Chapter 2

Kinematics: Deformation and Flow

2.1 Introduction

Lecture 5

We need a suitable mathematical framework in order to describe the behaviour of continua. Our everyday experience tells us that lumps of matter can both move (change in position) and deform (change in shape), so we need to be able to quantify both these effects. Hopefully, you are already be familiar with the kinematics of individual particles from basic mechanics. In classical particle mechanics, the *position* of the particle is described by a position vector $\boldsymbol{r}(t)$ measured from a chosen origin in three-dimensional Euclidean space, $\mathbb{E}^3$. The position is a function of one-dimensional continuous time, $t \in [0, \infty)$, and contains all the information that we need to describe the motion of the particle.

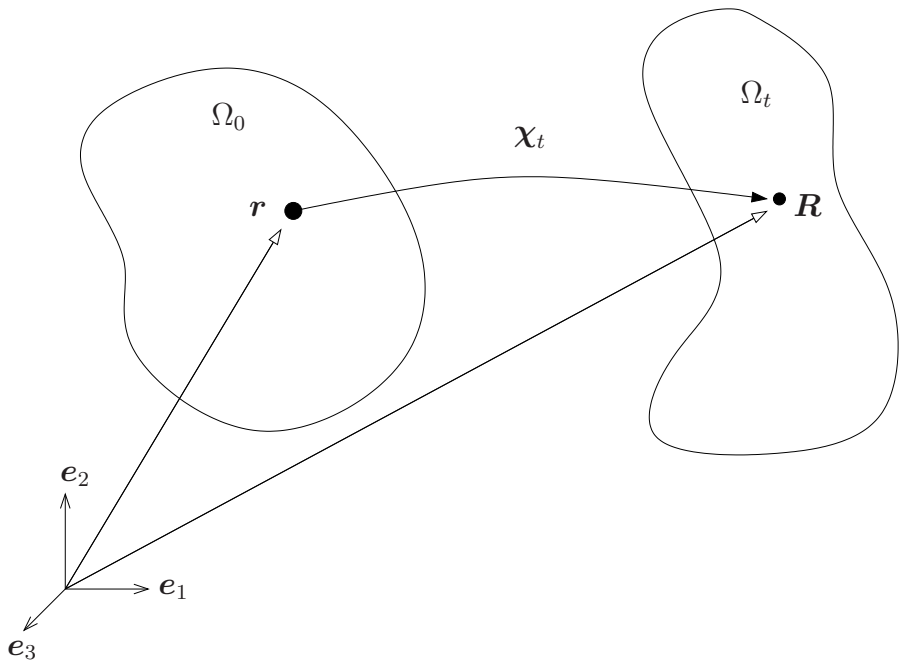

Figure 2.1: A region Ω_0 is mapped to the region Ω_t by the mapping $\boldsymbol{\chi}_t$ which carries the material point $\boldsymbol{r} \in \Omega_0$ to the point $\boldsymbol{R} \in \Omega_t$.

In continuum mechanics, we need to account for the motion of all the "particles" within the material. Consider a continuous body that initially occupies a region[1] Ω_0 of three-dimensional Euclidean space $\mathbb{E}^3$, with volume $\mathcal{V}_0$ and surface $\partial\mathcal{V}_0 \equiv \mathcal{S}_0$. The body can be regarded as a

[1] If you want to get fancy you can call this an open subset.

collection of particles, called material points, each of which[2] is described by a position vector $\boldsymbol{r} = x_K \boldsymbol{e}_K$ from our fixed origin in a Cartesian coordinate system[3]. We shall term the configuration Ω_0 the undeformed configuration. At a later time t, the same body occupies a different region of space, Ω_t, with volume $\mathcal{V}_t$ and surface $\mathcal{S}_t$. The material points within the region Ω_t are now described by a position vector $\boldsymbol{R} = X_K \boldsymbol{e}_K$ from the same origin in the same Cartesian coordinate system[4]. We shall term the configuration Ω_t the deformed configuration, see Figure 2.1.

The change in configuration from Ω_0 to Ω_t can be described by a function $\boldsymbol{\chi}_t : \Omega_0 \to \Omega_t$, sometimes called a *deformation map*. The position at time t is given by $\boldsymbol{R} = \boldsymbol{\chi}_t(\boldsymbol{r}) \equiv \boldsymbol{\chi}(\boldsymbol{r}, t)$, where $\boldsymbol{\chi} : \Omega_0 \times [0, \infty) \to \Omega_t$ is a continuous map. In other words, each material point[5] in the undeformed configuration Ω_0 is carried to a (material) point in the deformed configuration Ω_t.

On physical grounds, we expect that (a) matter cannot be destroyed and (b) matter does not interpenetrate. A deformation map will be consistent with these conditions if it is one-to-one and the Jacobian of the mapping remains non zero. The Jacobian of the mapping is the determinant of the matrix, $\mathsf{F} = \boldsymbol{\nabla}_r \boldsymbol{\chi}_t$, (note that the gradient is taken with respect to the undeformed coordinates $\boldsymbol{r}$) whose components in the Cartesian basis are given by

$$F_{IJ} = \frac{\partial X_I}{\partial x_J}, \tag{2.1}$$

and so our physical constraints demand that

$$\det \mathsf{F} \neq 0. \tag{2.2}$$

In fact, we impose the stronger condition that the Jacobian of the mapping remains positive, which ensures that material lines preserve their relative orientations: a body cannot be deformed onto its mirror image. If condition (2.2) is satisfied then a (local) inverse mapping can be constructed that gives the initial position as a function of the current position, $\boldsymbol{r} = \boldsymbol{\chi}_t^{-1}(\boldsymbol{R}) \equiv \boldsymbol{\chi}^{-1}(\boldsymbol{R}, t)$.

Example 2.1. An example deformation
Is the mapping given by

$$\boldsymbol{\chi}(\boldsymbol{x}) : \begin{cases} \chi_1 & = 2x_1 + 3x_2, \\ \chi_2 & = 1x_1 + 2x_2. \end{cases}$$

physically admissible? If so, sketch the deformed unit square $x_I \in [0, 1]$ that follows after application of the mapping.

Solution 2.1. The deformed position is given by

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and $\det F = 2 \times 2 - 3 \times 1 = 1 > 0$, so the mapping is physically admissible. The deformed unit square can be obtained by thinking about the deformation of the corners and the knowledge that

[2]We could label the position vector $\boldsymbol{r}$ by another index to distinguish different material points, but the notation is less cumbersome if we treat each specific value of $\boldsymbol{r}$ as the definition of the different material points.

[3]Even when we consider non-Cartesian coordinates later, it's easiest to define those coordinates relative to the fixed Cartesian system. Note that because this is a Cartesian coordinate system we have not bothered to indicate the contravariant nature of the components.

[4]For further generality, we should assume a different orthonormal coordinate system for the deformed configuration $\boldsymbol{R} = X_{\overline{K}} \boldsymbol{e}_{\overline{K}}$, but the notation becomes even more cluttered without really aiding understanding.

[5]From the classical particle mechanics viewpoint, there is a single material point (the particle) with position vector $\boldsymbol{x}(t) = \boldsymbol{R} = \chi_t(\boldsymbol{r}) = \chi_t(\boldsymbol{x}(0))$.

the deformation is homogeneous (the matrix entries are not themselves functions of $\boldsymbol{x}$). The corners map as follows

$$(0,0) \to (0,0); \quad (1,0) \to (2,1); \quad (0,1) \to (3,2); \quad (1,1) \to (5,3),$$

and the undeformed and deformed regions are shown in Figure 2.2.

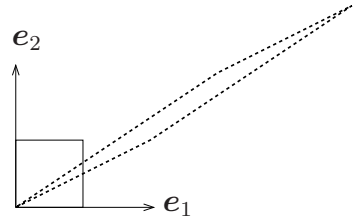

Figure 2.2: A unit square (solid boundary) is deformed into a quadrilateral region (dashed boundary) by the mapping $\boldsymbol{\chi}$.

2.2 Lagrangian (Material) and Eulerian (Spatial) Descriptions

Given the existence of the continuous mapping $\boldsymbol{\chi}$, we can write (each Cartesian coordinate of) a material point in the deformed position as a function of (the Cartesian coordinates of) the same material point in the undeformed position and the time t,

$$X_K(t) = \boldsymbol{\chi}(x_J, t) \cdot \boldsymbol{e_K}, \quad \text{which we can write in vector form as} \quad \boldsymbol{R}(\boldsymbol{r}, t). \tag{2.3}$$

In equation (2.3) the current position is treated as a function of the original position, which is the independent variable. This is called a *Lagrangian* or material description. Any other fields are also treated as functions of the original position. In the Lagrangian description, we follow the evolution of particular material particles with time[6]. It is more common to use a Lagrangian description in solid mechanics where the initial undeformed geometry is often simple, but the deformations become complex.

Alternatively, we could use the inverse mapping to write (each Cartesian coordinates of) the undeformed position of a material point as a function of (the Cartesian coordinates of) its current position

$$x_K(t) = \boldsymbol{\chi}^{-1}(X_J, t) \cdot \boldsymbol{e_K}, \quad \text{or} \quad \boldsymbol{r}(\boldsymbol{R}, t). \tag{2.4}$$

Here, the current position is the independent variable and this is called an *Eulerian* or spatial description. Any other fields are treated as functions of the fixed position $\boldsymbol{R}$. In the Eulerian

[6]The Lagrangian description is the viewpoint used in classical particle mechanics because we are only interested in the behaviour of each individual particle.

description, we observe the changes over time at a fixed point in space[7]. It is more common to use an Eulerian description in fluid mechanics, if the original location of particular fluid particles is not of interest.

The transformation of an object from an Eulerian (current) description to the Lagrangian (reference) description is sometimes called a **pullback** (terminology taken from differential geometry). The converse operation of transforming from the Lagrangian to the Eulerian description is called a **pushforward**.

For simplicity, we have defined the deformed and undeformed positions in the same global Cartesian coordinate system, but this is not necessary. It is perfectly possible to use different coordinate systems to represent the deformed and undeformed positions, which may be useful in certain special problems: *e.g.* a cube being deformed into a cylinder. However, the use of general coordinates would bring in the complication of covariant and contravariant transformations. We shall consider general curvilinear coordinates in what follows after the initial development in Cartesians. The curvilinear coordinates associated with the Lagrangian viewpoint will be denoted by ξ^i and those in the Eulerian $\chi^{\bar{i}}$.

2.3 Displacement, Velocity and Acceleration

Newton's laws of mechanics were originally formulated for individual particles and the displacement, velocity and acceleration of each particle are derived from its position as a function of time. In continuum mechanics we must also define these quantities based on the position of each material point as a function of time (the Lagrangian viewpoint), but we can then reinterpret them as functions of the absolute spatial position (the Eulerian viewpoint) if more convenient.

2.3.1 Displacement

The change in position of a material point within the body between configurations Ω_0 and Ω_t is given by the *displacement* vector field

$$\boldsymbol{u} = \boldsymbol{R} - \boldsymbol{r}, \tag{2.5}$$

which can be treated as a function of either $\boldsymbol{R}$ or $\boldsymbol{r}$; *i.e.* from the Eulerian or Lagrangian viewpoints, respectively. For consistency of notation we shall write $\boldsymbol{U}(\boldsymbol{R}, t) = \boldsymbol{R} - \boldsymbol{r}(\boldsymbol{R}, t)$ for the displacement in the Eulerian representation and $\boldsymbol{u}(\boldsymbol{r}, t) = \boldsymbol{R}(\boldsymbol{r}, t) - \boldsymbol{r}$ in the Lagrangian.

Example 2.2. Displacement of a moving block
A block of material that initially occupies the region $\boldsymbol{r} \in [0, 1] \times [0, 1] \times [0, 1]$ moves in such a way that its position is given by

$$\boldsymbol{R}(\boldsymbol{r}, t) = \boldsymbol{r}(1 + t), \tag{2.6}$$

at time $t \geq 0$. Find the displacement of the block in both the Eulerian and Lagrangian representations.

[7] The Eulerian description makes little sense in classical particle mechanics because in general motion the particle is unlikely to be located at our chosen fixed position for very long. Imagine trying to throw a ball in front of a fixed (Eulerian) video camera: the ball will only appear on a short section of the video, or not at all if we do not throw accurately. The reason why an Eulerian view makes more sense in continuum mechanics is that some part of the continuum will generally be located at our chosen fixed point, but exactly which part (which material points) changes with time.

Solution 2.2. We are given the relationship $\boldsymbol{R}(\boldsymbol{r},t)$, so the displacement in the Lagrangian viewpoint is straightforward to determine

$$\boldsymbol{u}(\boldsymbol{r},t) = \boldsymbol{R}(\boldsymbol{r},t) - \boldsymbol{r} = t\boldsymbol{r}.$$

For the Eulerian viewpoint, we need to invert the relationship (2.6) which gives $\boldsymbol{r} = \boldsymbol{R}/(1+t)$ and therefore the displacement is

$$\boldsymbol{U}(\boldsymbol{R},t) = \frac{t}{1+t}\boldsymbol{R}.$$

A potential problem with the Eulerian viewpoint is that it does not actually make sense unless the point $\boldsymbol{R}$ is within the body at time t.

