Chapter 0

Preliminaries

These notes cover the course MATH45061 (Continuum Mechanics) and are intended to supplement the lectures. The course does not follow any particular text, so you **do not need** to buy any text books. The notes should be sufficiently self-contained that you will be able to use them to understand the course material. That said, there are many excellent textbooks out there that present the concepts from slightly different perspectives.

Most textbooks tend to favour fluid mechanics or solid mechanics and there are only a few that treat these two continua in a unified manner. The subject is incredibly broad and it's impossible to do it justice in a 15-credit lecture course. However, I shall attempt to give a flavour of the methods and richness of the subject. The early stages of the course will be spent developing the necessary mathematical framework to study the mechanics of continua. Although the basic concepts are straightforward, the mathematics rapidly becomes cumbersome when developing a framework that will allow use of general coordinate systems.

One of the first problems that you will face is notation. There is no universally accepted notation in continuum mechanics. The problem is that for a general treatment it is effectively impossible to find a notation that doesn't look cluttered unless you suppress important bits of information. Conversely, if information is suppressed so that the notation looks clean, the results can be ambiguous or unclear. Of course, once you understand what is going on the notation doesn't matter, but it helps to have a notation that is as easy as possible to work with. I have not found the perfect notation, but I believe that I have found a notation that is consistent, complete and relatively easy to use. Please do spend time getting used to and working with the notation. Do the initial exercises as soon as you get a chance and go back to them if you get confused. Having a firm command of the notation will really help in following the lectures.

0.1 Things you should already know

The course is as self-contained as it can be, but you should already be confident with the basic calculus of scalar and vector fields (div, grad, curl, multiple integrals, divergence theorem, ...); Taylor series for functions of many variables; the solution of ordinary and partial differential equations (general methods for linear equations); as well as basic linear algebra (how to work with matrices, vectors, definitions of eigenvalues, linear independence, ...). If you do not immediately know the answers to the questions in section 0.1.1 (or at least how to find the answers) then I would suggest revising the appropriate material. I will not assume any knowledge of mechanics beyond basic particle mechanics and Newton's laws, but, of course, if you have already done courses in fluid or solid mechanics many of the concepts that we discuss should be familiar.

0.1.1 Pre-course fitness check

1. Two vectors are defined in components in a global Cartesian basis: $\boldsymbol{a} = (0, 1, 2)$, $\boldsymbol{b} = (3, 2, -1)$.
   (a) Find $\boldsymbol{a} \cdot \boldsymbol{b}$ and hence determine whether the two vectors are orthogonal.
   (b) Find two unit vectors $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$ that are parallel to $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively.

2. Is it always possible to find the inverse of a matrix? If so, prove it; if not, provide a counterexample.

3. In Cartesian coordinates (x, y, z), $f = xyz$ and $\boldsymbol{F} = (x, y, z)$.
   (a) Is f a scalar or vector field? What about $\boldsymbol{F}$?
   (b) Find $\boldsymbol{\nabla} f$, $\boldsymbol{\nabla} \cdot \boldsymbol{F}$, and $\boldsymbol{\nabla} \times \boldsymbol{F}$.
   (c) What does the notation ∇^2 mean?

4. Find the Taylor series of $\sin(xy)$ about the point $x = 0$, $y = \pi/2$.

5. A linear system of simultaneous equations is given by

$$\begin{aligned} 4x + 3y + 2z &= 1, \\ 2x + 7y &= 0, \\ 8x + 6y + 4z &= 5. \end{aligned}$$

   Write the system as a matrix equation. Does the system have a solution? If so, find the solution; if not, how could the system be changed to ensure that it does have a solution.

6. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 3 \\ 5 & 2 & 1 \end{pmatrix}.$$

7. Find the general solution $u(x)$ of the equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \omega^2 u = 0.$$

8. Find the solution $u(x, y)$ of the PDE $\boldsymbol{\nabla}^2 u = 0$ in a square domain $x, y \in [0, 1]$, subject to the boundary conditions $u = 1$ on the line $y = 0$, but $u = 0$ on all other boundaries.

