Chapter 3

Forces, Momentum & Stress

3.1 Newtonian mechanics: a very brief résumé

In classical Newtonian particle mechanics, particles (lumps of matter) only experience acceleration when acted on by external influences which are known as *forces*[1]. The resistance to acceleration is described by a constant of proportionality known as the mass[2], a fundamental property of matter, that is positive (or zero) and invariant under motion. Newton's second law, as originally formulated for a particle of mass m can be interpreted as a definition of force[3] $\boldsymbol{F}$,

$$m\boldsymbol{a} = \boldsymbol{F} \quad \Rightarrow \quad m\ddot{\boldsymbol{R}} = \boldsymbol{F},$$

where $\boldsymbol{a}$ is the (Lagrangian) acceleration of the particle; $\boldsymbol{R}$ is the current position; $\boldsymbol{F}$ is the resultant force applied to the particle; and the overdots represent (material) differentiation with respect to time. It is important to remember that Newton's second law is a postulate, or axiom, based on experimental observations.

Consider now a collection of particles, each one having mass $m_{(i)}$ and position $\boldsymbol{R}_{(i)}$ in the fixed global coordinate system. The index (i) is a Lagrangian marker to keep track of individual particles. We assume that there is mutual symmetric interaction between the particles that acts along the line between the particles[4] so that the particle i experiences a force $\boldsymbol{F}_{(ij)}$ due to the j-th particle and that $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{(ji)}$. Applying Newton's second law to each particle gives

$$m_{(i)}\ddot{\boldsymbol{R}}_{(i)} = \boldsymbol{F}^{\text{ext}}_{(i)} + \sum_{j\neq i} \boldsymbol{F}_{(ij)}, \tag{3.1}$$

where $\boldsymbol{F}^{\text{ext}}_{(i)}$ is the resultant external force on the i-th particle.

If we consider a subset (ensemble) of N particles, which we can define so that the index $i \in I = \{1, \dots N\}$, then we can define the total mass M, linear momentum $\boldsymbol{P}$, and angular momentum (or moment of momentum) about the origin $\boldsymbol{H}_{\mathbf{0}}$, of the ensemble by

$$M = \sum_{i=1}^{N} m_{(i)}, \quad \boldsymbol{P} = \sum_{i=1}^{N} m_{(i)}\dot{\boldsymbol{R}}_{(i)} \quad \text{and} \quad \boldsymbol{H}_{\mathbf{0}} = \sum_{i=1}^{N} \boldsymbol{R}_{(i)} \times m_{(i)}\dot{\boldsymbol{R}}_{(i)}. \tag{3.2}$$

[1]This is essentially Newton's first law.
[2]Strictly speaking this is the inertial mass.
[3]Although Truesdell dislikes this interpretation, arguing that we typically have a *priori* knowledge about forces.
[4]This is Newton's third law.

Using equation (3.1) and the fact that the mass of each particle in invariant, we can write

$$\frac{D}{Dt}\boldsymbol{P} = \sum_{i=1}^{N} \frac{\partial}{\partial t}\left(m_{(i)}\dot{\boldsymbol{R}}_{(i)}\right) = \sum_{i=1}^{N} m_{(i)}\ddot{\boldsymbol{R}}_{(i)} = \sum_{i=1}^{N}\left[\boldsymbol{F}^{\text{ext}}_{(i)} + \sum_{j\neq i}\boldsymbol{F}_{(ij)}\right],$$

but because $F_{(ij)} = -F_{(ji)}$ the internal forces between particles in the subset cancel and we have

$$\frac{D}{Dt}\boldsymbol{P} = \sum_{i=1}^{N}\left[\boldsymbol{F}^{\text{ext}}_{(i)} + \sum_{j\notin I}\boldsymbol{F}_{(ij)}\right] \equiv \boldsymbol{\mathcal{F}}, \tag{3.3}$$

Equation (3.3) states states that the material rate of change of the total linear momentum is given by the sum of the external forces and interactions from other particles not in the subset, which can be defined as the resultant external force, $\boldsymbol{\mathcal{F}}$. Equation (3.3) expresses the balance of linear momentum and also its conservation — in the absence of external forces, the material rate of change of linear momentum is zero. Note that although momentum is not conserved when external forces are present, the equation (3.3) is nonetheless often termed the conservation of momentum equation, or even just the momentum equation. In taking this sum, we are assuming that momenta can be summed even though they correspond to different spatial locations. The standard resolution of this concern is to recognise that

$$\frac{D}{Dt}\boldsymbol{P} = \sum_{i=1}^{N} m_{(i)} \frac{\sum_{i=1}^{N} m_{(i)}\ddot{\boldsymbol{R}}_i}{\sum_{i=1}^{N} m_{(i)}} = M\ddot{\overline{\boldsymbol{R}}},$$

where $\overline{\boldsymbol{R}} = (\sum_{i=1}^{N} m_{(i)}\boldsymbol{R}_{(i)})/M$ is called the centre of mass (the average position weighted by the mass distribution) and so equation (3.3) can be interpreted as an expression for the motion of the centre of mass.

