\chi^{\bar{j}} = X_{\bar{j}}$, so we have

$$t^{jk} \delta_{Ij} = p^{\bar{j}k} \delta_{I\bar{j}} \quad \Rightarrow \quad t^{Ik} = p^{Ik}.$$

Note that we do not have to choose Cartesian coordinates for the decomposition of the normal vector for the stress tensors to coincide, but we do need the decomposition of the stress vector to be in Cartesian coordinates. Thus, the expression found in many books for the governing equation in the Lagrangian viewpoint is

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{\nabla}_{\boldsymbol{r}} \cdot \mathsf{p} + \rho_0 \boldsymbol{f},$$

where $\boldsymbol{f}(\boldsymbol{r}, t) = \boldsymbol{F}(\boldsymbol{R}, t)$ is the body force represented in Lagrangian coordinates and p is the first Piola–Kirchhoff stress tensor. This view is somewhat simplistic however and only applies in Cartesian formulations. Moreover, the expression does not indicate how to take the divergence of a tensor.

4.3 Balance of angular momentum

In chapter 3, we showed that if there is no net torque acting on a body then the Cauchy stress tensor is symmetric. In fact, the Cauchy stress tensor is symmetric even if the the body is subject to a net angular acceleration. Once again, we return to the integral form of the balance law (3.7)

$$\frac{D\boldsymbol{H}_{\boldsymbol{Z}}}{Dt} = \frac{D}{Dt} \int_{\Omega_t} (\boldsymbol{R} - \boldsymbol{Z}) \times \rho \boldsymbol{V} \, \mathrm{d}\mathcal{V}_t = \boldsymbol{\mathcal{L}}_{\boldsymbol{Z}} = \int_{\partial\Omega_t} (\boldsymbol{R} - \boldsymbol{Z}) \times \boldsymbol{T} \mathrm{d}\mathcal{S}_t + \int_{\Omega_t} \rho (\boldsymbol{R} - \boldsymbol{Z}) \times \boldsymbol{F} \, \mathrm{d}\mathcal{V}_t.$$

The vector $\boldsymbol{Z}$ is a constant so it can be taken outside all integrals to yield

$$\frac{D}{Dt} \int_{\Omega_t} \boldsymbol{R} \times \rho \boldsymbol{V} \, \mathrm{d}\mathcal{V}_t - \int_{\partial\Omega_t} \boldsymbol{R} \times \boldsymbol{T} \mathrm{d}\mathcal{S}_t - \int_{\Omega_t} \rho \boldsymbol{R} \times \boldsymbol{F} \, \mathrm{d}\mathcal{V}_t =$$

$$\boldsymbol{Z} \times \left[\frac{D}{Dt} \int_{\Omega_t} \rho \boldsymbol{V} \, \mathrm{d}\mathcal{V}_t - \int_{\partial\Omega_t} \boldsymbol{T} \mathrm{d}\mathcal{S}_t - \int_{\Omega_t} \rho \boldsymbol{F} \, \mathrm{d}\mathcal{V}_t \right].$$

The term in square brackets is zero from the linear momentum balance in integral form (4.5) and therefore

$$\frac{D}{Dt}\int_{\Omega_t} \boldsymbol{R}\times\rho\boldsymbol{V}\,\mathrm{d}\mathcal{V}_t - \int_{\partial\Omega_t} \boldsymbol{R}\times\boldsymbol{T}\mathrm{d}\mathcal{S}_t - \int_{\Omega_t} \rho\boldsymbol{R}\times\boldsymbol{F}\,\mathrm{d}\mathcal{V}_t = \boldsymbol{0}.$$

Introducing the stress vector and using the divergence theorem as in chapter 3, see equation (3.18), we have

$$\frac{D}{Dt}\int_{\Omega_t} \rho\boldsymbol{R}\times\boldsymbol{V}\,\mathrm{d}\mathcal{V}_t = \int_{\Omega_t}\left[\frac{1}{\sqrt{G}}\frac{\partial}{\partial\chi^{\bar{i}}}\left(\boldsymbol{R}\times\boldsymbol{T}^{\bar{i}}\right) + \rho\boldsymbol{R}\times\boldsymbol{F}\right]\mathrm{d}\mathcal{V}_t. \tag{4.10}$$