9. State the divergence theorem and construct an example to verify it.

10. State Newton's three laws of motion. Use them to determine the position at which a cannonball fired at an angle of $\pi/4$ radians with a velocity of $1ms^{-1}$ returns to the ground, assuming uniform gravitational acceleration of magnitude g.

0.2 Nomenclature and notational conventions

Another difficulty that students have found with this material is simply remembering all the different symbols. I have tried to use a consistent notation throughout and so here are the "rules" of the notation. This will not make sense if this is the first time that you are reading these notes, but I hope that it will be a handy reference.

- Scalar variables are written in italics like this x, but constants are not, *e.g.* the base of natural logarithms e and imaginary number i.
- Vectors are written in bold face, $\boldsymbol{a}$, whereas matrices and tensors are in a san-serif font, M, so that we would write a linear system of equations as $\mathsf{M}\boldsymbol{a} = \boldsymbol{b}$.
- Round brackets are used if one variable is a function of another, $\boldsymbol{V}(\boldsymbol{R}, t)$.
- The standard notation is used for scalar $(\cdot)$, vector $(\times)$ and tensor $(\otimes)$ products.
- Subscripts or superscripts that denote components of a vector or tensor in a **Cartesian** coordinate system are uppercase Roman, x_I. Subscripts or superscripts that denote components of a vector or tensor in a **general** coordinate system are lower case Roman, a^i.
- Quantities with a lower case Roman **subscript** obey a covariant transformation law under change of coordinate system; *e.g.* when changing from ξ^i to $\chi^{\bar{i}}$, $a_{\bar{i}} = \frac{\partial \xi^j}{\partial \chi^{\bar{i}}} a_j$ (*c.f.* tangent base vectors or partial derivatives). Quantities with a lower case Roman **superscript** obey a contravariant transformation law under change of coordinate system; *e.g.* $a^{\bar{i}} = \frac{\partial \chi^{\bar{i}}}{\partial \xi^j} a^j$ (*c.f.* vector components or differential forms). Note that the position of the index reflects the location of the new coordinate within the partial derivative in the transformation rule. For orthonormal coordinate systems, the two transformations are identical so $a_I = a^I$.
- We use the summation convention that a subscript and superscript with the same index should be summed over that index
$$a^j b_j = \sum_{j=1}^{3} a^j b_j = a^1 b_1 + a^2 b_2 + a^3 b_3.$$
Note that this convention ensures that invariant quantities are easily identified.
- Coordinates in the reference (original) configuration of the continuum are denoted by x^I (Cartesian) or ξ^i (general). These are called Lagrangian coordinates.
- Coordinates in the deformed (current) configuration of the continuum are denoted by X^I (Cartesian) or $\chi^{\bar{i}}$ (general). These are called Eulerian coordinates.
- Note that in Cartesian formulations both the original and deformed coordinates refer to **the same** global Cartesian base vectors, hence the index is the same. In the general formulations, we allow the two coordinate systems to be different and hence we use the different indices.
- Uppercase quantities refer to the deformed (Eulerian) representation. Lowercase quantities refer to the undeformed (Lagrangian) viewpoint. **Many other textbooks use exactly the opposite convention so be careful.** This notation is oxymoronic for two-point tensors (quantities with on Eulerian index and one Lagrangian) and so following standard conventions these are uppercase for deformation measures F, H and lowercase for stress measures, p.

Health Warning

Any notation will eventually "break" and I shall try to point out any abuses of notation when they arise, but if you spot any inconsistencies or mistakes then please let me know.

Lists of symbols

In the tables below we list the most commonly used symbols in these lecture notes. One significant complication in the general theory is the need to distinguish between the Lagrangian (material) and Eulerian (spatial) treatments. For any specific problem, we can usually choose **either** the Eulerian **or** the Lagrangian viewpoint and can avoid an explosion of different symbols. However, whether we choose Eulerian or Lagrangian coordinates, we must still connect the deformed and undeformed configurations of the continuum. The natural choice is to use Eulerian coordinates to describe the deformed configuration and Lagrangian for the undeformed, but the relationship between the two coordinate systems is required to determine strain measures. We must therefore, in principle, consider equivalent quantities in deformed and undeformed configurations in both sets of coordinates. If we have tensor or vector quantities then we can easily convert between the two coordinate systems by the appropriate transformation rules, but we **cannot** convert between deformed and undeformed domains without knowing details of the deformation, *i.e.* $\boldsymbol{R}(\boldsymbol{r})$ or $\boldsymbol{r}(\boldsymbol{R})$.