Similarly, we can write

$$\frac{D}{Dt}\boldsymbol{H}_{\boldsymbol{0}} = \sum_{i=1}^{N}\dot{\boldsymbol{R}}_{(i)} \times m_{(i)}\dot{\boldsymbol{R}}_{(i)} + \sum_{i=1}^{N}\boldsymbol{R}_{(i)} \times m_{(i)}\ddot{\boldsymbol{R}}_{(i)} = \sum_{i=1}^{N}\boldsymbol{R}_{(i)} \times \left[\boldsymbol{F}^{\text{ext}}_{(i)} + \sum_{j\neq i}\boldsymbol{F}_{(ij)}\right],$$

because $\dot{\boldsymbol{R}}_{(i)} \times \dot{\boldsymbol{R}}_{(i)} = \boldsymbol{0}$. The mutual interaction between two particles i and j in the set is given by the two terms

$$\boldsymbol{R}_{(i)} \times \boldsymbol{F}_{(ij)} + \boldsymbol{R}_{(j)} \times \boldsymbol{F}_{(ji)} = \boldsymbol{R}_{(i)} \times \boldsymbol{F}_{(ij)} - \boldsymbol{R}_{(j)} \times \boldsymbol{F}_{(ij)} = (\boldsymbol{R}_{(i)} - \boldsymbol{R}_{(j)}) \times \boldsymbol{F}_{(ij)} = \boldsymbol{0},$$

because the line of action of the interaction is parallel to the line between the particles. Hence, the expression for the rate of change of the total angular momentum about the origin is

$$\frac{D}{Dt}\boldsymbol{H}_{\boldsymbol{0}} = \sum_{i=1}^{N}\boldsymbol{R}_{(i)} \times \left[\boldsymbol{F}^{\text{ext}}_{(i)} + \sum_{j\notin I}\boldsymbol{F}_{(ij)}\right] \equiv \boldsymbol{\mathcal{L}}_{\boldsymbol{0}}, \tag{3.4}$$

which is equal to the resultant external torque[5] (or moment) about the origin on the system, $\boldsymbol{\mathcal{L}}_{\boldsymbol{0}}$. The equation (3.4) expresses the balance of angular momentum, and its conservation when the external torque is zero.

[5]In classical mechanics the torque, or turning moment, is given by the cross product of the applied force and the distance from the point of application of the force to a pivot point, $\boldsymbol{L}_0 = \boldsymbol{R} \times \boldsymbol{F}$. Here, the pivot point is taken to be the origin and $\boldsymbol{F}$ is the applied force. The torque vector points along the axis of rotation and its sense is such that in a right-handed coordinate system, positive magnitudes correspond to anticlockwise rotations.

3.2 The continuum hypothesis

In the continuum hypothesis we assume that we can replace the discrete particles by a continuous distribution of matter, so that kinematic quantities such as acceleration, velocity and displacement are the averages of the individual particulate (molecular) motion. We introduce distributed masses and forces that are functions of position and the continuum hypothesis is essentially the assumption that such quantities exist.

3.2.1 Mass

For a fixed point in space $\boldsymbol{R}$, we define regions Ω_ϵ such that $\boldsymbol{R} \in \Omega_\epsilon$ and the volumes $\mathcal{V}_{\Omega_\epsilon}$ of the regions tend to zero as $\epsilon \to 0$. If the total mass of the continuum contained within the region Ω_ϵ is $\mathcal{M}_{\Omega_\epsilon}$, then the mass density field $\rho(\boldsymbol{R}, t)$ is defined by the limit

$$\rho(\boldsymbol{R}, t) = \lim_{\epsilon \to 0} \mathcal{M}_{\Omega_\epsilon} / \mathcal{V}_{\Omega_\epsilon}. \tag{3.5}$$

Under the continuum hypothesis we assume that this limit exists and is well defined and does not depend on the precise choice of volumes Ω_ϵ.

Assuming the existence of a mass density then the total mass within a deformed region Ω_t is given by

$$M(t) = \int_{\Omega_t} \rho(\boldsymbol{R}, t) \, \mathrm{d}\mathcal{V}_t,$$

which generalises the expression for the total mass in equation (3.2).

3.2.2 Momentum

Having asserted the existence of the mass density then the generalisation of the expression (3.2) for the total linear momentum in the deformed region Ω_t is simply

$$\boldsymbol{P}(t) = \int_{\Omega_t} \rho(\boldsymbol{R}, t) \boldsymbol{V}(\boldsymbol{R}, t) \, \mathrm{d}\mathcal{V}_t;$$

and the total angular momentum of the deformed region Ω_t about a point $\boldsymbol{Z}$ is given by

$$\boldsymbol{H}_{\boldsymbol{Z}}(t) = \int_{\Omega_t} (\boldsymbol{R} - \boldsymbol{Z}) \times [\rho(\boldsymbol{R}, t) \boldsymbol{V}(\boldsymbol{R}, t)] \, \mathrm{d}\mathcal{V}_t.$$

Note that these expressions may be further generalised to include the effects of additional point masses if required.

3.2.3 Balance of momenta

We can now generalise the equations (3.3) and (3.4) to apply to continua by writing

$$\frac{D\boldsymbol{P}}{Dt} = \frac{D}{Dt} \int_{\Omega_t} \rho \boldsymbol{V} \, \mathrm{d}\mathcal{V}_t = \boldsymbol{\mathcal{F}}, \tag{3.6}$$

where $\boldsymbol{\mathcal{F}}$ is the resultant force on the body; and

$$\frac{D\boldsymbol{H}_{\boldsymbol{Z}}}{Dt} = \frac{D}{Dt} \int_{\Omega_t} (\boldsymbol{R} - \boldsymbol{Z}) \times \rho \boldsymbol{V} \, \mathrm{d}\mathcal{V}_t = \boldsymbol{\mathcal{L}}_{\boldsymbol{Z}}, \tag{3.7}$$

where $\boldsymbol{\mathcal{L}}_{\boldsymbol{Z}}$ is the resultant torque about the point $\boldsymbol{Z}$. Again, we should note that these equations are postulates, or axioms, and cannot be formally derived from Newton's second law for particles, which is itself a postulate. Although we now know the (postulated) governing equations, we do not yet know how to express the forces and torques that act on continua, rather than particles.

3.2.4 Loads: Forces and Torques

The forces and torques that act on continua can be divided into three types:

(i) External body forces and torques that act on every mass within the region in question: Body forces include gravity, electromagnetic forces and the fictional centripetal/centrifugal forces that arise due to rotation if the governing equations are written in a rotating frame. A body force is usually expressed as a force density per unit mass, $\boldsymbol{F}(\boldsymbol{R}, t)$, or a force density per unit volume, $\widehat{\boldsymbol{F}} = \rho \boldsymbol{F}$ and under the continuum hypothesis we assume that such a densities exist. Hence, the total resultant body force acting on a volume Ω_t is

$$\boldsymbol{\mathcal{F}}^V = \int_{\Omega_t} \rho \boldsymbol{F} \, \mathrm{d}\mathcal{V}_t,$$

which is the generalisation of the first sum on the right-hand side of the equation of balance of linear momentum for systems of particles (3.3).