We now use the Reynolds transport theorem to write

$$\frac{D}{Dt}\int_{\Omega_t} \rho\boldsymbol{R}\times\boldsymbol{V}\,\mathrm{d}\mathcal{V}_t = \int_{\Omega_t}\left[\frac{D}{Dt}\left(\rho\boldsymbol{R}\times\boldsymbol{V}\right) + \rho\boldsymbol{R}\times\boldsymbol{V}\,\boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V}\right]\mathrm{d}\mathcal{V}_t.$$

$$= \int_{\Omega_t}\left[\rho\frac{D}{Dt}\left(\boldsymbol{R}\times\boldsymbol{V}\right) + \left(\boldsymbol{R}\times\boldsymbol{V}\right)\left(\frac{D\rho}{Dt} + \rho\,\boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V}\right)\right]\mathrm{d}\mathcal{V}_t.$$

The second term in the integrand vanishes through conservation of mass (4.4), so that

$$\frac{D}{Dt}\int_{\Omega_t} \rho\boldsymbol{R}\times\boldsymbol{V}\,\mathrm{d}\mathcal{V}_t = \int_{\Omega_t}\rho\frac{D}{Dt}\left(\boldsymbol{R}\times\boldsymbol{V}\right)\mathrm{d}\mathcal{V}_t = \int_{\Omega_t}\rho\left(\frac{D\boldsymbol{R}}{Dt}\times\boldsymbol{V} + \boldsymbol{R}\times\frac{D\boldsymbol{V}}{Dt}\right)\mathrm{d}\mathcal{V}_t;$$

but, by definition, $D\boldsymbol{R}/Dt = \boldsymbol{V}$ and $\boldsymbol{V}\times\boldsymbol{V} = \boldsymbol{0}$ from the definition of the cross product, so

$$\frac{D}{Dt}\int_{\Omega_t} \rho\boldsymbol{R}\times\boldsymbol{V}\,\mathrm{d}\mathcal{V}_t = \int_{\Omega_t}\rho\boldsymbol{R}\times\frac{D\boldsymbol{V}}{Dt}\,\mathrm{d}\mathcal{V}_t.$$

Thus, equation (4.10) becomes

$$\int_{\Omega_t}\left[\rho\boldsymbol{R}\times\frac{D\boldsymbol{V}}{Dt} - \frac{1}{\sqrt{G}}\frac{\partial}{\partial\chi^{\bar{i}}}\left(\boldsymbol{R}\times\boldsymbol{T}^{\bar{i}}\right) - \rho\boldsymbol{R}\times\boldsymbol{F}\right]\mathrm{d}\mathcal{V}_t = \boldsymbol{0};$$

and using the product rule to expand the derivative of the cross product yields

$$\int_{\Omega_t}\boldsymbol{R}\times\left[\rho\frac{D\boldsymbol{V}}{Dt} - \frac{1}{\sqrt{G}}\frac{\partial\boldsymbol{T}^{\bar{i}}}{\partial\chi^{\bar{i}}} - \rho\boldsymbol{F}\right]\mathrm{d}\mathcal{V}_t = \int_{\Omega_t}\frac{1}{\sqrt{G}}\boldsymbol{G}_{\bar{i}}\times\boldsymbol{T}^{\bar{i}}\,\mathrm{d}\mathcal{V}_t.$$

The left-hand side vanishes by the linear momentum balance in its local (differential equation) form, equation (4.7), so

$$\int_{\Omega_t}\frac{1}{\sqrt{G}}\boldsymbol{G}_{\bar{i}}\times\boldsymbol{T}^{\bar{i}}\,\mathrm{d}\mathcal{V}_t = \boldsymbol{0},$$

which is the same as the condition (3.19) obtained from the equilibrium equations in chapter 3. Hence the conclusion is the same, namely that the Cauchy stress tensor is symmetric.

For the Lagrangian viewpoint, we can transform the equation into the reference configuration

$$\int_{\Omega_0}\frac{1}{\sqrt{G}}\boldsymbol{G}_{\bar{i}}\times\boldsymbol{T}^{\bar{i}}\,J\,\mathrm{d}\mathcal{V}_0 = \boldsymbol{0},$$

and since the equation must be true for any volume we have that

$$\frac{1}{\sqrt{G}}\boldsymbol{G}_{\bar{i}}\times\boldsymbol{T}^{\bar{i}}\,J = \boldsymbol{G}_{\bar{i}}\times T^{\bar{i}\bar{j}}\,J\,\boldsymbol{G}_{\bar{j}} = \boldsymbol{0}.$$