Quantity	Undeformed Configuration	Deformed Configuration
Material region	Ω_0	Ω_t
Density	ρ_0	ρ
Volume of region	$\mathcal{V}_0$	$\mathcal{V}_t$
Infinitesimal volume element	$\mathrm{d}v$, $\mathrm{d}\mathcal{V}_0$	$\mathrm{d}V$, $\mathrm{d}\mathcal{V}_t$
Surface of region	$\partial\mathcal{V}_0 = \mathcal{S}_0$	$\partial\mathcal{V}_t = \mathcal{S}_t$
Unit normal to surface	$\boldsymbol{n}$	$\boldsymbol{N}$
Infinitesimal scalar surface element	$\mathrm{d}a$, $\mathrm{d}\mathcal{S}_0$	$\mathrm{d}A$, $\mathrm{d}\mathcal{S}_t$
Infinitesimal vector surface element	$\mathbf{d}\boldsymbol{a} = \boldsymbol{n}\mathrm{d}a$	$\mathbf{d}\boldsymbol{A} = \boldsymbol{N}\mathrm{d}A$
Position vector	$\boldsymbol{r}$	$\boldsymbol{R}$
Infinitesimal line element	$\mathbf{d}\boldsymbol{r}$	$\mathbf{d}\boldsymbol{R}$

Table 1: List of notation used to denote analogous geometric quantities for a material region of a continuum in its undeformed and deformed configurations; see chapter 2 for further details.

For vector quantities we distinguish the components referred to the different bases $\boldsymbol{g}_i$, $\boldsymbol{g}_{\bar{i}}$, $\boldsymbol{G}_i$ and $\boldsymbol{G}_{\bar{i}}$, see Table 2, as follows:

$$\boldsymbol{v}(\boldsymbol{r}, t) = v^i \boldsymbol{g}_i = v^{\bar{i}} \boldsymbol{g}_{\bar{i}} = \boldsymbol{V}(\boldsymbol{R}, t) = V^i \boldsymbol{G}_i = V^{\bar{i}} \boldsymbol{G}_{\bar{i}}.$$