Physical evidence suggests that body torques are not possible, but they can be included in the theory as a torque (or moment) per unit mass or per unit volume. The argument used to rule out the existence of body torques is that if the couple is pictured as a pair of equal and opposite forces connected by a rod, then if the rod length tends to zero and the forces remain bounded the torque must be zero[6].

Nonetheless, the body force can exert a net torque on the body about a point $\boldsymbol{Z}$, which is given by

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{Z}}^V = \int_{\Omega_t} (\boldsymbol{R} - \boldsymbol{Z}) \times \rho \boldsymbol{F} \, \mathrm{d}\mathcal{V}_t,$$

the generalisation of the first sum on the right-hand side of the equation of balance of angular momentum for systems of particles (3.4).

(ii) External surface forces and torques that act on the exposed surfaces of the body: These forces typically arise due to contact between bodies and usually consist of a tangential friction and normal pressure. If $\boldsymbol{R}$ is a point on the external surface of the body, we define a set of surrounding surface regions S_ϵ on the surface such that $S_\epsilon \subset \partial\Omega_t$, $\boldsymbol{R} \in S_\epsilon$ and the area of S_ϵ is given by $\mathcal{A}_\epsilon$. We assume that the areas $\mathcal{A}_\epsilon$ tend to zero as $\epsilon \to 0$. If the resultant force acting on the surface S_ϵ is given by $\boldsymbol{F}_\epsilon$ then the traction is defined by the limit

$$\boldsymbol{T}(\boldsymbol{R}, t) = \lim_{\epsilon \to 0} \boldsymbol{F}_\epsilon / \mathcal{A}_\epsilon, \tag{3.8}$$

and under the continuum hypothesis the limit exists. The traction field $\boldsymbol{T}$ is the force per unit area of the deformed surface. Hence the total resultant force on the volume Ω_t due to surface loads is

$$\boldsymbol{\mathcal{F}}^S = \int_{\partial\Omega_t} \boldsymbol{T} \, \mathrm{d}\mathcal{S}_t,$$

[6] There is always an exception of course, ferrofluids can be treated as suspensions of magnetic dipoles which do provide body torques, but the dipole does require unbounded forces.

which can be identified with the second sum on the right-hand side of the equation of balance of linear momentum for systems of particles (3.3) if we further assume that it is only particles on the "surface" of the subset that can be affected by the particles outside the subset. In other words, all long-range influences must be treated as body forces.

The physical evidence for surface torques is not clear, but in certain theories of granular materials (sands, powders, etc), it is assumed that different elements of the continuum can apply surface torques on each other due to irregularities of the particles. There is a body of work called Cosserat theory, or the theory of micro-polar materials, that deals with the inclusion of surface torques, but we shall not include them in the present development. As for body forces, a surface traction can exert a net torque on the body about a point $\boldsymbol{Z}$ which is given by

$$\boldsymbol{\mathcal{L}}^S_{\boldsymbol{Z}} = \int_{\partial\Omega_t} (\boldsymbol{R} - \boldsymbol{Z}) \times \boldsymbol{T} \, \mathrm{d}\mathcal{S}_t,$$

the generalisation of the second term on the right-hand side of equation (3.4) under the same assumptions about long-range interactions.

(iii) Internal forces that act between mass elements within the body: By Newton's third law mutual interaction between all internal masses is zero. Nonetheless, we may examine the state of the internal forces by considering an internal surface that divides a volume, see Figure 3.1. The force per unit area exerted on the material on one side of the surface by the material on the other can be represented by a stress vector or internal traction vector that is again defined by the limit of the resultant force acting on an area of of the internal surface as the area tends to zero.

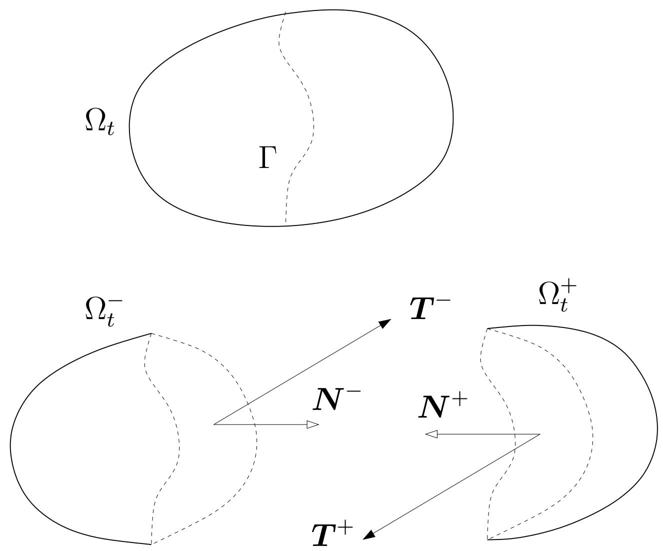

Figure 3.1: A region of a continuum Ω_t is divided by a nominal internal surface Γ into the region Ω_t^+ and Ω_t^-. A outer unit normal can be defined for Ω_t^+ and Ω_t^-, say $\boldsymbol{N}^+$ and $\boldsymbol{N}^-$ respectively. On Γ, $\boldsymbol{N}^+ = -\boldsymbol{N}^-$ and the associated stress vectors, $\boldsymbol{T}^+$ and $\boldsymbol{T}^-$ are also equal and opposite.