Transforming to the Lagrangian coordinates gives

$$\frac{\partial \xi^k}{\partial \chi^{\bar{i}}}\boldsymbol{G}_k \times T^{\bar{i}\bar{j}}\, J\, \frac{\partial \xi^l}{\partial \chi^{\bar{j}}}\boldsymbol{G}_l = \boldsymbol{G}_k \times \boldsymbol{G}_l \frac{\partial \xi^k}{\partial \chi^{\bar{i}}} T^{\bar{i}\bar{j}}\, J\, \frac{\partial \xi^l}{\partial \chi^{\bar{j}}} = \boldsymbol{G}_k \times \boldsymbol{G}_l s^{kl} = \boldsymbol{0},$$

where s^{kl} are the components of the second Piola–Kirchhoff stress tensor. It follows that

$$\frac{1}{2}\left[\boldsymbol{G}_k \times \boldsymbol{G}_l s^{kl} + \boldsymbol{G}_l \times \boldsymbol{G}_k s^{lk}\right] = \boldsymbol{0},$$

and by the antisymmetry of the cross product (or Levi-Civita symbol if we write the cross products in index notation)

$$\frac{1}{2}\boldsymbol{G}_k \times \boldsymbol{G}_l \left[s^{kl} - s^{lk}\right] = \boldsymbol{0},$$

from which we deduce that the second Piola–Kirchhoff stress tensor is symmetric $s^{kl} = s^{lk}$.

4.4 Conservation of energy

Developing a theory for the conservation of energy in continua is more complicated than for conservation of mass and momenta. The problem is that molecular fluctuations are not present in continuum theory, but these fluctuations may involve a significant amount energy. In order to include such effects we require a continuum theory of thermodynamics, which introduces the concepts of temperature and heat and their relation to the energy. The temperature can be thought of as a measure of the fluctuations of molecules within the body and heat is an energy associated with the temperature.

We also need to introduce the concepts of mechanical work done by external forces. The work done by a force in displacing a single particle is the total force exerted on the particle in moving it along a particular path, L, *i.e.* the line integral of the force acting tangentially to the path

$$\widehat{W} = \int_L \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{R} = \int_L \boldsymbol{F}\cdot\dot{\boldsymbol{R}}\,\mathrm{d}t.$$

The rate of work, or power, is the time (material) derivative of the work, so

$$\widehat{P} = \frac{\mathrm{d}\widehat{W}}{\mathrm{d}t} = \dot{\widehat{W}} = \boldsymbol{F}\cdot\dot{\boldsymbol{R}} = \boldsymbol{F}\cdot\boldsymbol{V}.$$

From Newton's second law applied to a particle

$$\dot{\widehat{W}} = \boldsymbol{F}\cdot\boldsymbol{V} = m\boldsymbol{a}\cdot\boldsymbol{V} = m\dot{\boldsymbol{V}}\cdot\boldsymbol{V},$$

and integrating in time, we find that the work done is equal to the kinetic energy of the particle

$$\widehat{W} = \frac{1}{2}m\boldsymbol{V}\cdot\boldsymbol{V} = \frac{1}{2}m|\boldsymbol{V}|^2.$$

Thus, the classical Newtonian mechanics approach to obtain an energy equation is to take the dot product of Newton's second law with $\boldsymbol{V}$ and integrate.

If you take a piece of rubber and stretch and compress it repeatedly it will get hot. It's more fun to do this experiment by smashing a squash ball against the wall several times. Alternatively, if you heat the rubber it will deform mechanically (the lazy squash player puts the ball on the radiator before playing). The fact the mechanical work can be converted into heat and vice versa is the essence of classical thermodynamics. Before we get into the thermodynamics, however, we must consider the mechanical work done by external forces on a continuum, rather than a single particle. An important point is that unlike a particle, a continuum can store energy internally by reconfiguration of the internal structure, which typically induces internal stresses. Internal energy may be also stored via chemical and electromagnetic effects.

4.4.1 The rate of work and work conjugacy

In a continuum body, the net rate of work, or net power, is the total power exerted by external forces that is not used to produce motion. If the body cannot store energy internally and none of the work is converted into heat, then the net rate of work must be zero because all the work is converted into motion. The net rate of work is therefore the sum of rate of work of the surface forces and the rate of work of the net body forces (the actual body forces minus the force required to produce the acceleration):

$$\frac{DW}{Dt} = \dot{W} = \int_{\partial\Omega_t} \boldsymbol{T} \cdot \boldsymbol{V} \,\mathrm{d}\mathcal{S}_t + \int_{\Omega_t} \left(\rho \boldsymbol{F} - \rho \dot{\boldsymbol{V}}\right) \cdot \boldsymbol{V} \,\mathrm{d}\mathcal{V}_t. \tag{4.11}$$