Quantity	Lagrangian	Eulerian
Global Cartesian basis	$\boldsymbol{e}_I$	
Undeformed Position vector	$\boldsymbol{r}$	$\boldsymbol{r}(\boldsymbol{R})$
Deformed Position vector	$\boldsymbol{R}(\boldsymbol{r})$	$\boldsymbol{R}$
Undeformed Cartesian coordinate	$x^I : \boldsymbol{r} = x^I \boldsymbol{e}_I$	$x^I(X^J)$
Deformed Cartesian coordinate	$X^I(x^J)$	$X^I : \boldsymbol{R} = X^I \boldsymbol{e}_I$
General Coordinate	$\xi^i : \boldsymbol{r}(\xi^i), \boldsymbol{R}(\xi^i)$	$\chi^{\bar{i}} : \boldsymbol{R}(\chi^{\bar{i}}), \boldsymbol{r}(\chi^{\bar{i}})$
Displacement field	$\boldsymbol{u}(\boldsymbol{r},t) = \boldsymbol{R} - \boldsymbol{r}$	$\boldsymbol{U}(\boldsymbol{R},t)$
Velocity field	$\boldsymbol{v}(\boldsymbol{r},t)$	$\boldsymbol{V}(\boldsymbol{R},t)$
Acceleration field	$\boldsymbol{a}(\boldsymbol{r},t)$	$\boldsymbol{A}(\boldsymbol{R},t)$
Partial derivative w.r.t. general coordinate	$f_{,i} = \frac{\partial a}{\partial \xi^i}$	$F_{,\bar{i}} = \frac{\partial A}{\partial \chi^{\bar{i}}}$
Undeformed covariant base vector	$\boldsymbol{g}_i = \boldsymbol{r}_{,i}$	$\boldsymbol{g}_{\bar{i}} = \boldsymbol{r}_{,\bar{i}}$
Deformed covariant base vector	$\boldsymbol{G}_i = \boldsymbol{R}_{,i}$	$\boldsymbol{G}_{\bar{i}} = \boldsymbol{R}_{,\bar{i}}$
Undeformed contravariant base vector	$\boldsymbol{g}^i : \quad \boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta^i_j$	$\boldsymbol{g}^{\bar{i}}$
Deformed contravariant base vector	$\boldsymbol{G}^i$	$\boldsymbol{G}^{\bar{i}}$
Undeformed covariant metric tensor	$\mathsf{g}^\flat : g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j$	$\overline{\mathsf{g}}^\flat : g_{\bar{i}\bar{j}} = \boldsymbol{g}_{\bar{i}} \cdot \boldsymbol{g}_{\bar{j}}$
Undeformed contravariant metric tensor	$\mathsf{g}^\sharp = (\mathsf{g}^\flat)^{-1}$	$\overline{\mathsf{g}}^\sharp = (\overline{\mathsf{g}}^\flat)^{-1}$
Deformed covariant metric tensor	$\mathsf{G}^\flat : G_{ij} = \boldsymbol{G}_i \cdot \boldsymbol{G}_j$	$\overline{\mathsf{G}}^\flat : G_{\bar{i}\bar{j}} = \boldsymbol{G}_{\bar{i}} \cdot \boldsymbol{G}_{\bar{j}}$
Deformed contravariant metric tensor	$\mathsf{G}^\sharp = (\mathsf{G}^\flat)^{-1}$	$\overline{\mathsf{G}}^\sharp = (\overline{\mathsf{G}}^\flat)^{-1}$
Determinant of undeformed covariant metric	$g = \vert\mathsf{g}^\flat\vert$	$\bar{g} = \vert\overline{\mathsf{g}}^\flat\vert$
Determinant of deformed covariant metric	$G = \vert\mathsf{G}^\flat\vert$	$\overline{G} = \vert\overline{\mathsf{G}}^\flat\vert$
Change in volume during deformation	$J = \sqrt{G}/\sqrt{g}$	$J = \overline{J} = \sqrt{\overline{G}}/\sqrt{\bar{g}}$
Undeformed gradient operator	$\boldsymbol{\nabla}_{\boldsymbol{r}} f = f_{,i} \boldsymbol{g}^i$	$\boldsymbol{\nabla}_{\boldsymbol{r}} f = f_{,\bar{i}} \boldsymbol{g}^{\bar{i}}$
Deformed gradient operator	$\boldsymbol{\nabla}_{\boldsymbol{R}} F = F_{,i} \boldsymbol{G}^i$	$\boldsymbol{\nabla}_{\boldsymbol{R}} F = F_{,\bar{i}} \boldsymbol{G}^{\bar{i}}$
Undeformed Christoffel symbol	$\Gamma^i_{jk} = \boldsymbol{g}^i \cdot \boldsymbol{g}_{j,k}$	$\Gamma^{\bar{i}}_{\bar{j}\bar{k}} = \boldsymbol{g}^{\bar{i}} \cdot \boldsymbol{g}_{\bar{j},\bar{k}}$
Undeformed covariant derivative	$\vert : f^i\vert_j = f^i_{,j} + \Gamma^i_{kj} f^k$	$\vert : f^{\bar{i}}\vert_{\bar{j}} = f^{\bar{i}}_{,\bar{j}} + \Gamma^{\bar{i}}_{\bar{k}\bar{j}} f^{\bar{k}}$
Deformed Christoffel symbol	$\overline{\Gamma}^i_{jk} = \boldsymbol{G}^i \cdot \boldsymbol{G}_{j,k}$	$\overline{\Gamma}^{\bar{i}}_{\bar{j}\bar{k}} = \boldsymbol{G}^{\bar{i}} \cdot \boldsymbol{G}_{\bar{j},\bar{k}}$
Deformed covariant derivative	$\vert\vert : F^i\vert\vert_j = F^i_{,j} + \overline{\Gamma}^i_{kj} F^k$	$\vert\vert : F^{\bar{i}}\vert\vert_{\bar{j}} = F^{\bar{i}}_{,\bar{j}} + \overline{\Gamma}^{\bar{i}}_{\bar{k}\bar{j}} F^{\bar{k}}$

Table 2: List of standard coordinates and common kinematic and geometric quantities in Lagrangian and Eulerian viewpoints; see chapter 2 for further details.