3.3 The stress tensor

3.3.1 Cauchy's Postulate

We can characterise a surface by a unit normal $\boldsymbol{N}$ taken in a certain sense. If the surface is external to the body then we can define the normal to be directed out of the volume. If it is internal, there is a certain ambiguity which can be resolved by treating the surface as one dividing the body into two regions, see Figure 3.1, and taking the outer unit normal in each region. Naturally we have to assume that the unit normal to the surface is well-defined.

Under Cauchy's postulate we assume that the surface traction depends only on the unit normal, $\boldsymbol{N}$, to the surface, *i.e.*

$$\boldsymbol{T}(\boldsymbol{R}) = \boldsymbol{T}(\boldsymbol{N}(\boldsymbol{R}), \boldsymbol{R}).$$

In particular, we are not assuming any dependence of the traction on higher derivatives of the surface, such as curvature. In fact, the hypothesis can be proved directly from the momentum equation in the absence of any surface couples, a result known as Noll's theorem.

If we assume that the resultant force due to surface loads on a region tends to zero as the volume of the region tends to zero, it can be shown by considering a small cylindrical volume (a pillbox) that

$$\boldsymbol{T}(\boldsymbol{N}(\boldsymbol{R}), \boldsymbol{R}) = -\boldsymbol{T}(-\boldsymbol{N}(\boldsymbol{R}, \boldsymbol{R}).$$

Hence, for an internal surface, the surface tractions exerted by one half of the material on the other are equal and opposite, see Figure 3.1

3.3.2 The existence of the stress tensor

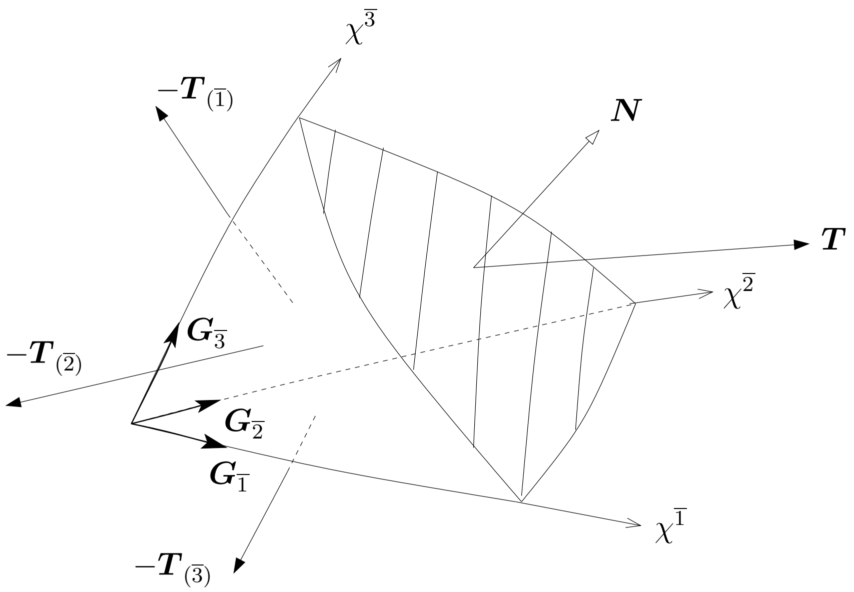

Figure 3.2: An infinitesimal tetrahedron in the deformed body aligned with the Eulerian coordinates. Three edges of the tetrahedron are directed along the covariant base vectors $\boldsymbol{G}_{\bar{i}}$, which means that normals to corresponding three faces are given by $\boldsymbol{G}^{\bar{j}}$.

We now consider the force balance on a small tetrahedral region in the deformed body with volume $\mathrm{d}V$, see Figure 3.2 The tetrahedron is chosen so that three edges are parallel to the coordinate directions $\chi^{\bar{i}}$ which means that three of its faces are aligned with planes of constant $\chi^{\bar{i}}$. The plane of constant $\chi^{\bar{i}}$ is spanned by the two covariant base vectors $\boldsymbol{G}_{\bar{j}}$ where $j \neq i$. Each tetrahedral face $\chi^{\bar{i}}$ is constant can be represented by a vector normal to the face directed out of the volume with magnitude equal to the area of the face,

$$\boldsymbol{S}_{(\bar{i})} = \boldsymbol{N}_{(\bar{i})}\,\mathrm{d}S_{(\bar{i})} = -\frac{\boldsymbol{G}^{\bar{i}}\,\mathrm{d}S_{(\bar{i})}}{\sqrt{G^{\bar{i}\bar{i}}}}, \quad \text{(no summation)}$$

where $\boldsymbol{G}^{\bar{i}}/\sqrt{(G^{\bar{i}\bar{i}})}$ is the unit vector normal to the face of constant $\chi^{\bar{i}}$ and the area of the face is $\mathrm{d}S_{(\bar{i})}$. We have used the contravariant (upper index) vector to ensure orthogonality to the face via an analogous result to equation (1.35).

The final face of the tetrahedron is not aligned with any coordinate direction and has the vector representation $\boldsymbol{N}\mathrm{d}S$. In the infinitesimal limit, the curves bounding each face become triangles and the edges of the final face can be written as differences between the covariant base vectors $\boldsymbol{G}_{\bar{i}}$. By taking cross-products of the bounding vectors to determine the vector representations of each face, it follows that the vector sum of the surfaces is zero

$$\boldsymbol{N}\mathrm{d}S - \sum_{\bar{i}} \frac{\boldsymbol{G}^{\bar{i}}\mathrm{d}S_{(\bar{i})}}{\sqrt{(G^{\bar{i}\bar{i}})}} = 0. \tag{3.9}$$

We decompose the normal into components in the contravariant basis $\boldsymbol{N} = N_{\bar{i}}\boldsymbol{G}^{\bar{i}}$, and then taking the dot product of equation (3.9) with $\boldsymbol{G}_{\bar{j}}$ gives

$$N_{\bar{j}}\,\mathrm{d}S = \frac{\mathrm{d}S_{(\bar{j})}}{\sqrt{(G^{\bar{j}\bar{j}})}} \quad \Rightarrow \quad N_{\bar{j}}\sqrt{(G^{\bar{j}\bar{j}})}\,\mathrm{d}S = \mathrm{d}S_{(\bar{j})} \quad \text{(no summation).}$$