Using the decomposition of the traction vector $\boldsymbol{T} = N_{\bar{i}} \boldsymbol{T}^{\bar{i}} / \sqrt{G}$ then the surface integral in equation (4.11) becomes

$$\int_{\partial\Omega_t} \boldsymbol{T} \cdot \boldsymbol{V} \,\mathrm{d}\mathcal{S}_t = \int_{\partial\Omega_t} \boldsymbol{T}^{\bar{i}} \cdot \boldsymbol{V} \frac{N_{\bar{i}}}{\sqrt{G}} \,\mathrm{d}\mathcal{S}_t$$

and using the divergence theorem in general coordinates gives

$$\dot{W} = \int_{\Omega_t} \frac{1}{\sqrt{G}} \frac{\partial}{\partial \chi^{\bar{i}}} \left(\boldsymbol{T}^{\bar{i}} \cdot \boldsymbol{V}\right) + \left(\rho \boldsymbol{F} - \rho \dot{\boldsymbol{V}}\right) \cdot \boldsymbol{V} \,\mathrm{d}\mathcal{V}_t,$$

$$= \int_{\Omega_t} \frac{1}{\sqrt{G}} \boldsymbol{T}^{\bar{i}} \cdot \frac{\partial \boldsymbol{V}}{\partial \chi^{\bar{i}}} + \left[\frac{1}{\sqrt{G}} \frac{\partial \boldsymbol{T}^{\bar{i}}}{\partial \chi^{\bar{i}}} + \rho \boldsymbol{F} - \rho \dot{\boldsymbol{V}}\right] \cdot \boldsymbol{V} \,\mathrm{d}\mathcal{V}_t,$$

The term in square brackets is zero because it is the simply governing differential equation for the linear momentum balance (4.7), so

$$\dot{W} = \int_{\Omega_t} \frac{1}{\sqrt{G}} \boldsymbol{T}^{\bar{i}} \cdot \frac{\partial \boldsymbol{V}}{\partial \chi^{\bar{i}}} \,\mathrm{d}\mathcal{V}_t = \int T^{\bar{i}\bar{j}} \boldsymbol{G}_{\bar{j}} \cdot \frac{\partial \boldsymbol{V}}{\partial \chi^{\bar{i}}} \,\mathrm{d}\mathcal{V}_t.$$

Expressing the derivative of the velocity in terms of the contravariant base vectors associated with the Eulerian coordinates gives

$$\dot{W} = \int_{\Omega_t} T^{\bar{i}\bar{j}} \boldsymbol{G}_{\bar{j}} \cdot V_{\bar{k}}||_{\bar{i}} \boldsymbol{G}^{\bar{k}} \,\mathrm{d}\mathcal{V}_t = \int_{\Omega_t} T^{\bar{i}\bar{j}} V_{\bar{j}}||_{\bar{i}} \,\mathrm{d}\mathcal{V}_t.$$

Hence, by symmetry of the stress tensor,

$$\dot{W} = \int_{\Omega_t} \frac{1}{2} \left[T^{\bar{i}\bar{j}} V_{\bar{j}}||_{\bar{i}} + T^{\bar{j}\bar{i}} V_{\bar{i}}||_{\bar{j}}\right] \,\mathrm{d}\mathcal{V}_t = \int_{\Omega_t} T^{\bar{i}\bar{j}} \frac{1}{2} \left[V_{\bar{i}}||_{\bar{j}} + V_{\bar{j}}||_{\bar{i}}\right] \,\mathrm{d}\mathcal{V}_t$$

$$= \int_{\Omega_t} T^{\bar{i}\bar{j}} D_{\bar{i}\bar{j}} \,\mathrm{d}\mathcal{V}_t = \int_{\Omega_t} \mathsf{T} : \mathsf{D} \,\mathrm{d}\mathcal{V}_t.$$

The net rate of work on the entire continuum is given by the integral of the contraction over both indices of the Cauchy stress and Eulerian rate of deformation tensor. Hence, the quantities T and D are said to be a work conjugate pair: the Cauchy stress is work conjugate to the Eulerian deformation rate. The quantity $\mathsf{T} : \mathsf{D}$ is called the stress power and represents the rate of work done by internal stresses per unit deformed volume of the body.