2.3.2 Velocity

The *velocity* of a material point is the rate of change of its displacement with time. The Lagrangian (material) coordinate of each material point remains fixed, so the velocity, $\boldsymbol{v}$, in the Lagrangian representation is simply

$$\boldsymbol{v}(\boldsymbol{r},t) = \left.\frac{\partial \boldsymbol{u}(\boldsymbol{r},t)}{\partial t}\right|_{\boldsymbol{r}\text{ fixed}} = \left.\frac{\partial(\boldsymbol{R}(\boldsymbol{r},t)-\boldsymbol{r})}{\partial t}\right|_{\boldsymbol{r}} = \left.\frac{\partial \boldsymbol{R}(\boldsymbol{r},t)}{\partial t}\right|_{\boldsymbol{r}}, \tag{2.7}$$

because $\boldsymbol{r}$ is held fixed during the differentiation[8]. Note that the velocity moves with the material points, which is exactly the same as in classical particle mechanics.

In the Eulerian framework, the velocity must be defined as a function of a specific fixed point in space. The problem is that different material points will pass through the chosen spatial point at different times. Thus, the Eulerian velocity must be calculated by finding the material coordinate that corresponds to the fixed spatial location at chosen instant in time, $\boldsymbol{r}(\boldsymbol{R},t)$, so that

$$\boldsymbol{V}(\boldsymbol{R},t) = \left.\frac{\partial \boldsymbol{U}(\boldsymbol{R},t)}{\partial t}\right|_{\boldsymbol{r}} = \left.\frac{\partial \boldsymbol{R}}{\partial t}\right|_{\boldsymbol{r}} = \boldsymbol{v}(\boldsymbol{r}(\boldsymbol{R},t),t). \tag{2.8}$$

Example 2.3. Calculation of the velocity for a rotating block
A cube of material that initially occupies the region $\boldsymbol{r} \in [-1,1]\times[-1,1]\times[-1,1]$ rotates about the x_3-axis with constant angular velocity so that its position at time t is given by

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{2.9}$$

Calculate both the Lagrangian and Eulerian velocities of the block.

Solution 2.3. The Lagrangian velocity is simple to calculate

$$\boldsymbol{v}(\boldsymbol{r},t) = \frac{\partial \boldsymbol{R}}{\partial t}, \quad \Rightarrow \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{2.10}$$

which is, as expected, the same as the velocity of a particle initially at (x_1, x_2) rotating about the origin in the $x_1 - x_2$ plane.

[8]This is indicated by the vertical bar next to the partial derivative, which should not be confused with the covariant derivative.

In order to convert to the Eulerian viewpoint we need to find $\boldsymbol{r}(\boldsymbol{R})$, which follows after inversion of the relationship (2.9):

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}. \tag{2.11}$$

Thus the Eulerian velocity $\boldsymbol{V}(\boldsymbol{R},t) = \boldsymbol{v}(\boldsymbol{r}(\boldsymbol{R},t),t)$ is obtained by substituting the relationship (2.11) into the expression for the Lagrangian velocity (2.10)

$$\begin{aligned} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} &= \begin{pmatrix} -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} X_2 \\ -X_1 \\ 0 \end{pmatrix}, \end{aligned}$$

which remains constant. In other words the velocity of the rotating block at a fixed point in space is constant. Different material points pass through our chosen point but the velocity of each is always the same when it does so.

2.3.3 Acceleration

The *acceleration* of a material point is the rate of change of its velocity with time. Once again, computing the acceleration in the Lagrangian representation is very simple

$$\boldsymbol{a}(\boldsymbol{r},t) = \left.\frac{\partial \boldsymbol{v}(\boldsymbol{r},t)}{\partial t}\right|_{\boldsymbol{r}} = \left.\frac{\partial^2 \boldsymbol{u}(\boldsymbol{r},t)}{\partial t^2}\right|_{\boldsymbol{r}} = \left.\frac{\partial^2 \boldsymbol{R}(\boldsymbol{r},t)}{\partial t^2}\right|_{\boldsymbol{r}}. \tag{2.12}$$

In the Eulerian framework, the velocity at a fixed point in space can change through two different mechanisms: (i) the material velocity changes with time; or (ii) the material point (with a specific velocity) is carried past the fixed point in space. The second mechanism is a consequence of the motion of the continuum and is known convection or advection. The two terms arise quite naturally in the computation of the material acceleration at a fixed spatial location because for a fixed material coordinate, the position $\boldsymbol{R}$ is also a function of time

$$\boldsymbol{A}(\boldsymbol{R},t) = \left.\frac{\partial \boldsymbol{V}(\boldsymbol{R},t)}{\partial t}\right|_{\boldsymbol{r}} = \left.\frac{\partial \boldsymbol{V}(\boldsymbol{R}(\boldsymbol{r},t),t)}{\partial t}\right|_{\boldsymbol{r}}. \tag{2.13}$$

After application of the chain rule, equation (2.13) becomes

$$\boldsymbol{A}(\boldsymbol{R},t) = \left.\frac{\partial \boldsymbol{V}}{\partial t}\right|_{\boldsymbol{R}} + \left.\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{R}}\right|_{t} \cdot \left.\frac{\partial \boldsymbol{R}}{\partial t}\right|_{\boldsymbol{r}} = \left.\frac{\partial \boldsymbol{V}}{\partial t}\right|_{\boldsymbol{R}} + \boldsymbol{V} \cdot \left.\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{R}}\right|_{t} = \frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V} \cdot \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{V}, \tag{2.14}$$

or in component form

$$A_I(\boldsymbol{R},t) = \frac{\partial V_I}{\partial t} + V_J \frac{\partial V_I}{\partial X_J}. \tag{2.15}$$

Note that the gradient is taken with respect to the Eulerian (deformed) coordinates $\boldsymbol{R}$. The first term $\partial \boldsymbol{V}/\partial t$ corresponds to the mechanism (i) above, whereas the second term $\boldsymbol{V} \cdot \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{V}$ corresponds to the mechanism (ii) and is the advective term. The advective term is nonlinear in $\boldsymbol{V}$, which leads to many of the complex phenomena observed in continua[9].

[9]Note that this nonlinearity only arises when observations are made in the Eulerian viewpoint, so it is definitely observer dependent.

2.3.4 Material Derivative

Lecture 6

The argument used to determine the acceleration is rather general and can be used to define the rates of change of any property that is carried with the continuum. For a scalar field $\phi(\boldsymbol{r},t) = \Phi(\boldsymbol{R}(\boldsymbol{r},t),t)$ the material derivative is the rate of change of the quantity keeping the Lagrangian coordinate $\boldsymbol{r}$ fixed. The usual notation for the material derivative is $D\phi/Dt$ and in the Lagrangian framework, the material derivative and the partial derivative coincide

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} \tag{2.16}$$

In the Eulerian framework, however,

$$\frac{D\Phi}{Dt} = \frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial X_K}\frac{\partial X_K}{\partial t} = \frac{\partial\Phi}{\partial t} + V_K\frac{\partial\Phi}{\partial X_K} = \frac{\partial\Phi}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}_{\boldsymbol{R}}\Phi. \tag{2.17}$$

Curvilinear coordinates

If we use general (time-independent) curvilinear coordinates $\chi^{\bar{i}}$ as as Eulerian coordinates, then $\boldsymbol{R}(\chi^{\bar{i}})$ and equation (2.17) becomes

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial\chi^{\bar{i}}}\frac{\partial\chi^{\bar{i}}}{\partial X_K}\frac{\partial X_K}{\partial t} = \frac{\partial\phi}{\partial t} + \frac{\partial\chi^{\bar{i}}}{\partial X_K}V_K\frac{\partial\phi}{\partial\chi^{\bar{i}}} = \frac{\partial\phi}{\partial t} + V^{\bar{i}}\frac{\partial\phi}{\partial\chi^{\bar{i}}},$$

where we have use equation (1.16b) to determine the components $V^{\bar{i}}$ in the covariant basis corresponding to the coordinates $\chi^{\bar{i}}$. The velocity vector is $\boldsymbol{V} = V^{\bar{i}}\boldsymbol{G}_{\bar{i}}$, where $\boldsymbol{G}_{\bar{i}} = \partial\boldsymbol{R}/\partial\chi^{\bar{i}}$ are the covariant base vectors in the deformed position with respect to Eulerian curvilinear coordinates $\chi^{\bar{i}}$. It follows that the acceleration in the Eulerian framework in general coordinates is given by

$$\frac{D(V^{\bar{j}}\boldsymbol{G}_{\bar{j}})}{Dt} = \frac{\partial(V^{\bar{j}}\boldsymbol{G}_{\bar{j}})}{\partial t} + V^{\bar{i}}\frac{\partial(V^{\bar{j}}\boldsymbol{G}_{\bar{j}})}{\partial\chi^{\bar{i}}}.$$

Assuming that the base vectors are fixed in time, which is to be expected for a (sensible) Eulerian coordinate system, we obtain

$$\frac{D\boldsymbol{V}}{Dt} = \left(\frac{\partial V^{\bar{i}}}{\partial t} + V^{\bar{j}}V^{\bar{i}}||_{\bar{j}}\right)\boldsymbol{G}_{\bar{i}},$$

where here $||_{\bar{j}}$ indicates covariant differentiation in the Eulerian viewpoint with respect to the Eulerian coordinates $\chi^{\bar{i}}$, i.e. using the the metric tensor $G_{\bar{i}\bar{j}} = \boldsymbol{G}_{\bar{i}}\cdot\boldsymbol{G}_{\bar{j}}$. As you would expect, this is simply the tensor component representation of the expression

$$\frac{D\boldsymbol{V}}{Dt} = \frac{\partial\boldsymbol{V}}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{V};$$

a coordinate-system-independent vector expression.

2.4 Deformation

Thus far, we have considered the extension of concepts in particle mechanics, displacement, velocity and acceleration, to the continuum setting, but the motion of isolated individual points does not tell

us whether the continuum has changed its shape, or deformed. The shape will change if material points move relative to one another, which means that we need to be able to measure distances. In the body moves rigidly then the distance between *every* pair of material points will remain the same. A body undergoes a deformation if the distance between *any* pair of material points changes. Hence, quantification of deformation requires the study of the evolution of material line elements, see Figure 2.3.

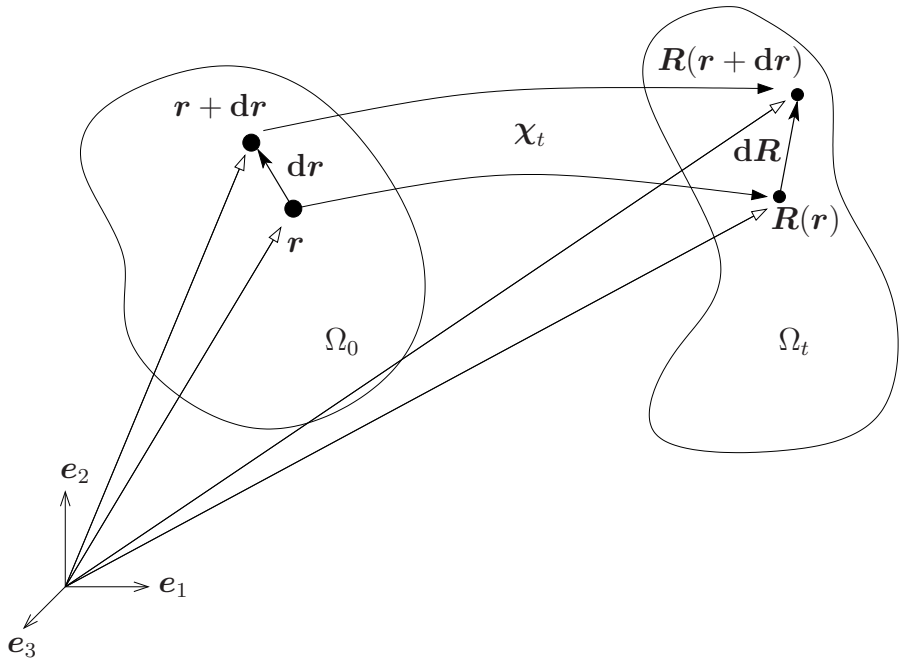

Figure 2.3: A region Ω_0 is mapped to the region Ω_t by the mapping $\boldsymbol{\chi}_t$ which carries the material point $\boldsymbol{r} \in \Omega_0$ to the point $\boldsymbol{R}(\boldsymbol{r}) \in \Omega_t$ and the material point $\boldsymbol{r}+\mathrm{d}\boldsymbol{r}$ to $\boldsymbol{R}(\boldsymbol{r}+\mathrm{d}\boldsymbol{r})$. The undeformed line element $\mathrm{d}\boldsymbol{r}$ is carried to the deformed line element $\mathrm{d}\boldsymbol{R}$.

Consider a line element $\mathrm{d}\boldsymbol{r}$ that connects two material points in the undeformed domain. If the position vector to one end of the line is $\boldsymbol{r}$, then the position vector to the other end is $\boldsymbol{r}+\mathrm{d}\boldsymbol{r}$. If we take the Lagrangian (material) viewpoint, the corresponding endpoints in the deformed domain are given by $\boldsymbol{R}(\boldsymbol{r})$ and $\boldsymbol{R}(\boldsymbol{r}+\mathrm{d}\boldsymbol{r})$, respectively[10]. Thus, the line element in the deformed domain is given by

$$\mathrm{d}\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{r}+\mathrm{d}\boldsymbol{r}) - \boldsymbol{R}(\boldsymbol{r}).$$

If we now assume that $|\mathrm{d}\boldsymbol{r}| \ll 1$, then we can use Taylor's theorem to write

$$\mathrm{d}\boldsymbol{R} \approx \boldsymbol{R}(\boldsymbol{r}) + \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{r}}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r} - \boldsymbol{R}(\boldsymbol{r}) = \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{r}}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r} \equiv \mathsf{F}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r}, \tag{2.18}$$

or in component form in the global Cartesian coordinates

$$\mathrm{d}R_I = \frac{\partial X_I}{\partial x_J}\,\mathrm{d}r_J \equiv F_{IJ}\,\mathrm{d}r_J. \tag{2.19}$$

The matrix F of components F_{IJ} is a representation of a quantity called the (material) deformation gradient tensor. It describes the mapping from undeformed line elements to deformed line elements. In fact in this representation F_{IJ} is something called a two-point tensor because the I-th index refers

[10]The dependence on time t will be suppressed in this section because it does not affect instantaneous measures of deformation.

to the deformed coordinate system, whereas the J-th index refers to the undeformed. We have chosen the same global coordinate system, so the distinction may seem irrelevant[11], but we must remember that transformation of the undeformed and deformed coordinates will affect different components of the tensor.