In general, three stress vectors $-\boldsymbol{T}_{(\bar{i})}$ will act on the faces of constant $\chi^{\bar{i}}$ and a stress vector $\boldsymbol{T}$ will act on the final face of the tetrahedron. If the tetrahedron of material obeys the balance of momentum (3.6) and if the tetrahedron is sufficiently small, the total resultant force is given by

$$\rho\boldsymbol{F}\,\mathrm{d}V + \boldsymbol{T}\mathrm{d}S - \sum_{\bar{i}} \boldsymbol{T}_{(\bar{i})}\,\mathrm{d}S_{(\bar{i})} = \rho\dot{\boldsymbol{V}}\,\mathrm{d}V,$$

where $\boldsymbol{F}$ is the resultant body force acting on the volume and $\rho\dot{\boldsymbol{V}}$ its acceleration. Assuming that the body force and the acceleration remains bounded then these terms are negligible because if a typical edge length is given by $\mathrm{d}\epsilon$, then $\mathrm{d}V \sim (\mathrm{d}\epsilon)^3$, whereas $\mathrm{d}S \sim (\mathrm{d}\epsilon)^2$. As $\mathrm{d}\epsilon \to 0$, $\mathrm{d}V$ is an order of magnitude smaller than $\mathrm{d}S$ and so the surface tractions are the only terms that remain

$$\boldsymbol{T}\mathrm{d}S - \sum_{\bar{i}} \boldsymbol{T}_{(\bar{i})}\mathrm{d}S_{(\bar{i})} = 0,$$

$$\boldsymbol{T}\mathrm{d}S - \sum_{\bar{i}} \boldsymbol{T}_{(\bar{i})}N_{\bar{i}}\sqrt{(G^{\bar{i}\bar{i}})}\mathrm{d}S = 0,$$

$$\Rightarrow \boldsymbol{T} = \sum_{\bar{i}} N_{\bar{i}}\sqrt{(G^{\bar{i}\bar{i}})}\,\boldsymbol{T}_{(\bar{i})}, \tag{3.10}$$

From Cauchy's postulate the traction vector $\boldsymbol{T}$ must remain invariant[7] provided the normal $\boldsymbol{N}$ does not change. By construction $N_{\bar{i}}$ are components of the normal vector that transform covariantly, so $\boldsymbol{T}_{(\bar{i})}\sqrt{(G^{\bar{i}\,\bar{i}})}$ must transform contravariantly in order for $\boldsymbol{T}$ to remain invariant. Hence, we can write

$$\boldsymbol{T}_{(\bar{i})}\sqrt{(G^{\bar{i}\,\bar{i}})} = T^{\bar{i}\,\bar{j}}\boldsymbol{G}_{\bar{j}},$$

which means that equation (3.10) becomes

$$\boldsymbol{T} = T^{\bar{i}\,\bar{j}}N_{\bar{i}}\boldsymbol{G}_{\bar{j}} \quad \Rightarrow \quad \boldsymbol{T}\cdot\boldsymbol{G}^{\bar{k}} = T^{\bar{i}\,\bar{j}}N_{\bar{i}}\boldsymbol{G}_{\bar{j}}\cdot\boldsymbol{G}^{\bar{k}} \quad \Rightarrow \quad T^{\bar{k}} = T^{\bar{i}\,\bar{j}}N_{\bar{i}}\,\delta^{\bar{k}}_{\bar{j}} = T^{\bar{i}\,\bar{k}}N_{\bar{i}}.$$

The quantity $T^{\bar{i}\,\bar{j}}$ is called the Cauchy stress tensor and represents the $\bar{j}$-th component (in the deformed covariant Eulerian basis) of the traction vector acting on a surface with normal component in the $\bar{i}$-th direction (also in the deformed covariant Eulerian basis). The above argument demonstrates that if $\boldsymbol{T}(\boldsymbol{N})$ then it is a linear function of $\boldsymbol{N}$, which means that

$$\boldsymbol{T} = \mathcal{T}(\boldsymbol{N}),$$

where $\mathcal{T}$ is a tensor[8].

The covariant base vectors are not necessarily of unit length and so the physical components of the stress tensor are obtained by expressing the stress vectors in terms of unit tangent vectors

$$\boldsymbol{T}_{(\bar{i})} = \sum_{\bar{j}} \sigma_{(\bar{i}\,\bar{j})}\boldsymbol{G}_{\bar{j}}/\sqrt{(G_{\bar{j}\,\bar{j}})} \quad \Rightarrow \quad \sigma_{(\bar{i}\,\bar{j})} = \sqrt{(G_{\bar{j}\,\bar{j}})/(G^{\bar{i}\,\bar{i}})}\;T^{\bar{i}\,\bar{j}} \quad \text{(no summation)}.$$

The components $\sigma_{(\bar{i}\,\bar{j})}$ are not, in general, a representation of a tensor unless the coordinate system $\chi^{\bar{i}}$ is orthonormal, in which case $G_{\bar{i}\,\bar{i}} = G^{\bar{j}\,\bar{j}} = 1$ (no summation) and so $\sigma_{(\bar{i}\,\bar{j})} = T^{\bar{i}\,\bar{j}}$. In the global Cartesian basis in the deformed configuration we can write

$$\sigma_{IJ} = \sigma^{IJ} = \frac{\partial X_I}{\partial \chi^{\bar{i}}}\,T^{\bar{i}\,\bar{j}}\frac{\partial X_J}{\partial \chi^{\bar{j}}}.$$

If we only use Cartesian coordinates, then $\xi^i = x^i$ and $\chi^{\bar{j}} = X^{\bar{j}}$ (capitals are not used for the Cartesian indices so that the equations are legal index notation), and

$$\sigma_{IJ} = \delta^{I}_{\bar{i}}\,T^{\bar{i}\,\bar{j}}\,\delta^{J}_{\bar{j}} = T^{IJ},$$

as claimed.