If we had worked in the Lagrangian coordinates, but in the deformed configuration, then we would have that

$$\dot{W} = \int_{\Omega_t} T^{ij} \frac{1}{2} \left[v_i||_j + v_j||_i \right] \mathrm{d}\mathcal{V}_t = \int_{\Omega_t} T^{ij} \dot{\gamma}_{ij} \, \mathrm{d}\mathcal{V}_t,$$

by using equation (2.47). Transforming to the reference configuration gives

$$\dot{W} = \int_{\Omega_0} T^{ij} \dot{\gamma}_{ij} J \, \mathrm{d}\mathcal{V}_0 = \int_{\Omega_0} s^{ij} \dot{\gamma}_{ij} \, \mathrm{d}\mathcal{V}_0, \tag{4.12}$$

which shows that the second Piola–Kirchhoff stress tensor and the material derivative of the Green–Lagrange strain tensor (the Green–Lagrange rate of strain tensor) are a work conjugate pair. In this case $\mathsf{s} : \dot{\mathsf{e}}$ is the work per unit undeformed volume of the body.

4.4.2 Temperature and heat

We assume that the temperature $\Theta(\boldsymbol{R}, t) > 0$ is a field defined throughout a continuum body. The heat, or thermal energy, is the energy associated specifically with the temperature. A simple, but helpful, viewpoint is that increasing the thermal energy, increases molecular fluctuations, which increases temperature.

The thermal energy of a continuum can be increased or decreased by sources or sinks of heat within the body[2]. In addition, heat can be transferred into or out of a body through conduction, convection or radiation at the surface of the body.

We define a heat supply per unit mass $B(\boldsymbol{R}, t)$ which is the rate per unit time per unit mass at which heat energy is supplied to the body. In addition, we assume that on the surface of the body, the heat transfer per unit deformed area is given by $H(\boldsymbol{R}, t)$ so that the total rate of heating of the body is

$$Q = \int_{\partial\Omega_t} H \, \mathrm{d}\mathcal{S}_t + \int_{\Omega_t} \rho B \, \mathrm{d}\mathcal{V}_t.$$

Assuming that the heat transfer depends only on the normal to the surface, a similar argument to the one used to derive the stress tensor can be used to show that

$$H(\boldsymbol{R}, t) = -\boldsymbol{Q}(\boldsymbol{R}, t) \cdot \boldsymbol{N}(\boldsymbol{R}, t),$$

where $\boldsymbol{N}$ is the outer unit normal to the surface and $\boldsymbol{Q}$ is the (Fourier–Stokes) heat flux vector. The negative sign is chosen so that heat flows into the body, $H > 0$, when the flux vector $\boldsymbol{Q}$ is also directed into the body. Hence the total rate of heating is

$$Q = -\int_{\partial\Omega_t} \boldsymbol{Q} \cdot \boldsymbol{N} \, \mathrm{d}\mathcal{S}_t + \int_{\Omega_t} \rho B \, \mathrm{d}\mathcal{V}_t.$$

[2]For example, these could be wires embedded within the continuum that are heated externally or heated via the passage of an electric current. Another possibility would be to embed chemical heat sources, a localised controlled exothermic reaction.

4.4.3 The first law of thermodynamics

The first law of thermodynamics expresses the notion of conservation of energy and states that doing work or heating increases the internal energy, U. The rate of change of internal energy, $\dot{U}$, is then equal to the sum of the rate of heating, Q, and the net rate of work[3]

$$\frac{DU}{Dt} = Q + \dot{W}. \tag{4.13}$$

Thus, the effects of heat are entirely separated from mechanical work, but the two effects combine through a simple sum to cause a change in internal energy. It is also possible to add additional types of energy input (*e.g.* chemical, electrical) to the balance.

We can make a connection to the more familiar form conservation of energy in classical mechanics by introducing the kinetic energy for a continuum

$$K = \int_{\Omega_t} \frac{\rho}{2} \boldsymbol{V} \cdot \boldsymbol{V} \, \mathrm{d}\mathcal{V}_t; \tag{4.14}$$

$$\Rightarrow \quad \dot{K} = \frac{D}{Dt} \int_{\Omega_t} \frac{\rho}{2} \boldsymbol{V} \cdot \boldsymbol{V} \, \mathrm{d}\mathcal{V}_t.$$

By applying the Reynolds transport theorem and using conservation of mass, as we have done several times already,

$$\dot{K} = \int_{\Omega_t} \frac{\rho}{2} \frac{D}{Dt} (\boldsymbol{V} \cdot \boldsymbol{V}) \, \mathrm{d}\mathcal{V}_t = \int_{\Omega_t} \rho \dot{\boldsymbol{V}} \cdot \boldsymbol{V} \, \mathrm{d}\mathcal{V}_t.$$