Quantity	Lagrangian	Eulerian
Deformation gradient tensor	$\mathsf{F} = (\boldsymbol{\nabla}_{\boldsymbol{r}} \otimes \boldsymbol{R})^T$	$\mathsf{H} = \mathsf{F}^{-1}$
Deformation tensor	$\mathsf{c} = \mathsf{F}^T\mathsf{F}$ (Cauchy–Green)	$\mathsf{C} = \mathsf{H}^T\mathsf{H}$ (Cauchy)
Deformation tensor	$\mathsf{b} = \mathsf{H}\mathsf{H}^T = \mathsf{c}^{-1}$ (Piola)	$\mathsf{B} = \mathsf{F}\mathsf{F}^T = \mathsf{C}^{-1}$ (Finger)
Cartesian strain tensor	$\mathsf{e} = (\mathsf{c} - \mathsf{I})/2$ (Green–Lagrange)	$\mathsf{E} = (\mathsf{I} - \mathsf{C})/2$ (Almansi)
General strain tensor	$\gamma_{ij} = (G_{ij} - g_{ij})/2$ (Green–Lagrange)	$\gamma_{\bar{i}\bar{j}} = (G_{\bar{i}\bar{j}} - g_{\bar{i}\bar{j}})/2$ (Almansi)
Velocity gradient tensor	$\mathsf{l} = (\boldsymbol{\nabla}_{\boldsymbol{r}} \otimes \boldsymbol{v})^T$	$\mathsf{L} = (\boldsymbol{\nabla}_{\boldsymbol{R}} \otimes \boldsymbol{V})^T$
Rate of deformation tensor	$\dot{\gamma}_{ij} = \frac{D\gamma_{ij}}{Dt}$	$\mathsf{D} = (\mathsf{L} + \mathsf{L}^T)/2$
Spin tensor	—	$\mathsf{W} = (\mathsf{L} - \mathsf{L}^T)/2$
Vorticity	—	$\boldsymbol{\omega}$

Table 3: Measures of strain and strainrate. The material deformation gradient tensor, F, connects the deformed and undeformed positions and is neither Eulerian nor Lagrangian. We have chosen to place it in the Lagrangian column for convenience and then its inverse, H, is then placed in the Eulerian column.

Quantity	Lagrangian Form	Eulerian Form
Body Force	$\boldsymbol{f}$	$\boldsymbol{F}$
Surface Traction	$\boldsymbol{t}$	$\boldsymbol{T}$
Stress tensor / Deformed Area	$\mathsf{T} = T^{ij}\boldsymbol{G}_i \otimes \boldsymbol{G}_j$ (Body)	$\mathsf{T} = T^{\bar{i}\bar{j}}\boldsymbol{G}_{\bar{i}} \otimes \boldsymbol{G}_{\bar{j}}$ (Cauchy)
Physical Stress (tensor)	—	$\sigma : \quad \sigma_{IJ} = T_{IJ}$
Stress / Undeformed Area	$\mathsf{s} = J\mathsf{F}^{-1}\sigma\mathsf{F}^{-T}$ (2nd Piola–Kirchhoff)	$\mathsf{p} = J\sigma^T\mathsf{F}^{-T}$ (1st Piola–Kirchhoff)

Table 4: Measures of force and stress. Note that the 1st Piola–Kirchoff stress is a two-point tensor, so does not naturally fit in the Eulerian or Lagrangian frameworks. We have placed it in the Eulerian column because if the stress vector is decomposed into Cartesian coordinates the Lagrangian formulation of the equations of motion use p.

Quantity	Lagrangian Form	Eulerian Form
Internal energy	ϕ	Φ
Heat flux	$\boldsymbol{q}$	$\boldsymbol{Q}$
Heat supply	b	B
Specific entropy	η_0	η
Temperature	θ	Θ
Helmholtz free energy	ψ	Ψ

Table 5: Thermodynamic and energetic quantities in Lagrangian and Eulerian representations.