The length of the undeformed line element is given by $\mathrm{d}s$ where $\mathrm{d}s^2 = \mathbf{d}\boldsymbol{r} \cdot \mathbf{d}\boldsymbol{r} = \mathrm{d}r_K \mathrm{d}r_K$ and the length of the deformed line element is $\mathrm{d}S$ where $\mathrm{d}S^2 = \mathbf{d}\boldsymbol{R} \cdot \mathbf{d}\boldsymbol{R} = \mathrm{d}R_K \mathrm{d}R_K$. Thus, a measure of whether the line element has changed in length is given by

$$\mathrm{d}S^2 - \mathrm{d}s^2 = F_{KI}\mathrm{d}r_I F_{KJ}\mathrm{d}r_J - \mathrm{d}r_K \mathrm{d}r_K = (F_{KI}F_{KJ} - \delta_{KI}\delta_{KJ})\,\mathrm{d}r_I \mathrm{d}r_J \equiv (c_{IJ} - \delta_{IJ})\,\mathrm{d}r_I \mathrm{d}r_J; \quad (2.20)$$

and the matrix $\mathsf{c} = \mathsf{F}^T\mathsf{F}$ is (a representation of) the right[12] Cauchy–Green deformation tensor. Its components represent the square of the lengths of the deformed material line elements relative to the undeformed lengths, *i.e.* from the Lagrangian viewpoint. Note that c is symmetric and positive definite because $\mathrm{d}r_I\, c_{IJ}\, \mathrm{d}r_J = \mathrm{d}S^2 > 0$, for all non-zero $\mathbf{d}\boldsymbol{r}$.

From equation (2.20) if the line element does not change in length then $c_{IJ} - \delta_{IJ} = 0$ for all I, J, which motivates the definition of the Green–Lagrange strain tensor

$$e_{IJ} = \frac{1}{2}\left(c_{IJ} - \delta_{IJ}\right) = \frac{1}{2}\left(\frac{\partial X_K}{\partial x_I}\frac{\partial X_K}{\partial x_J} - \delta_{IJ}\right). \quad (2.21)$$

The components of e_{IJ} represent the changes in lengths of material line elements from the Lagrangian perspective.

An alternative approach is to start from the Eulerian viewpoint, in which case

$$\mathbf{d}\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{R} + \mathbf{d}\boldsymbol{R}) - \boldsymbol{r}(\boldsymbol{R}),$$

and using Taylor's theorem as above we obtain

$$\mathbf{d}\boldsymbol{r} \approx \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{R}}(\boldsymbol{R}) \cdot \mathbf{d}\boldsymbol{R} \equiv \mathsf{H}(\boldsymbol{R}) \cdot \mathbf{d}\boldsymbol{R}, \quad (2.22)$$

or in component form

$$dr_I = \frac{\partial x_I}{\partial X_J} dR_J \equiv H_{IJ} dR_J. \quad (2.23)$$

The matrix H with components H_{IJ} is (a representation of) the (spatial) deformation gradient tensor. Comparing equations (2.19) and (2.23) shows that $\mathsf{H} = \mathsf{F}^{-1}$. Thus, the equivalent to equation (2.20) that quantifies the change in length is

$$dS^2 - ds^2 = dR_K dR_K - H_{KI} dR_I H_{KJ} dR_J = (\delta_{KI}\delta_{KJ} - H_{KI}H_{KJ})\, dR_I dR_J \equiv (\delta_{IJ} - C_{IJ})\, dR_I dR_J. \quad (2.24)$$

The matrix $\mathsf{C}{=}\mathsf{H}^T\mathsf{H} = \mathsf{F}^{-T}\mathsf{F}^{-1} = (\mathsf{F}\mathsf{F}^T)^{-1} = \mathsf{B}^{-1}$ is known as (a representation of) the Cauchy deformation tensor and is the inverse of the Finger deformation tensor[13]. The components of C represent the square of lengths of the undeformed material elements relative to the deformed lengths, *i.e.* from the Eulerian viewpoint. We can define the corresponding Eulerian (Almansi) strain tensor

$$E_{IJ} = \frac{1}{2}\left(\delta_{IJ} - C_{IJ}\right) = \frac{1}{2}\left(\delta_{IJ} - \frac{\partial x_K}{\partial X_I}\frac{\partial x_K}{\partial X_J}\right), \quad (2.25)$$

whose components represent the change in lengths of material line elements from the Eulerian perspective.

[11] The distinction is clear if we write $\boldsymbol{R} = X_{\overline{K}}\,\boldsymbol{e}_{\overline{K}}$, in which case $F_{\overline{I}\,J} = \frac{\partial X_{\overline{I}}}{\partial x_J}$, but there are already too many overbars in this section!

[12] The left Cauchy–Green deformation tensor is given by $\mathsf{B} = \mathsf{F}\mathsf{F}^T$ and is also called the Finger tensor.

[13] The final possible deformation tensor is $\mathsf{b} = \mathsf{c}^{-1} = \mathsf{H}\mathsf{H}^T$ and was introduced by Piola.

Curvilinear Coordinates

If the undeformed position is parametrised by a set of general Lagrangian coordinates $\boldsymbol{r}(\xi^i)$, the undeformed length of a material line will depend on the metric tensor. In general, lengths relative to general coordinates vary with position, whereas in Cartesian coordinates relative lengths are independent of absolute position.

The undeformed material line vector is given by

$$\mathbf{d}\boldsymbol{r} = \boldsymbol{r}(\xi^i + d\xi^i) - \boldsymbol{r}(\xi^i) \approx \frac{\partial \boldsymbol{r}}{\partial \xi^i} d\xi^i = \boldsymbol{g}_i d\xi^i,$$

where $\boldsymbol{g}_i$ is the covariant base vector in the undeformed configuration with respect to the Lagrangian coordinates. Similarly, the deformed material line vector is given by

$$\mathbf{d}\boldsymbol{R} = \boldsymbol{R}(\xi^i + d\xi^i) - \boldsymbol{R}(\xi^i) \approx \frac{\partial \boldsymbol{R}}{\partial \xi^i} d\xi^i = \boldsymbol{G}_i d\xi^i,$$

where $\boldsymbol{G}_i$ is the covariant base vector with respect to the Lagrangian coordinates in the deformed configuration[14]. Hence,

$$dS^2 - ds^2 = \mathbf{d}\boldsymbol{R} \cdot \mathbf{d}\boldsymbol{R} - \mathbf{d}\boldsymbol{r} \cdot \mathbf{d}\boldsymbol{r} = \boldsymbol{G}_i \cdot \boldsymbol{G}_j d\xi^i d\xi^j - \boldsymbol{g}_i \cdot \boldsymbol{g}_j d\xi^i d\xi^j = (G_{ij} - g_{ij})\, d\xi^i d\xi^j,$$

where $g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j$ is the (undeformed) Lagrangian metric tensor and $G_{ij} = \boldsymbol{G}_i \cdot \boldsymbol{G}_j$ is the (deformed) Lagrangian metric tensor. The Green–Lagrange strain tensor relative to these curvilinear coordinates is therefore

$$\gamma_{ij} = \frac{1}{2}\left(G_{ij} - g_{ij}\right); \tag{2.26}$$

and

$$\gamma_{ij} = \frac{1}{2}\left(\frac{\partial X_K}{\partial \xi^i}\frac{\partial X_K}{\partial \xi^j} - \frac{\partial x_K}{\partial \xi^i}\frac{\partial x_K}{\partial \xi^j}\right) = \frac{1}{2}\left(\frac{\partial X_K}{\partial x_I}\frac{\partial X_K}{\partial x_J} - \delta_{IJ}\right)\frac{\partial x_I}{\partial \xi^i}\frac{\partial x_J}{\partial \xi^j} = e_{IJ}\frac{\partial x_I}{\partial \xi^i}\frac{\partial x_J}{\partial \xi^j}, \tag{2.27}$$

which demonstrates that γ_{ij} are indeed the components of the Green–Lagrange tensor, after the appropriate (covariant) tensor transformation of the undeformed coordinates; and that the deformed metric tensor is obtained from the right Cauchy–Green deformation tensor under (covariant) transformation of the undeformed coordinates. Alternatively,

$$\gamma_{ij} = \frac{1}{2}\left(\delta_{IJ} - \frac{\partial x_K}{\partial X_I}\frac{\partial x_K}{\partial X_J}\right)\frac{\partial X_I}{\partial \xi^i}\frac{\partial X_J}{\partial \xi^j} = E_{IJ}\frac{\partial X_I}{\partial \xi^i}\frac{\partial X_J}{\partial \xi^j}, \tag{2.28}$$

and the (curvilinear) Green–Lagrange strain tensor can be obtained from the Almansi strain tensor after a covariant transformation of the deformed coordinates. Of course this means that the Cartesian Green–Lagrange strain tensor is the Almansi strain tensor after a change in coordinates from those in the deformed body to the undeformed, *i. e.* the Almansi tensor is the pushforward of the Green–Lagrange tensor:

$$E_{IJ} = e_{KL}\frac{\partial x_K}{\partial X_I}\frac{\partial x_L}{\partial X_J}, \quad \text{or} \quad \mathsf{E} = \mathsf{F}^{-T}\mathsf{e}\mathsf{F}^{-1}. \tag{2.29}$$

[14]Note that the components of the covariant base vector in the global Cartesian basis

$$[\boldsymbol{G}_j]^I = \frac{\partial X^I}{\partial \xi^j} = \frac{\partial X^I}{\partial x^J}\frac{\partial x^J}{\partial \xi^j} = F_{IJ}\frac{\partial x^J}{\partial \xi^j} = F_{Ij},$$

and are therefore equivalent to the deformation gradient tensor after covariant transformation of the undeformed coordinate (the second index).

The definition of F means that equation (1.30) defines the pushforward for covariant and contravariant tensors. Comparison with that equation reveals that E should be denoted $\mathsf{E}^\flat$ to avoid mistakes when working in general coordinates.

The fact that all these tensors are equivalent should be no surprise — they all represent the same physical measure: half the difference between the change in square lengths of material line elements.

Relationship between Cartesian and curvilinear formulations

It seems to be most common in the literature to work in terms of the tensors F, c and C (or b and B) because it is more compact; but the meaning can be obscured. For example, consider a vector field that is convected with the motion of the continuum so that it obeys the same transformation rules as line elements, see also §5.2.4. If the vector field in the undeformed configuration is given by $\boldsymbol{a}$ then by analogy with equation 2.19, the vector field in the deformed configuration, $\boldsymbol{A}$, is given by

$$A_I = F_{IJ} a_J, \quad \text{(in global Cartesian components) or} \quad \boldsymbol{A} = \mathsf{F}\boldsymbol{a}.$$

A natural question is how to represent this in general curvilinear coordinates using the established metric tensors. If we are completely explicit about the bases in our tensor formulation then it follows that

$$\mathsf{F} = F_{IJ}\boldsymbol{e}_I \otimes \boldsymbol{e}_J = \frac{\partial X_I}{\partial x_J}\boldsymbol{e}_I \otimes \boldsymbol{e}_J = \frac{\partial X_I}{\partial \xi^k}\frac{\partial \xi^k}{\partial x_J}\boldsymbol{e}_I \otimes \boldsymbol{e}_J = \left(\frac{\partial X_I}{\partial \xi^k}\boldsymbol{e}_I\right) \otimes \left(\frac{\partial \xi^k}{\partial x_J}\boldsymbol{e}_J\right)$$

$$\Rightarrow \quad \mathsf{F} = \boldsymbol{G}_k \otimes \boldsymbol{g}^k \quad \text{and hence} \quad \mathsf{F}^{-1} = \boldsymbol{g}_k \otimes \boldsymbol{G}^k,$$

which reveals the desired connection between the two formulations.

Returning to the transformation of our vector field we can write

$$A^i\boldsymbol{G}_i = \mathsf{F}(a^i\boldsymbol{g}_i) \quad \Rightarrow \quad A^i\boldsymbol{G}^j\boldsymbol{\cdot}\boldsymbol{G}_i = \boldsymbol{G}^j\boldsymbol{\cdot}\left(\boldsymbol{G}_k \otimes \boldsymbol{g}^k\right)\boldsymbol{\cdot}\boldsymbol{g}_i a^i \quad \Rightarrow \quad A^i\delta^j_i = \delta^j_k\delta^k_i a^i \quad \Rightarrow \quad A^j = a^j,$$

and we might have expected. The result indicates that a vector field convected with continuum line elements keeps identical values of its components, provided that covariant base vectors deform with the motion of the continuum, *i. e.* if $\boldsymbol{a} = a^i\boldsymbol{g}_i$, then $\boldsymbol{A} = a^i\boldsymbol{G}_i$.