Hence, from equation (4.11) and equation (4.13) the statement of conservation of energy becomes

$$\dot{U} = Q + \int_{\partial\Omega_t} \boldsymbol{T} \cdot \boldsymbol{V} \, \mathrm{d}\mathcal{S}_t + \int_{\Omega_t} \rho \boldsymbol{F} \cdot \boldsymbol{V} \, \mathrm{d}\mathcal{V}_t - \dot{K}. \tag{4.15}$$

If the power of the external forces can be written as the derivative of a potential energy[4] then

$$\int_{\partial\Omega_t} \boldsymbol{T} \cdot \boldsymbol{V} \, \mathrm{d}\mathcal{S}_t + \int_{\Omega_t} \rho \boldsymbol{F} \cdot \boldsymbol{V} \, \mathrm{d}\mathcal{V}_t = -\dot{G},$$

which means that

$$\frac{D}{Dt} (U + K + G) = Q.$$

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[3] In much of the literature the equation is not explicitly written as a rate equation so you will often see expressions of the form

$$\frac{DU}{Dt} = \mathcal{Q} + \mathcal{W},$$

where $\mathcal{Q}$ is called the net heating and $\mathcal{W}$ is the net working, which slightly obfuscates the fact that all terms are actually rates.

[4] For example, a gravitational body force corresponds to $\boldsymbol{F} = \boldsymbol{g}$, where $\boldsymbol{g}$ is a constant vector, and so the external power is

$$\int_{\Omega_t} \rho \boldsymbol{F} \cdot \boldsymbol{V} \, \mathrm{d}\mathcal{V}_t = \int_{\Omega_t} \rho \boldsymbol{g} \cdot \dot{\boldsymbol{R}} \, \mathrm{d}\mathcal{V}_t = \frac{D}{Dt} \int_{\Omega_t} \rho \boldsymbol{g} \cdot \boldsymbol{R} \, \mathrm{d}\mathcal{V}_t,$$

by using the Reynolds transport theorem and conservation of mass (in the reverse of our usual argument). Thus the potential energy in this case is

$$G = \int_{\Omega_t} \rho \boldsymbol{g} \cdot \boldsymbol{R} \, \mathrm{d}\mathcal{V}_t.$$

Note that the potential energy related to the external forces is not the same as the internal energy. In the absence of any heating then the total energy, the sum of the internal, kinetic and potential energies, is conserved.

In order to derive a partial differential equation that expresses conservation of energy we define an internal energy density per unit mass, $\Phi(\boldsymbol{R},t)$, so that

$$U = \int_{\Omega_t} \rho\Phi \, \mathrm{d}\mathcal{V}_t.$$

The integral form of conservation of energy (4.13) is thus

$$\frac{D}{Dt}\int_{\Omega_t} \rho\Phi \, \mathrm{d}\mathcal{V}_t = \int_{\Omega_t} \mathsf{T} : \mathsf{D} \, \mathrm{d}\mathcal{V}_t + \int_{\Omega_t} \rho B \, \mathrm{d}\mathcal{V}_t - \int_{\partial\Omega_t} \boldsymbol{Q} \cdot \boldsymbol{N} \, \mathrm{d}\mathcal{S}_t. \tag{4.16}$$

Using the Reynolds transport theorem combined with conservation of mass and the divergence theorem, we obtain

$$\int_{\Omega_t} \left[\rho\frac{D\Phi}{Dt} - \mathsf{T} : \mathsf{D} - \rho B + \boldsymbol{\nabla}_{\boldsymbol{R}} \cdot \boldsymbol{Q}\right] \mathrm{d}\mathcal{V}_t = 0,$$

and because the integral must be zero for every volume, we obtain the required differential equation

$$\rho\,\frac{D\Phi}{Dt} = \mathsf{T} : \mathsf{D} + \rho B - \boldsymbol{\nabla}_{\boldsymbol{R}} \cdot \boldsymbol{Q}. \tag{4.17}$$

In component form in the Eulerian coordinates the equation (4.17) becomes

$$\rho\,\frac{D\Phi}{Dt} = T^{\bar{i}\bar{j}} D_{\bar{i}\bar{j}} + \rho B - Q^{\bar{i}}||_{\bar{i}}. \tag{4.18}$$