3.3.3 Alternative forms of the stress tensor

We derived the components of the Cauchy stress tensor $T^{\bar{i}\,\bar{j}}$ by considering the force per unit area in the deformed configuration in which the traction vector and the normal to the surface are both represented in base vectors associated with the Eulerian coordinates, $\chi^{\bar{i}}$. Alternatively we can represent the components of this tensor in the base vectors associated with the Lagrangian

[7] We are actually assuming here that the traction is a vector that is invariant under coordinate transformation.

[8] The word tensor is derived from the Latin verb tendere meaning to tighten or or to stretch. The demonstration that the stress vector acting on a surface is a linear mapping of the normal vector is due to Cauchy and establishes the existence of an object that we now call a tensor.

coordinates, ξ^i, in the deformed position. Exactly the same tetrahedral argument can be applied to show that the traction is given by

$$\boldsymbol{T} = T^{ij} N_i \boldsymbol{G}_j,$$

where T^{ij} is the required tensor after a suitable coordinate transformation

$$T^{ij} = \frac{\partial \xi^i}{\partial \chi^{\bar{k}}} T^{\bar{k}\bar{l}} \frac{\partial \xi^j}{\partial \chi^{\bar{l}}}; \tag{3.11}$$

this is the stress tensor used by Green & Zerna and is sometimes called the body stress. Although it is given a different name the body stress represents the same quantity as the Cauchy stress, but with components in a different coordinate system.

A completely different family of stress tensors is formed by considering the force per unit area in the undeformed configuration, which essentially introduces multiplication by the Jacobian of the deformation gradient tensor J. The total force on an infinitesimal area is given by

$$\boldsymbol{F} = \boldsymbol{T}\,\mathrm{d}S = T^{\bar{i}\bar{j}} N_{\bar{i}} \boldsymbol{G}_{\bar{j}}\,\mathrm{dS} = T^{\bar{i}\bar{j}} \boldsymbol{G}_{\bar{j}}\,\mathrm{d}A_{\bar{i}},$$

and using Nanson's relation (2.43), we obtain

$$\boldsymbol{F} = T^{\bar{i}\bar{j}} \boldsymbol{G}_{\bar{j}} J \frac{\partial \xi^k}{\partial \chi^{\bar{i}}}\,\mathrm{d}a_k = J T^{\bar{i}\bar{j}} n_k \frac{\partial \xi^k}{\partial \chi^{\bar{i}}} \boldsymbol{G}_{\bar{j}}\,\mathrm{d}s = p^{\bar{j}k} n_k \boldsymbol{G}_{\bar{j}}\,\mathrm{d}s = \boldsymbol{t}\,\mathrm{d}s,$$

where $p^{\bar{j}k} = J T^{\bar{i}\bar{j}} \partial\xi^k / \partial\chi^{\bar{i}}$, are the components of a tensor often known as the first Piola–Kirchhoff[9], or nominal, stress tensor and $\boldsymbol{t}$ is the force per unit undeformed area. In the global Cartesian basis, the first Piola–Kirchhoff stress tensor is often written as

$$\mathsf{p} = J \sigma^T \mathsf{F}^{-T}. \tag{3.12}$$

The first Piola–Kirchhoff stress tensor can be interpreted as the force in the current configuration per unit area of the undeformed configuration. It is a two-point tensor because the normal is now the normal to the undeformed area which has now been decomposed into components in the undeformed Lagrangian base vectors, $\boldsymbol{g}^i$, but the traction itself is still decomposed into the deformed Eulerian base vectors, $\boldsymbol{G}_{\bar{j}}$. If instead we decompose the traction into components in the deformed Lagrangian base vectors, $\boldsymbol{G}_j$, then we obtain the second Piola–Kirchhoff stress tensor

$$\boldsymbol{t} = p^{\bar{j}k} n_k \frac{\partial \xi^l}{\partial \chi^{\bar{j}}} \boldsymbol{G}_l = J T^{\bar{i}\bar{j}} \frac{\partial \xi^k}{\partial \chi^{\bar{i}}} \frac{\partial \xi^l}{\partial \chi^{\bar{j}}} n_k \boldsymbol{G}_l = s^{kl} n_k \boldsymbol{G}_l = J T^{kl} n_k \boldsymbol{G}_l.$$

The components s^{kl} are the components of the second Piola–Kirchhoff stress tensor which is simply J multiplied by the body stress tensor[10]. In the global Cartesian basis, it is often written as

$$\mathsf{s} = J \mathsf{F}^{-1} \sigma \mathsf{F}^{-T}. \tag{3.13}$$

Comparison with (1.30) shows that s is the pullback of σ scaled by J, sometimes called the Piola transformation, and that the tensor is indeed contravariant, $\mathsf{s} = \mathsf{s}^\sharp$. The second Piola–Kirchhoff

[9]Note that in some texts the first Piola–Kirchhoff stress tensor is defined to be the transpose of the definition given here.

[10]We should note that unlike the body stress tensor the k component of the second Piola–Kirchhoff stress tensor refers to the Lagrangian basis in the undeformed, rather than the deformed configuration.

stress tensor can be interpreted as the force per unit area of the undeformed configuration decomposed into the Lagrangian base vectors in the deformed configuration. Alternatively, in the Cartesian basis, it is often interpreted as a fictional force (the force mapped back via the inverse deformation gradient tensor) per unit undeformed area. The interpretations are the same because mapping the components of a vector from the deformed to the undeformed configuration has the same effect as the same as mapping the undeformed Lagrangian base vectors into the deformed ones. Decomposing the force into the two different coordinate systems gives

$$\boldsymbol{F} = F^{\bar{i}}\boldsymbol{G}_{\bar{i}} = F^{j}\boldsymbol{G}_{j}, \quad \text{then} \quad F^{j} = \frac{\partial \xi^{j}}{\partial \chi^{\bar{i}}} F^{\bar{i}},$$

but converting from curvilinear to Cartesian coordinates in the deformed and undeformed positions, we obtain

$$F^{j} \equiv \frac{\partial \xi^{j}}{\partial x_{J}} f_{J} = \frac{\partial \xi^{j}}{\partial \chi^{\bar{i}}} \frac{\partial \chi^{\bar{i}}}{\partial X_{I}} F_{I} = \frac{\partial \xi^{j}}{\partial X_{I}} F_{I} \quad \Rightarrow \quad f_{J} = \frac{\partial x_{J}}{\partial X_{I}} F_{I},$$

which expresses the transformation from the components of the real force, F_I, to those of the fictional force, f_I in the global Cartesian basis.