If we have a vector field $\boldsymbol{m}$, in the undeformed configuration, with different transformation properties, *e. g.* the deformed configuration is given by $\boldsymbol{M} = \mathsf{F}^{-T}\boldsymbol{m}$, then

$$M^i\boldsymbol{G}_i = \mathsf{F}^{-T}(m_i\,\boldsymbol{g}^i) \quad \Rightarrow \quad M^i\boldsymbol{G}^j\boldsymbol{\cdot}\boldsymbol{G}_i = \boldsymbol{G}^j\boldsymbol{\cdot}\left(\boldsymbol{G}^k \otimes \boldsymbol{g}_k\right)\boldsymbol{\cdot}\boldsymbol{g}^i\,m_i \quad \Rightarrow \quad M^j = G^{jk}m_k,$$

a different type of transformation.

2.4.1 The connection between deformation and displacement

From equation (2.5), the deformed position can be written as the vector sum of the undeformed position and the displacement Lecture 7

$$\boldsymbol{R} = \boldsymbol{r} + \boldsymbol{u},$$

which means that

$$\mathbf{d}\boldsymbol{R} \approx \frac{\partial(\boldsymbol{r}+\boldsymbol{u})}{\partial \xi^i}\,\mathrm{d}\xi^i = (\boldsymbol{r}_{,i} + \boldsymbol{u}_{,i})\,\mathrm{d}\xi^i = \boldsymbol{g}_i\,\mathrm{d}\xi^i + \boldsymbol{u}_{,i}\,\mathrm{d}\xi^i = \mathbf{d}\boldsymbol{r} + \boldsymbol{u}_{,i}\,\mathrm{d}\xi^i. \tag{2.30}$$

The first term in equation (2.30) is the undeformed line element and so represents a rigid-body translation; the second term contains all the information about the strain and rotation.

The covariant base vectors in the deformed configuration are

$$\boldsymbol{G}_i = \boldsymbol{R}_{,i} = \boldsymbol{g}_i + \boldsymbol{u}_{,i},$$

so the Green–Lagrange strain tensor becomes

$$\gamma_{ij} = \frac{1}{2}\left(\boldsymbol{G}_i{\cdot}\boldsymbol{G}_j - \boldsymbol{g}_i{\cdot}\boldsymbol{g}_j\right) = \frac{1}{2}\left[(\boldsymbol{g}_i + \boldsymbol{u}_{,i}) \cdot (\boldsymbol{g}_j + \boldsymbol{u}_{,j}) - g_{ij}\right] = \frac{1}{2}\left[\boldsymbol{g}_i{\cdot}\boldsymbol{u}_{,j} + \boldsymbol{u}_{,i}{\cdot}\boldsymbol{g}_j + \boldsymbol{u}_{,i}{\cdot}\boldsymbol{u}_{,j}\right].$$

In order to simplify the scalar products, we write the displacement using the undeformed base vectors

$$\boldsymbol{u} = u_j\,\boldsymbol{g}^j, \quad \Rightarrow \quad \boldsymbol{u}_{,i} = u_k|_i\,\boldsymbol{g}^k,$$

where $|$ represents the covariant derivative in the Lagrangian viewpoint with respect to Lagrangian coordinates, *i.e.* using the metric tensor g_{ij}. Then the strain tensor becomes

$$\gamma_{ij} = \frac{1}{2}\left[\boldsymbol{g}_i{\cdot}\boldsymbol{g}^k u_k|_j + u_k|_i\boldsymbol{g}^k{\cdot}\boldsymbol{g}_j + u_k|_i\boldsymbol{g}^k{\cdot}\boldsymbol{g}^l u_l|_j\right] = \frac{1}{2}\left[\delta_i^k u_k|_j + u_k|_i\delta_j^k + u_k|_i u_l|_j g^{kl}\right]$$

and so

$$\gamma_{ij} = \frac{1}{2}\left[u_i|_j + u_j|_i + u^k|_i\,u_k|_j\right].$$

The quantity with components $u_i|_j$ is known as the displacement gradient tensor and can also be written $\boldsymbol{\nabla}_r \otimes \boldsymbol{u}$. In our global Cartesian coordinate system, the Green–Lagrange strain tensor has the form

$$e_{IJ} = \frac{1}{2}\left[u_{I,J} + u_{J,I} + u_{K,I}\,u_{K,J}\right], \tag{2.31}$$

which is often seen in textbooks.

2.4.2 Interpretation of the deformed metric (right Cauchy–Green deformation) tensor G_{ij}

We can scale the infinitesimal increments in the general coordinates so that $\mathrm{d}\xi^i = n^i\,\mathrm{d}\epsilon$, where n^i represents the direction of the increment and is chosen so that the vector $\boldsymbol{n} = n^i\,\boldsymbol{g}_i$ has unit length,

$$\boldsymbol{n}\cdot\boldsymbol{n} = n^i\,\boldsymbol{g}_i{\cdot}n^j\boldsymbol{g}_j = n^i g_{ij} n^j = 1.$$

The undeformed and deformed line elements are then functions of the direction $\boldsymbol{n}$

$$\mathbf{d}\boldsymbol{r}(\boldsymbol{n}) = \boldsymbol{g}_i n^i \mathrm{d}\epsilon \quad \text{and} \quad \mathbf{d}\boldsymbol{R}(\boldsymbol{n}) = \boldsymbol{G}_i n^i \mathrm{d}\epsilon.$$

We define the stretch in the direction $\boldsymbol{n}$ to be the ratio of the length of the deformed line element to the undeformed line element:

$$\lambda(\boldsymbol{n}) = \frac{|\mathbf{d}\boldsymbol{R}(\boldsymbol{n})|}{|\mathbf{d}\boldsymbol{r}(\boldsymbol{n})|} = \frac{\sqrt{n^i G_{ij} n^j}\mathrm{d}\epsilon}{\sqrt{n^i g_{ij} n^j}\mathrm{d}\epsilon} = \sqrt{n^i G_{ij} n^j} = \sqrt{n_I c_{IJ} n_J}, \tag{2.32}$$

where the last equality is obtained on transformation from the general coordinates to the global Cartesian coordinates. Note that the stretch is invariant and is well-defined because we have already established that $n_I c_{IJ} n_J > 0$ for all non-zero $\boldsymbol{n}$.

If we define $\boldsymbol{n}^{(i)}$ to be a unit vector in the ξ^i direction, then $n_i^{(i)} = 1/\sqrt{g_{ii}}$ (not summed) $n_j^{(i)} = 0$, $j \neq i$, so

$$\lambda(\boldsymbol{n}^{(i)}) = \sqrt{G_{ii}/g_{ii}} \quad \text{(not summed)},$$

which means that the diagonal entries of the deformed metric tensor are proportional to the squares of the stretch in the coordinate directions. If the undeformed coordinates are Cartesian coordinates, for which $g_{ii} = 1$ (not summed), then the diagonal entries of the deformed metric tensor are exactly the squares of the stretches in coordinate directions.

We now consider two distinct line elements in the undeformed configuration

$$\mathbf{d}\boldsymbol{r} = n^i \boldsymbol{g}_i \, \mathrm{d}\epsilon \quad \text{and} \quad \mathbf{d}\boldsymbol{q} = m^i \boldsymbol{g}_i \, \mathrm{d}\epsilon,$$

where $n^i \boldsymbol{g}_i$ and $m^i \boldsymbol{g}_i$ are both unit vectors. The corresponding line elements in the deformed configuration are

$$\mathbf{d}\boldsymbol{R} = n^i \boldsymbol{G}_i \, \mathrm{d}\epsilon \quad \text{and} \quad \mathbf{d}\boldsymbol{Q} = m^i \boldsymbol{G}_i \, \mathrm{d}\epsilon.$$

The dot product of the two deformed line elements is

$$n^i G_{ij} m^j (\mathrm{d}\epsilon)^2 = |\mathbf{d}\boldsymbol{R}||\mathbf{d}\boldsymbol{Q}| \cos\Theta = \sqrt{n^i G_{ij} n^j} \, \mathrm{d}\epsilon \, \sqrt{m^i G_{ij} m^j} \, \mathrm{d}\epsilon \cos\Theta,$$

where Θ is the angle between the two deformed line elements. Thus,

$$\cos\Theta = \frac{n^i G_{ij} m^j}{\sqrt{n^i G_{ij} n^j} \sqrt{m^i G_{ij} m^j}}; \tag{2.33}$$

and if θ is the angle between the two undeformed line elements

$$\cos\theta = n^i g_{ij} m^j.$$

Thus, the information about change in length and relative rotation of two line elements is entirely contained within the deformed metric tensor.

Example 2.4. Rigid body motion
If the body undergoes a rigid-body motion show that every component of the Green–Lagrange strain tensor is zero.

Solution 2.4. In a rigid-body motion the line elements do not change lengths and the angles between any two line elements remain the same. Hence, for any unit vectors $\boldsymbol{n}$ and $\boldsymbol{m}$ in the undeformed configuration

$$\lambda(\boldsymbol{n}) = n^i G_{ij} n^j = 1, \qquad \lambda(\boldsymbol{m}) = m^i G_{ij} m^j = 1, \tag{2.34a}$$

and $\cos\Theta = \cos\theta$, which means that

$$n^i G_{ij} m^j = n^i g_{ij} m_j \sqrt{n^i G_{ij} n^j} \sqrt{m^i G_{ij} m^j}. \tag{2.34b}$$

Using equation (2.34a) in equation (2.34b), we obtain the condition

$$n^i G_{ij} m^j = n^i g_{ij} m^j \quad \Rightarrow \quad n^i \left(G_{ij} - g_{ij}\right) m^j = 0,$$

which must be true for all possible unit vectors $\boldsymbol{n}$ and $\boldsymbol{m}$ and so by picking the nine different combinations on non-zero components, it follows that $G_{ij} - g_{ij} = 0$ for all i, j. Hence, every component of the Green–Lagrange strain tensor is zero.

2.4.3 Strain Invariants and Principal Stretches

At any point, the direction of maximum stretch is given by

$$\max_{\boldsymbol{n}} \lambda(\boldsymbol{n}) \quad \text{subject to the constraint} \quad |\boldsymbol{n}| = 1,$$

which can be solved by the method of Lagrange multipliers. We seek the stationary points of the function

$$L(\boldsymbol{n}, \mu) = \lambda^2(\boldsymbol{n}) - \mu \left\{ g_{ij} n^i n^j - 1 \right\} = n^i n^j G_{ij} - \mu \left\{ g_{ij} n^j n^i - 1 \right\},$$

where μ is an unknown Lagrange multiplier and n^i are the components of the unit vector in the basis $\boldsymbol{g}_i$. The condition $\partial L / \partial \mu = 0$ recovers the constraint and the three other partial derivative conditions give

$$\frac{\partial L}{\partial n^i} = 0 \quad \Rightarrow \quad G_{ij} n^j - \mu g_{ij} n^j = (G_{ij} - \mu g_{ij}) n^j = 0. \tag{2.35}$$

Thus, the maximum (or minimum) stretch is given by non-trivial solutions of the equation (2.35), which is an equation that defines the eigenvalues and eigenvectors of the deformed metric tensor.

The deformed metric tensor has real components and is symmetric which means that it has real eigenvalues and mutually orthogonal eigenvectors[15]. The eigenvectors $\boldsymbol{v}$ are non-trivial solutions of the equation

$$\mathcal{G}(\boldsymbol{v}) = \mu \boldsymbol{v} \quad \text{or} \quad G_{ij} v^j = \mu g_{ij} v^j = \mu v_i, \quad \text{etc}, \tag{2.36}$$

where $\mathcal{G}$ represents the deformed metric tensor as a linear map and the scalars μ are the associated eigenvalues. Note that because the eigenvalues are scalars they do not depend on the coordinate system, but, of course, the components of the eigenvectors are coordinate-system dependent. The eigenvalues (or in fact any functions of the eigenvalues) are, therefore, scalar invariants of the deformed metric tensor.

Equation (2.36) only has non-trivial solutions if

$$\det(G_{ij} - \mu\, g_{ij}) = 0.$$

and expansion of the determinant gives a cubic equation for μ, so there are three real eigenvalues and, hence, three distinct invariants. The invariants are not unique, but those most commonly used in the literature follow from the expansion of the determinant of the mixed deformed metric tensor in which the index is raised by multiplication with undeformed covariant metric tensor[16] which means that the undeformed metric tensor becomes the Kronecker delta:

$$|g^{ik} G_{kj} - \mu \delta^i_j| = -\mu^3 + I_1 \mu^2 - I_2 \mu + I_3. \tag{2.37}$$

Expanding the determinant using the alternating symbol gives

$$e^{ijk} (G^1_i - \mu\, \delta^1_i)(G^2_j - \mu\, \delta^2_j)(G^3_k - \mu\, \delta^3_k)$$

$$= e^{ijk} \left[G^1_i G^2_j G^3_k - \mu \left(\delta^1_i G^2_j G^3_k + \delta^2_j G^1_i G^3_k + \delta^3_k G^1_i G^2_j \right) + \mu^2 \left(\delta^1_i \delta^2_j G^3_k + \delta^1_i \delta^3_k G^2_j + \delta^2_j \delta^3_k G^1_i \right) - \mu^3 \delta^1_i \delta^2_j \delta^3_k \right],$$

[15] This is a fundamental result in linear algebra is easily proved for distinct eigenvalues by considering the appropriate dot products between different eigenvectors and also between an eigenvector and its complex conjugate. For repeated eigenvalues the proof proceeds by induction on subspaces of successively smaller dimension.