The equation (4.17) is a scalar equation that expresses the rate of change of internal energy in Eulerian coordinates. For the Lagrangian viewpoint, we transform the integrals in equation (4.16) from the deformed to the undeformed domain, so that

$$\frac{D}{Dt}\int_{\Omega_0} \rho\Phi\, J \, \mathrm{d}\mathcal{V}_0 = \int_{\Omega_0} \mathsf{T} : \mathsf{D}\, J \, \mathrm{d}\mathcal{V}_0 + \int_{\Omega_0} \rho B \, \mathrm{d}\mathcal{V}_0 - \int_{\partial\Omega_0} J\, \boldsymbol{Q} \cdot \mathsf{F}^{-T} \boldsymbol{n} \, \mathrm{d}\mathcal{S}_0, \tag{4.19}$$

where we have used Nanson's relation (2.42) to transform the surface integral. From conservation of mass $\rho_0 = \rho\, J$ and because the undeformed domain is fixed the material derivative passes under the integral sign. We also use equation (4.12) to express the net rate of work in terms of the second Piola–Kirchhoff stress tensor so that equation (4.19) becomes

$$\int_{\Omega_0} \rho_0 \frac{\partial\phi}{\partial t} - s^{ij}\dot{\gamma}_{ij} - \rho_0 b \, \mathrm{d}\mathcal{V}_0 = -\int_{\partial\Omega_0} \boldsymbol{q} \cdot \boldsymbol{n} \, \mathrm{d}\mathcal{S}_0,$$

where $\phi(\boldsymbol{r},t) = \Phi(\boldsymbol{R},t)$ is the internal energy in the Lagrangian representation; $b(\boldsymbol{r},t) = B(\boldsymbol{R},t)$ is the Lagrangian heat supply and $\boldsymbol{q}$ is the heat flux per unit area of undeformed surface, $\boldsymbol{q} = J\boldsymbol{Q}F^{-T}$ or $q^j = J\frac{\partial\xi^j}{\partial\chi^{\bar{i}}}Q^{\bar{i}}$. Using the divergence theorem and making our usual argument about the integral being zero for every volume gives the required equation in the Lagrangian representation

$$\rho_0\frac{\partial\phi}{\partial t} = s^{ij}\dot{\gamma}_{ij} + \rho_0 b - q^i|_i. \tag{4.20}$$

4.4.4 The second law of thermodynamics

The essence of the second law of thermodynamics is that there is a physical limit to the rate at which heat can be absorbed by a body, but no limit to the rate at which it can be released. In its simplest form, the law states that

$$Q \leq \mathcal{B}, \tag{4.21}$$

where $\mathcal{B}$ is the least upper bound for the net heating. From the first law (4.13) in the absence of any net working,

$$\frac{DU}{Dt} = Q \leq \mathcal{B},$$

which gives another interpretation of the second law as stating that there is an upper bound to the rate at which a body can store internal energy without doing mechanical work.

For a body at uniform temperature, Θ, we define a quantity called the entropy[5], $\mathcal{H}$, such that

$$\frac{D\mathcal{H}}{Dt} = \frac{\mathcal{B}}{\Theta}. \tag{4.22}$$

The rate of change of entropy represents the ability of a particular material to absorb heat. Combining the second law (4.21) with the definition of entropy (4.22) gives the classic Clausius–Planck inequality

$$\Theta\dot{\mathcal{H}} \geq Q. \tag{4.23}$$

Thus, if there is no heat input, then the total entropy of the body cannot decrease

$$\dot{\mathcal{H}} \geq 0, \quad \text{when} \quad Q = 0,$$

which leads to the common statement of the second law that "entropy increases".

More generally, we assume that there exists a specific entropy η such that for a continuum

$$\mathcal{H} = \int_{\Omega_t} \rho\eta \, \mathrm{d}\mathcal{V}_t.$$

In addition, we assume that the rate of change in total entropy is greater than or equal to the net heating per unit temperature

$$\dot{\mathcal{H}} \geq -\int_{\partial\Omega_t} \frac{\boldsymbol{Q}\cdot\boldsymbol{N}}{\Theta(\boldsymbol{R},t)} \, \mathrm{d}\mathcal{S}_t + \int_{\Omega_t} \frac{\rho B}{\Theta(\boldsymbol{R},t)} \, \mathrm{d}\mathcal{V}_t,$$

which is known as the Clausius–Duhem inequality. Note that this does not follow directly from equation (4.23) unless the temperature is uniform.