3.3.4 Equations of equilibrium

If a body is in equilibrium then there is no change in momentum or angular momentum, in other words all acceleration terms, both linear and angular are zero. An alternative interpretation is that all forces and torques balance so that there is no net force or torque acting on the body.

Linear momentum

In the absence of any acceleration terms, momentum is conserved and equation (3.6) applied to a region in the deformed configuration yields the equation

$$\int_{\partial \Omega_t} \boldsymbol{T} \mathrm{d}\mathcal{S}_t + \int_{\Omega_t} \rho \boldsymbol{F}\, \mathrm{d}\mathcal{V}_t = \boldsymbol{0}, \tag{3.14}$$

and using equation (3.10) to replace the traction vector, we have

$$\int_{\partial \Omega_t} \sum_{\bar{i}} N_{\bar{i}} \sqrt{(G^{\bar{i}\bar{i}})}\, \boldsymbol{T}_{(\bar{i})} \mathrm{d}\mathcal{S}_t + \int_{\Omega_t} \rho \boldsymbol{F}\, \mathrm{d}\mathcal{V}_t = \boldsymbol{0}.$$

In order to simplify application of the divergence theorem we introduce the vector $\boldsymbol{T}^{\bar{i}} = \boldsymbol{T}_{(\bar{i})} \sqrt{(\overline{G} G^{\bar{i}\bar{i}})}$, which transforms contravariantly. Here $\overline{G}$ is the determinant of the covariant metric tensor $G_{\bar{i}\bar{j}}$. The conservation of momentum equation becomes

$$\Rightarrow \quad \int_{\partial \Omega_t} \frac{N_{\bar{i}}}{\sqrt{\overline{G}}} \boldsymbol{T}^{\bar{i}}\, \mathrm{d}\mathcal{S}_t + \int_{\Omega_t} \rho \boldsymbol{F}\, \mathrm{d}\mathcal{V}_t = \boldsymbol{0}. \tag{3.15}$$

We now use the divergence theorem (1.59) in the Eulerian curvilinear coordinates,

$$\int_{\partial \Omega_t} A^{\bar{i}} N_{\bar{i}}\, \mathrm{d}\mathcal{S}_t = \int_{\Omega_t} \frac{1}{\sqrt{\overline{G}}} \frac{\partial}{\partial \chi^{\bar{i}}} (A^{\bar{i}} \sqrt{\overline{G}})\, \mathrm{d}\mathcal{V}_t$$

to transform equation (3.15) into

$$\int_{\Omega_t} \frac{1}{\sqrt{G}} \frac{\partial \boldsymbol{T}^{\bar{i}}}{\partial \chi^{\bar{i}}} + \rho \boldsymbol{F} \mathrm{d}\mathcal{V}_t = \mathbf{0}. \tag{3.16}$$

The equation must be valid for any volume, which means that the integrand must be zero

$$\frac{1}{\sqrt{G}} \frac{\partial \boldsymbol{T}^{\bar{i}}}{\partial \chi^{\bar{i}}} + \rho \boldsymbol{F} = \mathbf{0}, \tag{3.17}$$

which is the governing differential equation of motion known as Cauchy's equation. If we express the stress vector, $\boldsymbol{T}^{\bar{i}}$ in terms of the Cauchy stress tensor, we have

$$\frac{1}{\sqrt{G}} \frac{\partial}{\partial \chi^{\bar{i}}} \left(\sqrt{G}\, T^{\bar{i}\bar{j}}\, \boldsymbol{G}_{\bar{j}} \right) + \rho \boldsymbol{F} = \mathbf{0},$$

which can be written as

$$\left(T^{\bar{i}\bar{j}}||_{\bar{i}} + \rho F^{\bar{j}} \right) \boldsymbol{G}_{\bar{j}} = \mathbf{0},$$

where, as usual, the symbol $||$ refers to covariant differentiation in the deformed configuration. This step could be interpreted as the definition of the divergence of a second-order tensor[11]. The equation is often written as

$$\boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \mathsf{T} + \rho \boldsymbol{F} = \mathbf{0},$$

which looks nice and compact but requires knowledge of how to take the divergence of a tensor and, in particular, over which index the contraction is to be taken.

In components in the basis $G_{\bar{j}}$, we have the governing equation

$$T^{\bar{i}\bar{j}}||_{\bar{i}} + \rho F^{\bar{j}} = 0,$$

and in Cartesian coordinates the equation reduces to

$$T_{IJ,I} + \rho F_J = 0,$$

which starts to look a lot less intimidating.