[16] In other words we are defining $G^i_j = g^{ik} G_{kj}$, which is a slight abuse of notation: if we worked solely in the deformed metric then we would define $G^i_j = G^{ik} G_{jk} = \delta^i_j$. The fact that our definition of $G^i_j \neq \delta^i_j$ is why we can reuse the notation without ambiguity, but it has the potential to be confusing.

and on comparison with equation (2.37) we obtain

$$I_1 = G^1_1 + G^2_2 + G^3_3 = G^i_i = \text{trace}(\mathcal{G}), \tag{2.38a}$$

$$\begin{aligned} I_2 &= G^2_2 G^3_3 - G^2_3 G^3_2 + G^1_1 G^3_3 - G^1_3 G^3_1 + G^1_1 G^2_2 - G^1_2 G^2_1, \\ &= \frac{1}{2}\left[G^i_i G^j_j - G^i_j G^j_i\right] = \frac{1}{2}\left[(\text{trace}(\mathcal{G}))^2 - \text{trace}(\mathcal{G}^2)\right]. \end{aligned} \tag{2.38b}$$

$$I_3 = e^{ijk} G^1_i G^2_j G^3_k = |G^i_j| = |g^{ik} G_{kj}| = |g^{ik}||G_{kj}| = G/g, \tag{2.38c}$$

where $G = |G_{ij}|$ is the determinant of the deformed covariant metric tensor and $g = |g_{ij}|$ is the determinant of the undeformed covariant metric tensor.

The use of invariants will be an essential part of constitutive modelling, because the behaviour of a material should not depend on the coordinate system.

The mutual orthogonality of the eigenvectors means that they form a basis of the Euclidean space and if we choose the eigenvectors to have unit length we can write the eigenbasis as $\boldsymbol{v}_{\widehat{I}} = \boldsymbol{v}^{\widehat{I}}$. The circumflexes on the indices are used to indicate the eigenbasis rather than the standard Cartesian basis. The components of the deformed metric tensor in the eigenbasis are given by

$$G_{\widehat{I}\widehat{J}} = \boldsymbol{v}_{\widehat{I}} \cdot \mathcal{G}(\boldsymbol{v}_{\widehat{J}}) = \boldsymbol{v}_{\widehat{I}} \cdot \mu_{(\widehat{J})} \boldsymbol{v}_{\widehat{J}} = \mu_{(\widehat{J})} \boldsymbol{v}_{\widehat{I}} \cdot \boldsymbol{v}_{\widehat{J}} \quad (\widehat{J} \text{ not summed}),$$

where $\mu_{(\widehat{J})}$ is the eigenvalue associated with the eigenvector $\boldsymbol{v}_{\widehat{J}}$. Thus the components of the deformed metric tensor in the eigenbasis form a diagonal matrix with the eigenvalues as diagonal entries

$$G_{\widehat{I}\,\widehat{J}} = \begin{cases} \mu_{(\widehat{I})} & \widehat{I} = \widehat{J}, \\ 0 & \widehat{I} \neq \widehat{J}, \end{cases} = \mu_{(\widehat{I})}\, \delta_{\widehat{I}\,\widehat{J}} \quad (\text{not summed}).$$

or writing the components in the eigenbasis in matrix form

$$\mathcal{G} = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \tag{2.39}$$

or

$$\mathcal{G} = \sum_{\widehat{I}=1}^{3} \mu_{(\widehat{I})} \boldsymbol{v}_{\widehat{I}} \otimes \boldsymbol{v}_{\widehat{I}}.$$

These eigenvalues are related to the stretch because if we decompose a unit vector into the eigenbasis, $\boldsymbol{n} = n_{\widehat{I}} \boldsymbol{v}_{\widehat{I}}$, then from equation (2.32)

$$\lambda(\boldsymbol{n}) = \sqrt{n_{\widehat{I}} G_{\widehat{I}\,\widehat{J}} n_{\widehat{J}}} = \sqrt{\sum_{\widehat{I}=1}^{3} n_{\widehat{I}} \mu_{(\widehat{I})} n_{\widehat{I}}},$$

and so the stretches in the direction of the eigenvectors are the square-roots[17] of the associated eigenvalues

$$\lambda(\boldsymbol{v}_{\widehat{I}}) \equiv \lambda_{(\widehat{I})} = \sqrt{\mu_{(\widehat{I})}}. \tag{2.40}$$

The three quantities $\lambda_{(\widehat{I})}$ are called the principal stretches and the associated eigenvectors are the principal axes of stretch.

[17] We could have anticipated this result because the metric tensor was formed by taking the square of the lengths of line elements.

These results motivate the definition of another symmetric tensor in which the eigenvectors remain the same, but the eigenvalues coincide exactly with the stretches

$$\mathcal{U} = \sum_{\widehat{I}=1}^{3} \lambda_{(\widehat{I})} \boldsymbol{v}_{\widehat{I}} \otimes \boldsymbol{v}_{\widehat{I}} = \sum_{\widehat{I}=1}^{3} \sqrt{\mu_{(\widehat{I})}} \boldsymbol{v}_{\widehat{I}} \otimes \boldsymbol{v}_{\widehat{I}} = \sqrt{\mathcal{G}}.$$

The deformed metric tensor can be written as the product of the deformation gradient tensor and its transpose, so in matrix form in Cartesian coordinates (although the coordinates don't matter because the quantities are all tensors)

$$\mathsf{c} = \mathsf{F}^T \mathsf{F} = \mathsf{U}^2.$$

Recall that the matrix c is the Cauchy–Green deformation tensor, which is the deformed metric tensor when the Lagrangian and Eulerian coordinates are both Cartesian. We define

$$\mathsf{R} = \mathsf{F}\mathsf{U}^{-1} \quad \Rightarrow \quad \mathsf{F} = \mathsf{R}\mathsf{U},$$

and note that because U is symmetric $\mathsf{U}^{-1} = \mathsf{U}^{-T}$ and so

$$\mathsf{R}^T\mathsf{R} = (\mathsf{F}\mathsf{U}^{-1})^T(\mathsf{F}\mathsf{U}^{-1}) = \mathsf{U}^{-T}\mathsf{F}^T\mathsf{F}\mathsf{U}^{-1} = \mathsf{U}^{-1}\mathsf{U}^2\mathsf{U}^{-1} = \mathsf{I},$$

so R is orthogonal. Hence, F can be written as the product of an orthogonal transformation (rotation) and U which is known as the (right) stretch tensor[18]. The interpretation is that, in addition to rigid-body translation, any deformation consists (locally) of a rotation and three mutually orthogonal stretches — the principal stretches, which are the square-roots of the eigenvalues of the deformed metric tensor.

2.4.4 Alternative measures of strain

The strain is zero whenever the principal stretches are all of unit length, so in the eigenbasis representation the matrix of components of the right stretch tensor is $\mathsf{U} = \mathsf{I}$. However, the eigenbasis is orthonormal and therefore any other Cartesian basis is obtained via orthogonal transformation so that the matrix representation of the stretch tensor becomes,

$$\overline{\mathsf{U}} = \mathsf{Q}\mathsf{U}\mathsf{Q}^T = \mathsf{Q}\mathsf{I}\mathsf{Q}^T = \mathsf{Q}\mathsf{Q}^T = \mathsf{I}.$$

Hence, if the strain is zero then $\mathsf{U} = \mathsf{I}$ when the stretch tensor is represented in **any** orthonormal coordinate system. Transforming to a general coordinate system would give $U_{ij} = g_{ij}$ in the zero strain case, where g_{ij} is the undeformed metric tensor.

We can define general strain measures using tensor functions of the right stretch tensor $\mathcal{F}(\mathcal{U})$. In the eigenbasis $\mathcal{U}$ is diagonal and we define $\mathcal{F}(\mathcal{U})$ by

$$\mathcal{F}(\mathcal{U}) = \sum_{\widehat{I}=1}^{3} f\left(\lambda_{(\widehat{I})}\right) \boldsymbol{v}_{\widehat{I}} \otimes \boldsymbol{v}_{\widehat{I}},$$

where $f(x)$ is any monotonically increasing function such that $f(1) = 0$ and $f'(1) = 1$. The monotonicity ensures that an increase in stretch leads to an increase in strain; the condition $f(1) = 0$

[18]This result is often called the polar decomposition. It is unique and the orthogonal transformation does not involve any reflection because $\det \mathsf{R} > 0$, which follows from $\det \mathsf{U} > 0$ and $\det \mathsf{F} > 0$.

means that when there is no stretch there is no strain; and the condition $f'(1) = 1$ is a normalisation to ensure that all strain measures are the same when linearised and consistent with the theory of small deformations. Admissible functions include

$$f(x) = \frac{1}{m}(x^m - 1) \quad (m \neq 0) \quad \text{and} \quad \ln x \quad (m = 0).$$

In orthonormal coordinate systems, we can then write the corresponding strain measures as

$$\frac{1}{m}(\mathsf{U}^m - \mathsf{I}) \quad (m \neq 0) \quad \text{and} \quad \ln \mathsf{U} \quad (m = 0).$$

The case $m = 2$ corresponds to the Green–Lagrange strain tensor; the case $m = -2$ is the Almansi strain tensor; the case $m = 1$ is called the Biot strain tensor and the case $m = 0$ is called the Hencky or incremental strain tensor.

Note that it is only the Green–Lagrange and Almansi strain tensors that can be formed without prior knowledge of the eigenvectors, so these are most commonly used in practice.

2.4.5 Deformation of surface and volume elements

Lecture 8

Having characterised the deformation of material line elements, we will also find it useful to consider the deformation of material surfaces and volumes. This will be particularly important when developing continuum conservation, or balance, laws.

Volume elements

We already know from equation (1.46) that a volume element in the undeformed configuration is given by $\mathrm{d}\mathcal{V}_0 = \sqrt{g}\,\mathrm{d}\xi^1\mathrm{d}\xi^2\mathrm{d}\xi^3$. The corresponding volume element in the deformed configuration is given by $\mathrm{d}\mathcal{V}_t = \sqrt{G}\,\mathrm{d}\xi^1\mathrm{d}\xi^2\mathrm{d}\xi^3$. Hence, the relationship between the two volume elements is

$$\mathrm{d}\mathcal{V}_t = \sqrt{G/g}\,\mathrm{d}\mathcal{V}_0. \tag{2.41a}$$

From equation (2.19)

$$F_{IJ} = \frac{\partial X_I}{\partial x_J} = \frac{\partial X_I}{\partial \xi^i}\frac{\partial \xi^i}{\partial x_J},$$

which means that $J = \det \mathsf{F} = \sqrt{G}/\sqrt{g}$ from equation (1.44). Thus we can also write the change in volume using the determinant of the deformation gradient tensor

$$\mathrm{d}\mathcal{V}_t = \det \mathsf{F}\,\mathrm{d}\mathcal{V}_0 = J\,\mathrm{d}\mathcal{V}_0. \tag{2.41b}$$

Surface elements

If we now consider a vector surface element given by $\mathbf{d}\boldsymbol{a} = \boldsymbol{n}\,\mathrm{d}a$ in the undeformed configuration, where $\mathrm{d}a$ is the area and $\boldsymbol{n}$ is a unit normal to the surface. We can form a volume element by taking the dot product of the vector surface element and a line element $\mathbf{d}\boldsymbol{r}$, so

$$\mathrm{d}\mathcal{V}_0 = \mathrm{d}v = \mathbf{d}\boldsymbol{r}\cdot\mathbf{d}\boldsymbol{a} = n_I\,\mathrm{d}r_I\,\mathrm{d}a.$$

From equation (2.41b) the corresponding deformed volume element is

$$\mathrm{d}\mathcal{V}_t = \mathrm{d}V = Jn_I\,\mathrm{d}r_I\,\mathrm{d}a = N_I\,\mathrm{d}R_I\,\mathrm{d}A = \mathbf{d}\boldsymbol{R}\cdot\mathbf{d}\boldsymbol{A},$$

where $\mathbf{d}\boldsymbol{A} = \boldsymbol{N}\,\mathrm{d}A$ is the deformed vector surface element. From equation (2.19) $\mathrm{d}R_I = F_{IJ}\,\mathrm{d}r_J$, so

$$Jn_I\,\mathrm{d}r_I\,\mathrm{d}a = N_I F_{IJ}\,\mathrm{d}r_J\,\mathrm{d}A, \quad \Rightarrow \quad Jn_J\,\mathrm{d}a = N_I F_{IJ}\,\mathrm{d}A$$

$$\Rightarrow \quad \mathrm{d}A_I = N_I\,\mathrm{d}A = Jn_J F^{-1}_{JI}\,\mathrm{d}a = JF^{-1}_{JI}\mathrm{d}a_J \quad \Rightarrow \quad \boldsymbol{N}\,\mathrm{d}A = J\mathsf{F}^{-T}\boldsymbol{n}\,\mathrm{d}a, \tag{2.42}$$

a result known as Nanson's relation.

Converting to general tensor notation, we decompose the deformed area vector $\mathbf{d}\boldsymbol{A} = \boldsymbol{N}\,\mathrm{d}A$ into the deformed contravariant basis in the Eulerian coordinates $\chi^{\bar{i}}$, to obtain

$$\mathrm{d}A_{\bar{i}} = \mathrm{dA}_I \frac{\partial X^I}{\partial \chi^{\bar{i}}} = J\frac{\partial x^J}{\partial X^I}\,\mathrm{d}a_J\,\frac{\partial X^I}{\partial \chi^{\bar{i}}} = J\frac{\partial x^J}{\partial \chi^{\bar{i}}}\,\mathrm{d}a_J = J\frac{\partial \xi^j}{\partial \chi^{\bar{i}}}\frac{\partial x^J}{\partial \xi^j}\,\mathrm{d}a_J = J\frac{\partial \xi^j}{\partial \chi^{\bar{i}}}\,\mathrm{d}a_j, \tag{2.43}$$

where the undeformed area vector has been decomposed into the undeformed contravariant basis in Lagrangian coordinates, $\mathrm{d}\boldsymbol{a} = \mathrm{d}a_i\boldsymbol{g}^i$. Thus, the transformation is similar to the usual covariant transformation between Eulerian and Lagrangian coordinate systems, but with an additional scaling by J to compensate for the change in area.