Using the Reynolds transport theorem, conservation of mass and the divergence theorem, we obtain

$$\int_{\Omega_t} \left[\rho\dot{\eta} + \boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\left(\frac{\boldsymbol{Q}}{\Theta}\right) - \rho\frac{B}{\Theta}\right] \mathrm{d}\mathcal{V}_t \geq 0.$$

[5]The introduction of the entropy was originally motivated in equilibrium thermodynamics by the desire to write the change in heat energy in the first law as an exact differential $Q\mathrm{d}t = \Theta\,\mathrm{d}\mathcal{H}$, so that $\mathrm{d}H = Q\mathrm{d}t/\Theta$. The problem is that this concept does not easily extend to nonequilibrium thermodynamics, because the value of a function at a single point (the limit of an equilibrium) does not determine the function. The approach taken here (which is standard) avoids this issue by treating the entropy as a fundamental property of matter.

If the integral must be non-negative for any volume, no matter how small, then the integrand must be non-negative, so

$$\rho\dot{\eta} \geq -\boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\left(\frac{\boldsymbol{Q}}{\Theta}\right) + \rho\frac{B}{\Theta}.$$

Expanding the divergence term and multiplying through by Θ, we obtain the equation

$$\rho\Theta\dot{\eta} + \boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{Q} - \rho B - \frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!\boldsymbol{R}}\Theta \geq 0, \tag{4.24}$$

or

$$D - \frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!\boldsymbol{R}}\Theta \geq 0, \tag{4.25}$$

where

$$D = \rho\Theta\dot{\eta} + \boldsymbol{\nabla}_{\!\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{Q} - \rho B. \tag{4.26}$$

is called the internal dissipation per unit deformed volume. The internal dissipation is the increase in specific entropy that is not due to net heating. It represents the losses in other parts of the system. From the local form of the conservation of energy (4.17), we can write the dissipation as

$$D = \rho\Theta\dot{\eta} - \rho\dot{\Phi} + \mathsf{T} : \mathsf{D}, \tag{4.27}$$

which indicates that the dissipation is the difference between the rate of change of specific entropy and the change in internal energy that is not due to mechanical work. In the absence of temperature gradients, the dissipation must be non-negative

$$D \geq 0,$$

which is sometimes called Planck's inequality. Alternatively, if $D = 0$, then

$$-\frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!\boldsymbol{R}}\Theta \geq 0,$$

which is the statement that heat flows from regions of high temperature to low temperature in the absence of an internal energy supply[6]. We will find it convenient to introduce the Helmholtz free energy per unit mass

$$\Psi(\boldsymbol{R},t) = \Phi(\boldsymbol{R},t) - \eta(\boldsymbol{R},t)\Theta(\boldsymbol{R},t), \tag{4.28}$$

which from equation (4.27) is the energy available to do mechanical work in a non-dissipative (reversible) system at constant temperature. Using the definition of the free energy (4.28) in equations (4.25) and (4.26), we obtain another form of the entropy inequality

$$-\rho\dot{\Psi} - \rho\eta\dot{\Theta} - \frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!\boldsymbol{R}}\Theta + \mathsf{T} : \mathsf{D} \geq 0. \tag{4.29}$$

In Lagrangian form, by the usual methods, we obtain the inequality

$$\rho_0\dot{\eta}_0 \geq -\boldsymbol{\nabla}_{\!\boldsymbol{r}}\boldsymbol{\cdot}\left(\frac{\boldsymbol{q}}{\theta}\right) + \rho_0\frac{b}{\theta},$$

where in Lagrangian coordinates $\eta_0(\boldsymbol{r},t) = \eta(\boldsymbol{R},t)$ is the specific entropy and $\boldsymbol{\theta}(\boldsymbol{r},t) = \Theta(\boldsymbol{R},t)$. Introducing the free energy field

$$\psi(\boldsymbol{r},t) = \phi(\boldsymbol{r},t) - \eta_0(\boldsymbol{r},t)\theta(\boldsymbol{r},t),$$

we obtain the equivalent form of the inequality

$$-\rho_0\dot{\psi} - \rho_0\eta_0\dot{\theta} - \frac{1}{\theta}\boldsymbol{q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!\boldsymbol{r}}\theta + \mathsf{s} : \dot{\mathsf{e}} \geq 0.$$

[6]Truesdell notes that the form of equation (4.26) allows heat to flow from regions of low temperature to high temperature if a sufficiently large internal energy is supplied near a cold spot, or if sinks of energy are present at hot spots.