Angular momentum

In the absence of any angular acceleration, conservation of angular momentum applied to a region in the deformed configuration yields

$$\int_{\partial\Omega_t} (\boldsymbol{R} - \boldsymbol{Z}) \times \boldsymbol{T} \mathrm{d}\mathcal{S}_t + \int_{\Omega_t} \rho(\boldsymbol{R} - \boldsymbol{Z}) \times \boldsymbol{F}\, \mathrm{d}\mathcal{V}_t = \mathbf{0}.$$

The vector $\boldsymbol{Z}$ is a constant, so it can be taken outside the integrals, to give

$$\int_{\partial\Omega_t} \boldsymbol{R} \times \boldsymbol{T} \mathrm{d}\mathcal{S}_t + \int_{\Omega_t} \rho \boldsymbol{R} \times \boldsymbol{F}\, \mathrm{d}\mathcal{V}_t - \boldsymbol{Z} \times \left[\int_{\partial\Omega_t} \boldsymbol{T} \mathrm{d}\mathcal{S}_t + \int_{\Omega_t} \rho \boldsymbol{F}\, \mathrm{d}\mathcal{V}_t \right] = \mathbf{0},$$

[11] It is consistent with the covariant derivative of a tensor, as it should be. Using the definition of the Christoffel symbol (1.49) and the derivative of the square-root of the determinant (1.57), we have

$$\frac{1}{\sqrt{G}} \frac{\partial}{\partial \chi^{\bar{i}}} \left(\sqrt{G}\, T^{\bar{i}\bar{j}}\, \boldsymbol{G}_{\bar{j}} \right) = T^{\bar{i}\bar{j}}_{\ ,\bar{i}} \boldsymbol{G}_{\bar{j}} + T^{\bar{i}\bar{j}} \overline{\Gamma}^{\bar{k}}_{\bar{i}\bar{j}} \boldsymbol{G}_{\bar{k}} + \overline{\Gamma}^{\bar{k}}_{\bar{k}\bar{i}} T^{\bar{i}\bar{j}} \boldsymbol{G}_{\bar{j}} = \left(T^{\bar{i}\bar{j}}_{\ ,\bar{i}} + \overline{\Gamma}^{\bar{j}}_{\bar{i}\bar{k}} T^{\bar{i}\bar{k}} + \overline{\Gamma}^{\bar{k}}_{\bar{k}\bar{i}} T^{\bar{i}\bar{j}} \right) \boldsymbol{G}_{\bar{j}} = T^{\bar{i}\bar{j}}||_{\bar{i}} \boldsymbol{G}_{\bar{j}}.$$

$$\Rightarrow \int_{\partial\Omega_t} \boldsymbol{R} \times \boldsymbol{T} \,\mathrm{d}\mathcal{S}_t + \int_{\Omega_t} \rho \boldsymbol{R} \times \boldsymbol{F} \,\mathrm{d}\mathcal{V}_t = \boldsymbol{0},$$

from the equations of equilibrium (3.14).

Introducing the vector $\boldsymbol{T}^{\bar{i}}$, such that $\boldsymbol{T} = N_{\bar{i}} \boldsymbol{T}^{\bar{i}}/\sqrt{G}$ as above, the equation becomes

$$\int_{\partial\Omega_t} \boldsymbol{R} \times \frac{1}{\sqrt{G}} \boldsymbol{T}^{\bar{i}} N_{\bar{i}} \,\mathrm{d}\mathcal{S}_t + \int_{\Omega_t} \rho \boldsymbol{R} \times \boldsymbol{F} \,\mathrm{d}\mathcal{V}_t = \boldsymbol{0},$$

and using the divergence theorem gives

$$\int_{\Omega_t} \left[\frac{1}{\sqrt{G}} \frac{\partial}{\partial \chi^{\bar{i}}} \left(\boldsymbol{R} \times \boldsymbol{T}^{\bar{i}} \right) + \rho \boldsymbol{R} \times \boldsymbol{F} \right] \mathrm{d}\mathcal{V}_t = \boldsymbol{0}. \tag{3.18}$$

Expanding the derivative of the cross product gives

$$\int_{\Omega_t} \frac{1}{\sqrt{G}} \boldsymbol{G}_{\bar{i}} \times \boldsymbol{T}^{\bar{i}} + \boldsymbol{R} \times \left[\frac{1}{\sqrt{G}} \frac{\partial \boldsymbol{T}^{\bar{i}}}{\partial \chi^{\bar{i}}} + \rho \boldsymbol{F} \right] \mathrm{d}\mathcal{V}_t = \boldsymbol{0},$$

but the term in square brackets is equal to zero by Cauchy's equation (3.17) and so

$$\int_{\Omega_t} \frac{1}{\sqrt{G}} \boldsymbol{G}_{\bar{i}} \times \boldsymbol{T}^{\bar{i}} \,\mathrm{d}\mathcal{V}_t = \boldsymbol{0}. \tag{3.19}$$

Once again the equation must be valid for any volume, which is only possible if the integrand is zero

$$\frac{1}{\sqrt{G}} \boldsymbol{G}_{\bar{i}} \times \boldsymbol{T}^{\bar{i}} = \boldsymbol{G}_{\bar{i}} \times (T^{\bar{i}\bar{j}} \boldsymbol{G}_{\bar{j}}) = \epsilon_{\bar{i}\bar{j}\bar{k}} T^{\bar{i}\bar{j}} \boldsymbol{G}^{\bar{k}} = \boldsymbol{0}.$$

Each component of the vector equation must be zero,

$$\epsilon_{\bar{i}\bar{j}\bar{k}} T^{\bar{i}\bar{j}} = 0 \quad \Rightarrow \quad \frac{1}{2} \left[\epsilon_{\bar{i}\bar{j}\bar{k}} T^{\bar{i}\bar{j}} + \epsilon_{\bar{j}\bar{i}\bar{k}} T^{\bar{j}\bar{i}} \right] = 0,$$

and using the antisymmetry property of the Levi-Civita symbol, we have

$$\frac{1}{2} \epsilon_{\bar{i}\bar{j}\bar{k}} \left[T^{\bar{i}\bar{j}} - T^{\bar{j}\bar{i}} \right] = 0.$$

It follows that the Cauchy stress tensor must be symmetric

$$T^{\bar{i}\bar{j}} = T^{\bar{j}\bar{i}}.$$

From their definitions, it follows that the body stress (3.11) and second Piola–Kirchhoff (3.13) stress tensors are also symmetric, but the first Piola–Kirchhoff stress tensor (3.12) is not, which is because it is a two-point tensor.