2.4.6 Special classes of deformation

If the deformation gradient tensor, F_{IJ}, is a function only of time (does not vary with position) then the deformation is said to be **homogeneous**. In this case, the equation (2.19)

Required Reading (not covered in lectures)

$$\mathrm{d}R_I(t) = F_{IJ}(t)\,\mathrm{d}r_J,$$

and integrating between two material points, at positions $\boldsymbol{a}$ and $\boldsymbol{b}$ in the undeformed configuration, gives

$$A_I(t) - B_I(t) = F_{IJ}(t)\,(a_I - b_I)\,, \tag{2.44}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are the corresponding positions of the material points in the deformed configuration. Equation (2.44) demonstrates that straight lines are carried to straight lines in a homogeneous deformation. Furthermore, any homogeneous deformation can be written in the form

$$X_I = A_I(t) + F_{IJ}(t)\,x_J.$$

- **Pure strain** is such that the principal axes of strain are unchanged, which means that the rotation tensor in the polar decomposition is the identity $\mathsf{R} = \mathsf{I}$.
- **Rigid-body deformations** are such that the distance between every pair of points remains unchanged and we saw in Example 2.4 that the strain is zero in such cases. We can also show that in this case the invariants are $I_1 = I_2 = 3$ and $I_3 = 1$.
- **Isochoric deformation** is such that the volume does not change, which means that $J = \sqrt{I_3} = 1$.
- **Uniform dilation** is a homogeneous deformation in which $F_{IJ} = \lambda\delta_{IJ}$, which means that all the principal stretches are λ. In other words, the material is stretched equally in all directions.
- **Uniaxial strain** is a homogeneous deformation in which one of the principal stretches is λ and the other two are 1, *i.e.* the material is stretched (and therefore strained) in only one direction.
- **Simple shear** is a homogeneous deformation in which all diagonal elements of F are 1 and one off-diagonal element is nonzero.

2.4.7 Strain compatibility conditions

Optional Reading

The strain tensors are symmetric tensors which means that they have six independent components, but they are determined from the displacement vector which has only three independent components. If we interpret the six equations (2.31) as a set of differential equations for the displacement, assuming that we are given the strain components, then we have an overdetermined system. It follows that there must be additional constraints satisfied by the components of a strain tensor to ensure that a meaningful displacement field can be recovered.

One approach to determine these conditions is to eliminate the displacement components from the equations (2.31) by partial differentiation and elimination, which is tedious particularly in curvilinear coordinates. A more sophisticated approach is to invoke a theorem due to Riemann that the Riemann–Christoffel tensor formed from a symmetric tensor g_{ij} should be zero if the tensor is a metric tensor of Euclidean space. The Riemann–Christoffel tensor is motivated by considering second covariant derivatives of vectors

$$v_i|_{jr} = (v_i|_j)|_r = (v_i|_j)_{,r} - \Gamma^k_{ir}(v_k|_j) - \Gamma^k_{jr}(v_i|_k),$$

see Example Sheet 1, q. 10. Writing out the covariant derivatives gives

$$v_i|_{jr} = v_{i,jr} - \Gamma^l_{ij,r}v_l - \Gamma^l_{ij}v_{l,r} - \Gamma^k_{ir}v_{k,j} + \Gamma^k_{ir}\Gamma^l_{kj}v_l - \Gamma^k_{jr}v_{i,k} + \Gamma^k_{jr}\Gamma^l_{ik}v_l. \tag{2.45a}$$

Interchanging r and j gives

$$v_i|_{rj} = v_{i,rj} - \Gamma^l_{ir,j}v_l - \Gamma^l_{ir}v_{l,j} - \Gamma^k_{ij}v_{k,r} + \Gamma^k_{ij}\Gamma^l_{kr}v_l - \Gamma^k_{rj}v_{i,k} + \Gamma^k_{rj}\Gamma^l_{ik}v_l, \tag{2.45b}$$

and subtracting equation (2.45b) from equation (2.45a) gives

$$v_i|_{jr} - v_i|_{rj} = \left[\Gamma^l_{ir,j} - \Gamma^l_{ij,r} + \Gamma^k_{ir}\Gamma^l_{kj} - \Gamma^k_{ij}\Gamma^l_{kr}\right] v_l \equiv R^l_{.ijr}\, v_l,$$

after using the symmetry properties of the Christoffel symbol and the partial derivative. The quantities on the left are the components of the type (0,3) tensors and v_l are the components of a type (0,1) tensor, which means that $R^l_{.ijr}$ is a type (1,3) tensor called the Riemann–Christoffel tensor. Thus if $R^l_{.ijr} = 0$, then covariant derivatives commute. The tensor is a measure of the curvature of space and in Cartesians coordinate in an Eulerian space the Christoffel symbols are all zero, which means that $R^l_{.ijr} = 0$ in any coordinate system in our Eulerian space.

A physical interpretation for compatibility conditions is that a deformed body must "fit together" so that a continuous region of space is filled. Thus, the tensor G_{ij} associated with the deformed material points must be a metric tensor of our Euclidean space and the required conditions are that the Riemann–Christoffel symbol associated with G_{ij} must be zero. We can use the relationship

$$\gamma_{ij} = \frac{1}{2}(G_{ij} - g_{ij}) \quad \Rightarrow \quad G_{ij} = 2\gamma_{ij} + g_{ij},$$

to determine explicit equations for the conditions on the components of the Green–Lagrange strain tensor γ_{ij}, but these are somewhat cumbersome.

2.5 Deformation Rates

In many materials, particularly fluids, it is the rate of deformation, rather than the deformation itself that is of most importance. For example if we take a glass of water at rest and shake it

(gently) then wait, the water will come to rest and look as it did before we shook it. However, the configuration of the material points will almost certainly have changed. Thus, the fluid will have deformed significantly, but this does not affect its behaviour. The obvious "rates" are material time derivatives of objects that quantify the deformation. The general approach when computing material derivatives of complicated objects is to work in the undeformed configuration, by using (or transforming to) Lagrangian coordinates, in which case the time-derivative is equal to the partial derivative and the coordinates are fixed in time. We shall use such an approach throughout the remainder of this chapter.

A measure of the rate of deformation of material lines is given by

$$\frac{D}{Dt}\left(|\mathbf{d}\boldsymbol{R}|^2\right) = \left.\frac{\partial}{\partial t}\right|_{\boldsymbol{r}}\left(|\mathbf{d}\boldsymbol{R}|^2\right) = \left.\frac{\partial}{\partial t}\right|_{\xi}\left(G_{ij}\,\mathrm{d}\xi^i\mathrm{d}\xi^j\right) = \left(\left.\frac{\partial \boldsymbol{G}_i}{\partial t}\right|_{\xi}\cdot\boldsymbol{G}_j + \boldsymbol{G}_i\cdot\left.\frac{\partial \boldsymbol{G}_j}{\partial t}\right|_{\xi}\right)\mathrm{d}\xi^i\mathrm{d}\xi^j,$$

because the Lagrangian coordinates ξ^i are fixed on material line and are independent of time. The fact that time and the Lagrangian coordinates are independent means that

$$\frac{D\boldsymbol{G}_i}{Dt} = \left.\frac{\partial \boldsymbol{G}_i}{\partial t}\right|_{\xi} = \frac{\partial^2 \boldsymbol{R}}{\partial t\,\partial \xi^i} = \frac{\partial}{\partial \xi^i}\frac{D\boldsymbol{R}}{Dt} = \boldsymbol{v}_{,i} = \left(V_k\boldsymbol{G}^k\right)_{,i} = V_k||_i\boldsymbol{G}^k, \tag{2.46}$$

where $||$ indicates covariant differentiation in the Lagrangian deformed basis with covariant base vectors $\boldsymbol{G}_i$ and contravariant base vectors $\boldsymbol{G}^i$. Thus,

$$\frac{D}{Dt}\left(|\mathbf{d}\boldsymbol{R}|^2\right) = \left(V_k||_i\boldsymbol{G}^k\cdot\boldsymbol{G}_j + V_k||_j\boldsymbol{G}_i\cdot\boldsymbol{G}^k\right)\mathrm{d}\xi^i\mathrm{d}\xi^j = \left(V_i||_j + V_j||_i\right)\mathrm{d}\xi^i\mathrm{d}\xi^j.$$

Now because the undeformed line element $\mathbf{d}\boldsymbol{r}$ has constant length, we can write

$$\frac{D}{Dt}\left(|\mathbf{d}\boldsymbol{R}|^2\right) = \frac{D}{Dt}\left(|\mathbf{d}\boldsymbol{R}|^2 - |\mathbf{d}\boldsymbol{r}|^2\right) = \frac{D}{Dt}\left(2\gamma_{ij}\,\mathrm{d}\xi^i\mathrm{d}\xi^j\right) = 2\dot{\gamma}_{ij}\,\mathrm{d}\xi^i\mathrm{d}\xi^j,$$

where

$$\dot{\gamma}_{ij} = \frac{D\gamma_{ij}}{Dt} = \frac{1}{2}\left(V_i||_j + V_j||_i\right), \tag{2.47}$$

is the (Lagrangian) rate of deformation tensor and is the material derivative of the Green–Lagrange strain tensor.

Alternatively, we could have decomposed the velocity into the (curvilinear) Eulerian coordinates $\chi^{\bar{j}}$, in which case

$$\frac{\partial \boldsymbol{v}}{\partial \xi^i} = \frac{\partial \chi^{\bar{l}}}{\partial \xi^i}\frac{\partial \boldsymbol{V}}{\partial \chi^{\bar{l}}} = \frac{\partial \chi^{\bar{l}}}{\partial \xi^i}V_{\bar{k}}||_{\bar{l}}\boldsymbol{G}^{\bar{k}},$$

and then

$$\begin{aligned}\frac{D}{Dt}\left(|\mathbf{d}\boldsymbol{R}|^2\right) &= \left(\frac{\partial \chi^{\bar{l}}}{\partial \xi^i}V_{\bar{k}}||_{\bar{l}}\boldsymbol{G}^{\bar{k}}\cdot\boldsymbol{G}_j + \frac{\partial \chi^{\bar{l}}}{\partial \xi^j}V_{\bar{k}}||_{\bar{l}}\boldsymbol{G}^{\bar{k}}\cdot\boldsymbol{G}_i\right)\mathrm{d}\xi^i\mathrm{d}\xi^j,\\ &= \left(\frac{\partial \chi^{\bar{l}}}{\partial \xi^i}V_{\bar{k}}||_{\bar{l}}\frac{\partial \chi^{\bar{k}}}{\partial \xi^m}\boldsymbol{G}^m\cdot\boldsymbol{G}_j + \frac{\partial \chi^{\bar{l}}}{\partial \xi^j}V_{\bar{k}}||_{\bar{l}}\frac{\partial \chi^{\bar{k}}}{\partial \xi^m}\boldsymbol{G}^m\cdot\boldsymbol{G}_i\right)\mathrm{d}\xi^i\mathrm{d}\xi^j,\\ &= \left(V_{\bar{k}}||_{\bar{l}} + V_{\bar{l}}||_{\bar{k}}\right)\frac{\partial \chi^{\bar{k}}}{\partial \xi^i}\frac{\partial \chi^{\bar{l}}}{\partial \xi^j}\mathrm{d}\xi^i\mathrm{d}\xi^j = 2D_{\bar{k}\bar{l}}\,\mathrm{d}\chi^{\bar{k}}\mathrm{d}\chi^{\bar{l}},\end{aligned} \tag{2.48}$$

where

$$D_{\bar{i}\bar{j}} = \frac{1}{2}\left(V_{\bar{i}}||_{\bar{j}} + V_{\bar{j}}||_{\bar{i}}\right),$$

is the Eulerian rate of deformation tensor in components in the Eulerian curvilinear basis.

At a given time, we can recover the Lagrangian rate of deformation tensor from the Eulerian by simply applying the appropriate covariant transformation; $\dot{\gamma}_{ij}$ is the pullback of $\mathsf{D}^\flat$ and is therefore equal to D_{ij} componentwise.

However, an important point is that although the Lagrangian rate of deformation tensor is the material time-derivative of the (Green-Lagrange) strain tensor. The Eulerian rate of deformation tensor, in general, does not coincide with the spatial or material time-derivative of an Eulerian strain tensor because the Eulerian coordinates themselves vary with time.

