Once again there was no choice in the exam and so all questions were attempted by all students. In general, students answered the questions well. There were varying degrees of clarity of explanation, but the vast majority had the correct idea of what to do.

1 This question tested understanding of objectivity and the fact that the (time) derivative of an objective quantity is not objective, in general. Almost all students correctly showed that

$$\overset{\circ}{\boldsymbol{A}}{}^{*} = \mathsf{Q}\overset{\circ}{\boldsymbol{A}},$$

which is the required relationship.

A query was raised in the exam (and on one exam script) about whether the partial derivative should have been a material derivative. The question is correct: it was intended to be a partial derivative. In this case it doesn't matter because for the most general objective transformation the orthogonal matrix $\mathsf{Q}(t)$ is independent of space. This means that the material derivative of Q with respect to time is equal to the partial derivative of Q with respect to time. (Many students didn't spot the subtlety, but weren't penalised.)

2 This question tested the derivation of conservation laws. For (i), some students were unsure how to include R_A and R_B in the integral balance, nor how to handle the fact that there were two chemicals. Mass of each chemical must be conserved, so the starting point should have been two conservation equations

$$\frac{D}{Dt}\int_{\Omega_t} A \,\mathrm{d}\mathcal{V}_t = \int_{\Omega_t} R_A \,\mathrm{d}\mathcal{V}_t \quad \text{and} \quad \frac{D}{Dt}\int_{\Omega_t} B \,\mathrm{d}\mathcal{V}_t = \int_{\Omega_t} R_B \,\mathrm{d}\mathcal{V}_t.$$

This leads to two very similar looking partial differential equations: one for A and one for B. Most students correctly answered this part, however,

For (ii), if the total mass is conserved than

$$\frac{D}{Dt}\int_{\Omega_t} A + B \,\mathrm{d}\mathcal{V}_t = 0,$$

from which it follows that $R_A + R_B = 0$. Approximately 50% of students got this part correct.

For (iii), if the mass of A does not change then from (ii) the mass of B cannot change and this is achieved if the evaporation rate exactly balances the internal source. R_B and E do not have the same units, however, because E is per unit (deformed) area and R_B is per unit deformed volume. The required integral relationship is that

$$\int_{\Omega_t} R_B \,\mathrm{d}\mathcal{V}_t = \int_{\partial\Omega_t} E \,\mathrm{d}\mathcal{S}_t,$$

and because R_B and E are constant, we have

$$E = (R_B \mathcal{V}_t)/\mathcal{S}_t,$$

where $\mathcal{V}_t$ is the volume of the domain and $\mathcal{S}_t$ is the domain's surface area. Note that it is not possible to use the divergence theorem to transform the surface integral of E into a volume integral because E is **only** defined on the surface of the domain. No students got this final relationship, but a few correctly wrote down the integral form of the balance and got full marks for doing so.

3 This question tested understanding of invariants and objectivity as well as whether students could use tensor algebra. Part (i) was probably the least well answered. The point is that we can identify the original direction field with an infinitesimal vector in the undeformed configuration: $\mathrm{d}x_I = a_I \mathrm{d}\epsilon$, where $\mathrm{d}\epsilon \ll 1$. From standard results the corresponding vector in the deformed configuration is

$$\mathrm{d}X_J = F_{JI}\mathrm{d}x_I \Rightarrow A_J \mathrm{d}\epsilon = F_{JI} a_I \mathrm{d}\epsilon,$$

giving the result after cancelling the scalar length $\mathrm{d}\epsilon$.

All students that attempted it correctly answered (ii): the explanation being that the scalar W must be independent of orientation of the fibres and material so W must be independent of any orthogonal transformation applied to its tensor arguments.

For (iii), it was necessary to show that I_1,I_2, I_3, I_4 and I_5 are **invariant**, *i.e.* do not change their values under orthogonal transformation of c and $\boldsymbol{a} \otimes \boldsymbol{a}$. For example

$$I_1^* = Q_{IJ} c_{JK} Q^T_{KI} = Q_{IJ} Q_{IK} c_{JK} = \delta_{JK} c_{JK} = c_{jj} = I_1,$$

because Q is an orthogonal matrix. A number of students tried to establish the relationship $I_1^* = \mathsf{Q} I_1 \mathsf{Q}^T$, which is the transformation for a second-rank tensor. In fact, the invariants are scalar and so $\mathsf{Q} I_1 \mathsf{Q}^T = I_1 \mathsf{Q}\mathsf{Q}^T = I_1 \mathsf{I}$, but that still leaves a tensor, rather than a scalar quantity!

4 This question tested whether students could determine strain invariants and work with the constitutive laws for general hyperelastic materials (given in the formula sheet) for a specific problem in solid mechanics.

(i) Most people correctly argued that the restriction to a spherical shape and incompressibility means that the radius cannot change.

(ii) The required result is that

$$\boldsymbol{X} = r \sin\theta \cos(\phi + f(r))\boldsymbol{e}_X + r \sin\theta \sin(\phi + f(r))\boldsymbol{e}_Y + r \cos\theta \boldsymbol{e}_Z,$$

which most students wrote down.

(iii) Everybody used the correct formulæ to find the strain invariants. The most common mistakes were in calculation of G_{ij}, which then lead to incorrect values of the invariants. The correct solution is that

$$I_1 = I_2 = 3 + r^2 \sin^2\theta (f')^2, \quad I_3 = 1,$$

where $f' = \frac{\mathrm{d}f(r)}{\mathrm{d}r}$. Students were awarded a mark if they had calculated the wrong value of I_3 but stated that it had to be wrong because they knew that I_3 should be 1 from incompressibility.