In the Eulerian representation (in Cartesian coordinates for convenience)

$$\frac{D}{Dt}\left(|\mathbf{d}\boldsymbol{R}|^2\right) = \frac{D}{Dt}\left(2E_{IJ}\,\mathrm{d}X_I\mathrm{d}X_J\right) = 2\frac{DE_{IJ}}{Dt}\,\mathrm{d}X_I\,\mathrm{d}X_J + 2E_{IJ}\frac{D(\mathrm{d}X_I)}{Dt}\mathrm{d}X_J + 2E_{IJ}\mathrm{d}X_I\frac{D(\mathrm{d}X_J)}{Dt},$$

and the material time derivative of the Eulerian line element is not zero in general. In fact, making use of Lagrangian coordinates,

$$\frac{D(\mathrm{d}X_I)}{Dt} = \frac{D}{Dt}\left(\frac{\partial X_I}{\partial x_J}\,\mathrm{d}x_J\right) = \frac{\partial}{\partial x_J}\left(\frac{DX_I}{Dt}\right)\,\mathrm{d}x_J,$$

because the material derivative and the Lagrangian coordinates are independent. By definition, the material derivative of the Eulerian position is the Eulerian velocity, which in Cartesian coordinates is given by $\boldsymbol{V} = V_I\boldsymbol{e}_I$, so

$$\frac{D(\mathrm{d}X_I)}{Dt} = \frac{\partial V_I}{\partial x_J}\,\mathrm{d}x_J = \frac{\partial V_I}{\partial X_K}\frac{\partial X_K}{\partial x_J}\,\mathrm{d}x_J = \frac{\partial V_I}{\partial X_K}\,\mathrm{d}X_K.$$

Thus,

$$\begin{aligned}\frac{D}{Dt}\left(2E_{IJ}\,\mathrm{d}X_I\mathrm{d}X_J\right) &= 2\left[\frac{DE_{IJ}}{Dt} + E_{IK}V_{K,J} + E_{KJ}V_{K,I}\right]\mathrm{d}X_I\,\mathrm{d}X_J \\ &= 2D_{\bar{i}\bar{j}}\,\mathrm{d}\chi^{\bar{i}}\,\mathrm{d}\chi^{\bar{j}} = 2D_{IJ}\,\mathrm{d}X_I\mathrm{d}X_J,\end{aligned}$$

after transformation to Cartesian coordinates. The equation is valid for arbitrary $\mathrm{d}X_I$ and $\mathrm{d}X_J$, so

$$\dot{E}_{IJ} = D_{IJ} - E_{IK}V_{K,J} - E_{KJ}V_{K,I}. \tag{2.49}$$

Hence, if the Eulerian rate of deformation D_{IJ} is zero, but the material is strained $E_{IJ} \neq 0$, an Eulerian observer will perceive a rate of strain of

$$\dot{E}_{IJ} = -E_{IK}V_{K,J} - E_{KJ}V_{K,I},$$

which is a consequence of the movement of the material lines in the fixed Eulerian coordinate system.

2.5.1 Rates of stretch, spin & vorticity

The Eulerian velocity gradient tensor is given by

Required Reading (briefly covered in lectures)

$$L_{\bar{i}\bar{j}} = V_{\bar{i}}||_{\bar{j}}, \quad \text{or} \quad L_{IJ} = \frac{\partial V_I}{\partial X_J},$$

which means that the (Eulerian) deformation rate tensor is the symmetric part of the (Eulerian) velocity gradient tensor,

$$D_{\bar{i}\bar{j}} = \frac{1}{2}\left(L_{\bar{i}\bar{j}} + L_{\bar{j}\bar{i}}\right). \tag{2.50}$$

The antisymmetric part of the (Eulerian) velocity gradient tensor is called the spin tensor

$$W_{\bar{i}\bar{j}} = \frac{1}{2}\left(L_{\bar{i}\bar{j}} - L_{\bar{j}\bar{i}}\right),$$

and so by construction

$$L_{\bar{i}\bar{j}} = D_{\bar{i}\bar{j}} + W_{\bar{i}\bar{j}}.$$

The vorticity vector is a vector associated with the spin tensor and is defined by

$$\omega^{\bar{k}} = \epsilon^{\bar{k}\bar{i}\bar{j}} W_{\bar{j}\bar{i}} = \frac{1}{2}\epsilon^{\bar{k}\bar{i}\bar{j}}\left(V_{\bar{j}}||_{\bar{i}} - V_{\bar{i}}||_{\bar{j}}\right) = \epsilon^{\bar{k}\bar{i}\bar{j}} V_{\bar{j}}||_{\bar{i}},$$

where $\epsilon^{\bar{i}\bar{j}\bar{k}}$ is the Levi-Civita symbol in the Eulerian curvilinear coordinates $\chi^{\bar{i}}$, which means that

$$\boldsymbol{\omega} = \mathrm{curl}_{\boldsymbol{R}} \boldsymbol{V},$$

in the Eulerian framework.

We can now use the same arguments as in §2.4.2 to demonstrate that the the diagonal entries D_{II} (not summed) are the instantaneous rates of stretch along Cartesian coordinate axes and that the off-diagonal entries represent the instantaneous rate of change in angle between two coordinate lines, or one-half the rate of shearing.

In addition, equation (2.46) expresses the material derivative of our Lagrangian base vectors, *i.e.* how lines tangent to the Lagrangian coordinates evolve with the deformation. Expressing the equation (2.46) in Cartesian coordinates gives

$$\frac{D\boldsymbol{G}_i}{Dt} = V_k||_i \boldsymbol{G}^i \quad \Rightarrow \quad \frac{DF_{Ii}}{Dt}\boldsymbol{e}_I = V_k||_i \frac{\partial \xi^k}{\partial X^I}\boldsymbol{e}_I,$$

$$\Rightarrow \quad \frac{DF_{Ii}}{Dt} = v_k||_i \frac{\partial \xi^k}{\partial X_I} = V_I||_i = V_{I,K}\frac{\partial X_K}{\partial \xi^i} = V_{I,K}F_{Ki} \quad \Rightarrow \quad \frac{DF_{IJ}}{Dt} = V_{I,K}F_{KJ}, \tag{2.51}$$

which demonstrates that the material derivative of the deformation gradient tensor is simply the velocity gradient tensor composed with the deformation gradient. A physical interpretation is that it is gradients (differences) in velocity that lead to changes in material deformation, which explains the above identification of the rate of deformation with the velocity gradient.

We next introduce the relative deformation gradient tensor, in which we measure the deformation from the initial state to a time τ and then the deformation from time τ to a subsequent time t. Then

$$F_{IJ}(t) = \frac{\partial X_I(t)}{\partial x_J} = \frac{\partial X_I(t)}{\partial X_K(\tau)}\frac{\partial X_K(\tau)}{\partial x_J} = \tilde{F}_{IK}(t,\tau)F_{KJ}(\tau),$$

where $\tilde{F}_{IK}(t,\tau)$ is called the relative deformation gradient tensor. The deformation gradient at the fixed time τ is independent of t so that

$$\frac{DF_{IJ}}{Dt} = \frac{D\tilde{F}_{IK}(t,\tau)}{Dt}F_{KJ}(\tau), \tag{2.52}$$

and taking the limit $\tau \to t$ of equation (2.52) and comparing to equation (2.51) we see that

$$\left.\frac{D\tilde{F}_{IK}(t,\tau)}{Dt}\right|_{\tau=t} = V_{I,K}, \quad \text{or written as matrices} \quad \left.\frac{D\tilde{\mathsf{F}}(t,\tau)}{Dt}\right|_{\tau=t} = \mathsf{L},$$

and the instantaneous relative deformation tensor is given by the Eulerian velocity gradient tensor.

We can now use the polar decomposition to write

$$\frac{D\tilde{\mathsf{F}}}{Dt} = \left.\frac{D(\tilde{\mathsf{R}}\tilde{\mathsf{U}})}{Dt}\right|_{\tau=t} = \mathsf{L} = \mathsf{D} + \mathsf{W}$$

$$\Rightarrow \quad \tilde{\mathsf{R}}\Big|_{\tau=t} \left.\frac{D\tilde{\mathsf{U}}}{Dt}\right|_{\tau=t} + \left.\frac{D\tilde{\mathsf{R}}}{Dt}\right|_{\tau=t} \tilde{\mathsf{U}}\Big|_{\tau=t} = \mathsf{D} + \mathsf{W}.$$

Now the relative rotation $\tilde{\mathsf{R}}$ and stretch $\tilde{\mathsf{U}}$ tensors both tend to the identity as $\tau \to t$ and there is difference between the state at τ and t. Thus,

$$\left.\frac{D\tilde{\mathsf{U}}}{Dt}\right|_{\tau=t} + \left.\frac{D\tilde{\mathsf{R}}}{Dt}\right|_{\tau=t} = \mathsf{D} + \mathsf{W}.$$

The (right) stretch tensor U is symmetric which means that

$$\left.\frac{D\tilde{\mathsf{U}}}{Dt}\right|_{\tau=t} = \mathsf{D} \quad \text{and} \quad \left.\frac{D\tilde{\mathsf{R}}}{Dt}\right|_{\tau=t} = \mathsf{W},$$

with the interpretation that the instantaneous material rate of change of the stretch tensor, *i.e.* the rate of stretch is given by the Eulerian deformation rate tensor; and the instantaneous material rate of rotation is given by the spin tensor.

2.5.2 Material derivative of volume and area elements

Lecture 8 (ctd)

A volume element in the current (deformed) position is given by

$$\mathrm{d}\mathcal{V}_t = \sqrt{G}\,\mathrm{d}\xi^1\mathrm{d}\xi^2\mathrm{d}\xi^3,$$

so the material derivative of the volume gives the dilation rate

$$\frac{D\,\mathrm{d}\mathcal{V}_t}{Dt} = \frac{D\sqrt{G}}{Dt}\,\mathrm{d}\xi^1\mathrm{d}\xi^2\mathrm{d}\xi^2.$$

We use a similar approach to that in the proof of the divergence theorem to determine

$$\frac{D\sqrt{G}}{Dt} = \frac{1}{2\sqrt{G}}\frac{DG}{Dt} = \frac{1}{2\sqrt{G}}\frac{\partial G}{\partial G_{ij}}\frac{DG_{ij}}{Dt}.$$

From comparison with equation (1.56) we have

$$\frac{\partial G}{\partial G_{ij}} = GG^{ij}, \tag{2.53}$$

and using equation (2.46)

$$\frac{DG_{ij}}{Dt} = \frac{D\boldsymbol{G}_i}{Dt}\cdot\boldsymbol{G}_j + \frac{D\boldsymbol{G}_j}{Dt}\cdot\boldsymbol{G}_i = V_i||_j + V_j||_i.$$

Hence,

$$\frac{D\sqrt{G}}{Dt} = \frac{1}{2\sqrt{G}} G G^{ij} \left(V_i||_j + V_j||_i\right) = \frac{1}{2}\sqrt{G}\left(V^j||_j + V^i||_i\right) = \sqrt{G}\, V^i||_i = \sqrt{G}\, V^{\bar{i}}||_{\bar{i}}, \tag{2.54}$$

after change of coordinates from Lagrangian to Eulerian. In other words, the rate of change of a volume element is given by

$$\frac{D\,\mathrm{d}\mathcal{V}_t}{Dt} = V^{\bar{i}}||_{\bar{i}}\sqrt{G}\,\mathrm{d}\xi^1\mathrm{d}\xi^2\mathrm{d}\xi^3 = V^{\bar{i}}||_{\bar{i}}\,\mathrm{d}\mathcal{V}_t,$$

and the relative volume change, or dilation, is

$$\frac{1}{\mathrm{d}\mathcal{V}_t}\frac{D\mathrm{d}\,\mathcal{V}_t}{Dt} = V^{\bar{i}}||_{\bar{i}} = \mathrm{div}_{\boldsymbol{R}}\boldsymbol{V}, \tag{2.55}$$

the divergence of the Eulerian velocity field.

The material derivative of an area element can be obtained by using the equation (2.43) to map the area element back into the Lagrangian coordinates

$$\frac{D\,\mathrm{d}A_{\bar{i}}}{Dt} = \frac{D}{Dt}\left(J\frac{\partial\xi^j}{\partial\chi^{\bar{i}}}\,\mathrm{d}a_j\right) = \left[\frac{DJ}{Dt}\frac{\partial\xi^j}{\partial\chi^{\bar{i}}} + J\frac{D}{Dt}\left(\frac{\partial\xi^j}{\partial\chi^{\bar{i}}}\right)\right]\mathrm{d}a_j.$$

$$= V^{\bar{k}}||_{\bar{k}}J\frac{\partial\xi^j}{\partial\chi^{\bar{i}}}\mathrm{d}a_j - V^{\bar{l}}||_{\bar{i}}J\frac{\partial\xi^j}{\partial\chi^{\bar{l}}}\mathrm{d}a_j = V^{\bar{k}}||_{\bar{k}}\mathrm{d}A_{\bar{i}} - V^{\bar{l}}||_{\bar{i}}\mathrm{d}A_{\bar{l}}. \tag{2.56}$$

The first term is obtained on using the result (2.54) divided by $\sqrt{g}$ because $J = \sqrt{G/g}$ and $\sqrt{g}$ is constant under material differentiation. The second term uses the result that

$$\frac{D}{Dt}\left(\frac{\partial\xi^j}{\partial\chi^{\bar{i}}}\right) = -V^{\bar{l}}||_{\bar{i}}\frac{\partial\xi^j}{\partial\chi^{\bar{l}}}, \tag{2.57}$$

which has a non-trivial derivation in curvilinear coordinates[19] because we need to be careful about what is being held fixed when computing all the derivatives. Firstly, we note that

$$\frac{D}{Dt}\left(\frac{\partial\chi^{\bar{i}}}{\partial\xi^j}\right) = \frac{D}{Dt}\left(\chi^{\bar{i}}_{,j}\right) = \left.\frac{\partial\chi^{\bar{i}}_{,j}}{\partial t}\right|_{\boldsymbol{r}} + V^{\bar{l}}\left.\frac{\partial\chi^{\bar{i}}_{,j}}{\partial\chi^{\bar{l}}}\right|_t + V^{\bar{l}}\,\overline{\Gamma}^{\bar{i}}_{\bar{l}\,\overline{m}}\,\overline{\chi}^{\overline{m}}_{,j}, \tag{2.58}$$

[19]The derivation in Cartesians is even simpler because the complexity of covariant differentiation is finessed,

$$\frac{D}{Dt}\left(\frac{\partial X_J}{\partial x_I}\right) = \frac{\partial}{\partial x_I}\frac{DX_J}{Dt} = \frac{\partial V_J}{\partial x_I} = \frac{\partial V_J}{\partial X_K}\frac{\partial X_K}{\partial x_I} = V_{J,K}\frac{\partial X_K}{\partial x_I}, \quad \text{or} \quad \frac{D\mathsf{F}}{Dt} = \mathsf{LF};$$

and then

$$\frac{D}{Dt}\left(\frac{\partial X_J}{\partial x_I}\frac{\partial x_I}{\partial X_L}\right) = \frac{D}{Dt}\left(\frac{\partial X_J}{\partial x_I}\right)\frac{\partial x_I}{\partial X_L} + \frac{\partial X_J}{\partial x_I}\frac{D}{Dt}\left(\frac{\partial x_I}{\partial X_L}\right) = \frac{D}{Dt}\left(\delta_{JL}\right) = 0,$$

$$\Rightarrow \quad \frac{\partial X_J}{\partial x_I}\frac{D}{Dt}\left(\frac{\partial x_I}{\partial X_L}\right) = -\frac{D}{Dt}\left(\frac{\partial X_J}{\partial x_I}\right)\frac{\partial x_I}{\partial X_L} = -V_{J,K}\frac{\partial X_K}{\partial x_I}\frac{\partial x_I}{\partial X_L} = -V_{J,K}\delta_{KL} = -V_{J,L}$$

$$\Rightarrow \quad \frac{D}{Dt}\left(\frac{\partial x_K}{\partial X_L}\right) = -V_{J,L}\frac{\partial x_K}{\partial X_J}, \quad \text{or} \quad \frac{D\mathsf{F}^{-1}}{Dt} = -\mathsf{F}^{-1}\mathsf{L}.$$

because although the quantity $\chi^{\bar{i}}_{,j}$ is a two-point tensor, the Lagrangian component is unaffected by material differentiation. Thus, the material derivative of $\chi^{\bar{i}}_{,j}$ is the same as for a contravariant vector component. The velocity is given by

$$\boldsymbol{V} = \left.\frac{\partial \boldsymbol{R}(\chi^{\bar{i}}(\boldsymbol{r},t))}{\partial t}\right|_{\boldsymbol{r}} = \left.\frac{\partial \boldsymbol{R}}{\partial \chi^{\bar{i}}}\frac{\partial \chi^{\bar{i}}}{\partial t}\right|_{\boldsymbol{r}} = \left.\frac{\partial \chi^{\bar{i}}}{\partial t}\right|_{\boldsymbol{r}} \boldsymbol{G}_{\bar{i}},$$

which means that the components of the Eulerian velocity vector in the basis $\boldsymbol{G}_{\bar{i}}$ are given by $V^{\bar{i}} = \frac{\partial \chi^{\bar{i}}}{\partial t}|_{\boldsymbol{r}}$. Thus, equation (2.58) can be rewritten as

$$\frac{D}{Dt}\left(\frac{\partial \chi^{\bar{i}}}{\partial \xi^j}\right) = \left.\frac{\partial \chi^{\bar{i}}_{,j}}{\partial t}\right|_{\boldsymbol{r}} + \left.\frac{\partial \chi^{\bar{l}}}{\partial t}\right|_{\boldsymbol{r}} \left.\frac{\partial \chi^{\bar{i}}_{,j}}{\partial \chi^{\bar{l}}}\right|_{t} + V^{\bar{l}}\,\overline{\Gamma}^{\bar{i}}_{\bar{l}\,\overline{m}}\,\overline{\chi}^{\overline{m}}_{,j} = \frac{\mathrm{d}\chi^{\bar{i}}_{,j}}{\mathrm{d}t} + V^{\bar{l}}\,\overline{\Gamma}^{\bar{i}}_{\bar{l}\,\overline{m}}\,\overline{\chi}^{\overline{m}}_{,j}, \tag{2.59}$$

where $\mathrm{d}/\mathrm{d}t$ is the total derivative with respect to time and, in the Eulerian framework, $\chi^{\bar{i}}_{,j}$ is a function of $\chi^{\bar{i}}(t)$ and t. Now,

$$\frac{\mathrm{d}\chi^{\bar{i}}_{,j}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \chi^{\bar{i}}(\xi^j,t)}{\partial \xi^j}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \chi^{\bar{i}}}{\partial \xi^j}\right) = \frac{\partial}{\partial \xi^j}\left(\frac{\partial \chi^{\bar{i}}}{\partial t}\right) = \frac{\partial V^{\bar{i}}}{\partial \xi^j},$$

because the Lagrangian coordinates are not functions of time and so the partial time derivative is equal to the material derivative. Equation (2.59) can therefore be written as

$$\frac{D}{Dt}\left(\frac{\partial \chi^{\bar{i}}}{\partial \xi^j}\right) = V^{\bar{i}}_{,j} + V^{\bar{l}}\,\overline{\Gamma}^{\bar{i}}_{\bar{l}\overline{m}}\,\overline{\chi}^{\overline{m}}_{,j} = \left[V^{\bar{i}}_{,\overline{m}} + V^{\bar{l}}\,\overline{\Gamma}^{\bar{i}}_{\bar{l}\overline{m}}\right]\overline{\chi}^{\overline{m}}_{,j} = V^{\bar{i}}||_{\overline{m}}\,\chi^{\overline{m}}_{,j},$$

after using the chain rule. The right-hand side has the appropriate two-point tensor transformation properties.

In order to derive equation (2.57), we use the fact that

$$\frac{D}{Dt}\left(\frac{\partial \xi^k}{\partial \chi^{\bar{i}}}\frac{\partial \chi^{\bar{i}}}{\partial \xi^j}\right) = \frac{D}{Dt}\left(\delta^k_j\right) = 0,$$

to write

$$\frac{D}{Dt}\left(\frac{\partial \xi^k}{\partial \chi^{\bar{i}}}\right)\frac{\partial \chi^{\bar{i}}}{\partial \xi^j} + \frac{\partial \xi^k}{\partial \chi^{\bar{i}}}\frac{D}{Dt}\left(\frac{\partial \chi^{\bar{i}}}{\partial \xi^j}\right) = 0$$

$$\Rightarrow \quad \frac{D}{Dt}\left(\frac{\partial \xi^k}{\partial \chi^{\bar{i}}}\right)\frac{\partial \chi^{\bar{i}}}{\partial \xi^j} = -\frac{\partial \xi^k}{\partial \chi^{\bar{i}}}\frac{D}{Dt}\left(\frac{\partial \chi^{\bar{i}}}{\partial \xi^j}\right) = -\frac{\partial \xi^k}{\partial \chi^{\bar{i}}}V^{\bar{i}}||_{\overline{m}}\frac{\partial \chi^{\overline{m}}}{\partial \xi^j}$$

and multiplying both sides by $\partial \xi^j/\partial \chi^{\bar{l}}$ gives

$$\Rightarrow \quad \frac{D}{Dt}\left(\frac{\partial \xi^k}{\partial \chi^{\bar{l}}}\right) = -\frac{\partial \xi^k}{\partial \chi^{\bar{i}}}V^{\bar{i}}||_{\bar{l}},$$

as required.

We can interpret the result for the transformation of the area element as follows

$\frac{D\,\mathrm{d}A_{\bar{i}}}{Dt}$	$=$	$V^{\bar{k}}\|\|_{\bar{k}}\,\mathrm{d}A_{\bar{i}}$	$-$	$V^{\bar{l}}\|\|_{\bar{i}}\,\mathrm{d}A_{\bar{l}},$
Material Derivative of Area	$=$	Volume Expansion	$-$	Expansion normal to Area Element.

2.6 The Reynolds Transport Theorem

The Reynolds transport theorem concerns the time derivative of integrals over material volumes (although it can be generalised to arbitrary moving volumes by considering a new set of coordinates that move with the volume in question rather than with the fluid)

$$\frac{\partial \mathcal{I}(t)}{\partial t} = \frac{\mathrm{d}\mathcal{I}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \phi \,\mathrm{d}\mathcal{V}_t.$$

Integrating $\phi(\boldsymbol{R}, t)$ over space means that $\mathcal{I}$ is a function only of t so the total derivative is equal to the partial derivative. Firstly we transform the material volume to the equivalent undeformed volume

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0} \phi\sqrt{G/g}\,\sqrt{g}\,\mathrm{d}\xi^1\mathrm{d}\xi^2\mathrm{d}\xi^3 = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0} \phi\sqrt{G/g}\,\mathrm{d}\mathcal{V}_0,$$

and because the undeformed volume is fixed, we can take the time derivative under the integral so that

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}t} = \int_{\Omega_0} \frac{\partial}{\partial t}\left(\phi\sqrt{G/g}\right)\mathrm{d}\mathcal{V}_0 = \int_{\Omega_0}\left(\frac{\partial \phi}{\partial t}\sqrt{G/g} + \phi\,\frac{\partial\sqrt{G/g}}{\partial t}\right)\mathrm{d}\mathcal{V}_0.$$

In the undeformed configuration we take the partial time derivative with $\boldsymbol{r}$ held constant, *i.e.* the material derivative $D/Dt = \partial/\partial t$, and

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}t} = \int_{\Omega_0}\left(\frac{D\phi}{Dt}\sqrt{\frac{G}{g}} + \frac{\phi}{\sqrt{g}}\,\frac{D\sqrt{G}}{Dt}\right)\mathrm{d}\mathcal{V}_0 = \int_{\Omega_0}\left(\frac{D\phi}{Dt} + \phi\, V^{\bar{i}}||_{\bar{i}}\right)\sqrt{\frac{G}{g}}\,\mathrm{d}\mathcal{V}_0 = \int_{\Omega_t}\left(\frac{D\phi}{Dt} + \phi\, V^{\bar{i}}||_{\bar{i}}\right)\mathrm{d}\mathcal{V}_t,$$

after using equation (2.54) and transforming back to the material volume. Thus, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\phi\,\mathrm{d}\mathcal{V}_t = \int_{\Omega_t}\left(\frac{D\phi}{Dt} + \phi\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V}\right)\mathrm{d}\mathcal{V}_t,$$

or expanding the material derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\phi\,\mathrm{d}\mathcal{V}_t = \int_{\Omega_t}\left(\frac{\partial\phi}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}_{\boldsymbol{R}}\phi + \phi\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V}\right)\mathrm{d}\mathcal{V}_t = \int_{\Omega_t}\left(\frac{\partial\phi}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}}\cdot(\phi\boldsymbol{V})\right)\mathrm{d}\mathcal{V}_t;$$

and using the divergence theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\phi\,\mathrm{d}\mathcal{V}_t = \int_{\Omega_t}\frac{\partial\phi}{\partial t}\,\mathrm{d}\mathcal{V}_t + \int_{\partial\Omega_t}\phi\boldsymbol{V}\cdot\boldsymbol{N}\,\mathrm{d}\mathcal{S}_t,$$

which means that the time rate of change of a quantity within a material volume is the sum of its rate of change obtained by treating the volume as fixed and the flux of the quantity across the boundaries (due to the movement of the boundaries).

In fact, the result is the multi-dimensional generalisation of the Leibnitz rule for differentiating under the integral sign when the limits are functions of the integration variable. Consider

$$\frac{\partial I}{\partial t} = \frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial}{\partial t}\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x = \lim_{\delta t\to 0}\frac{\int_{a(t+\delta t)}^{b(t+\delta t)} f(x,t+\delta t)\,\mathrm{d}x - \int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x}{\delta t},$$

from the fundamental definition of the partial derivative. Splitting up the domain of integration for the first integral gives

$$\frac{\partial I}{\partial t} = \lim_{\delta t \to 0} \frac{\int_{a(t+\delta t)}^{a(t)} f(x, t+\delta t)\,\mathrm{d}x + \int_{a(t)}^{b(t)} f(x, t+\delta t)\,\mathrm{d}x + \int_{b(t)}^{b(t+\delta t)} f(x, t+\delta t)\,\mathrm{d}x - \int_{a(t)}^{b(t)} f(x, t)\,\mathrm{d}x}{\delta t}.$$

Now, assuming that all the limits exist we can write

$$\frac{\partial I}{\partial t} = \int_{a(t)}^{b(t)} \lim_{\delta t \to 0} \frac{[f(x, t+\delta t) - f(x, t)]}{\delta t}\,\mathrm{d}x$$

$$+ \lim_{\delta \to 0} \frac{1}{\delta t} \left[\int_{a(t+\delta t)}^{a(t)} f(x, t+\delta t)\,\mathrm{d}x + \int_{b(t)}^{b(t+\delta t)} f(x, t+\delta t)\,\mathrm{d}x \right].$$

The first integral is simply the integral of $\partial f/\partial t$ and the others can be written using the mean value theorem as the product of the length of the integration interval and the function evaluated at some point within the interval, so

$$\frac{\partial I}{\partial t} = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\,\mathrm{d}x + \lim_{\delta t \to 0} \frac{1}{\delta t} \left\{ [a(t) - a(t+\delta t)] f(x, \tau_a) + [b(t+\delta t) - b(t)] f(x, \tau_b) \right\},$$

where $a(t) < \tau_a < a(t+\delta t)$ and $b(t) < \tau_b < b(t+\delta t)$. Using Taylor's theorem to expand the terms $a(t+\delta t)$ and $b(t+\delta t)$ we obtain

$$\frac{\partial I}{\partial t} = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\,\mathrm{d}x + \lim_{\delta t \to 0} \left\{ -a'(t) f(x, \tau_a) + b'(t) f(x, \tau_b) \right\} = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\,\mathrm{d}x - a'(t) f(x, a) + b'(t) f(x, b),$$

because $\tau_a \to a$ and $\tau_b \to b$ as $\delta t \to 0$, where the last two terms can be identified as the flux of f out of the moving domain.