(iv) Even given the wrong values of the invariants, many spotted that the required condition is that $f' = 0$, which means that f must be a constant. The deformation in this case is a rigid body rotation. A number of students forgot to write down the description of the deformation.

5 This question tested thermodynamic modelling and use of the Clausius–Duhem inequality in an unseen context. In this case using a 1D model problem means that the algebra simplifies considerably. The question was answered well despite their being errors in the question, sorry. These errors did not affect the majority of students and marks were awarded for correct use of methods to obtain either the answers shown on the paper or the correct answers.

(i) Using the definition of the second Piola–Kirchhoff stress tensor from the formula sheet and deducing that $J = \lambda$ gives the result. The vast majority of students wrote that $E = \frac{1}{2}(\lambda - 1)$, but this is incorrect. The Green–Lagrange strain tensor is a measure of the difference in **square** lengths $E = \frac{1}{2}\left(\lambda^2 - 1\right)$.

(ii) This part followed from the standard methods applied to the Clausius–Duhem inequality. The two points to note are that $T = P$ and that, as most students deduced, the rate of strain tensor used was $D = \dot{\lambda}$ (assuming uniform stretch). Following the standard Coleman-Noll procedure using the Clausius–Duhem inequality and treating λ and Θ as independent variables leads to the desired formulæ. Unfortunately the presented result is incorrect. D is the **Eulerian** rate-of-strain tensor which means $D = \dot{\lambda}/\lambda$. The working is as follows $X = \lambda x$ and so $V = \dot{X} = \dot{\lambda} x$, but $D = \frac{\partial V}{\partial X}$ so we write $V = \dot{\lambda} X/\lambda$ and so $D = \dot{\lambda}/\lambda$. In that case the formula for the entropy is the same, but

$$P = \lambda \rho \frac{\partial \Psi}{\partial \lambda} = \rho_0 \frac{\partial \Psi}{\partial \lambda}. \tag{1}$$

In the following sections I shall discuss the solution using the incorrect formula in the paper first, which is the formula that all students used, and then point out the correct solution afterwards. Note that the only difference is a factor of λ in a few key places.

(iii) Most students correctly spotted that using the formulæ in (ii) we have

$$\frac{\partial}{\partial\Theta}\left(\frac{P}{\rho}\right)=\frac{\partial^2\Psi}{\partial\Theta\partial\lambda}=\frac{\partial^2\Psi}{\partial\lambda\partial\Theta}=\frac{\partial}{\partial\lambda}\left(\frac{\partial\Psi}{\partial\Theta}\right)=-\frac{\partial\eta}{\partial\lambda}.$$

The connection to S is found by using the result that $P/\rho = S/(\lambda\rho)$ and that because mass is conserved $\rho_0=\rho\lambda=1$ and so

$$\frac{\partial}{\partial\Theta}\left(\frac{P}{\rho}\right)=\frac{\partial S}{\partial\Theta}.$$

Note that if the correct formula for P, equation (1), is used then there is an additional factor of λ and

$$\frac{1}{\lambda}\left.\frac{\partial S}{\partial\Theta}\right|_\lambda=-\left.\frac{\partial\eta}{\partial\lambda}\right|_\Theta.$$

(iv) Very few students knew how to attempt this part. The key point is that for an isentropic material $\mathrm{d}\eta=0$ and combined with the hint, this gives

$$\mathrm{d}\Theta=-\frac{\frac{\partial\eta}{\partial\lambda}}{\frac{\partial\eta}{\partial\theta}}\,\mathrm{d}\lambda.$$

This result can be used to replace $\mathrm{d}\Theta$ in the expression

$$\mathrm{d}S=\left(\frac{\partial S}{\partial\lambda}\right)\mathrm{d}\lambda+\left(\frac{\partial S}{\partial\Theta}\right)\mathrm{d}\Theta.$$

If the result from (iii) is used to replace $\frac{\partial\eta}{\partial\lambda}$ then we obtain

$$\mathrm{d}S=\left[\frac{\partial S}{\partial\lambda}+\frac{\left(\frac{\partial S}{\partial\Theta}\right)^2}{\frac{\partial\eta}{\partial\Theta}}\right]\mathrm{d}\lambda.$$

Finally, we use the result that because $E=(\lambda^2-1)/2$ then $\mathrm{d}E=\lambda\,\mathrm{d}\lambda$ and $\frac{\partial}{\partial\lambda}=\frac{\partial E}{\partial\lambda}\frac{\partial}{\partial E}=\lambda\frac{\partial}{\partial E}$ to deduce that $\mathrm{d}\lambda\frac{\partial}{\partial\lambda}=\mathrm{d}E\frac{\partial}{\partial E}$ and so

$$\mathrm{d}S=\left[\frac{\partial S}{\partial E}+\frac{1}{\lambda}\frac{\left(\frac{\partial S}{\partial\Theta}\right)^2}{\frac{\partial\eta}{\partial\Theta}}\right]\mathrm{d}E.$$

The result in the paper had an incorrect factor of 2 in the denominator. If the correct formula for P is used then the final result it

$$\mathrm{d}S=\left[\frac{\partial S}{\partial E}+\frac{1}{\lambda^2}\frac{\left(\frac{\partial S}{\partial\Theta}\right)^2}{\frac{\partial\eta}{\partial\Theta}}\right]\mathrm{d}E.$$

The sought interpretation is that the first term is the elastic stress and the second term is the thermal stress.

6 This question tested working with fluid constitutive laws and establishing objectivity of constitutive laws. The first part (i) was standard and very similar to questions in previous exams. Almost all students got full marks on this part. Part (i) required substitution of the given velocity field into the constitutive law to determine the stress. Most students correctly calculated that $L_{12}=\epsilon$ with all other components being 0 and hence that $D_{12}=D_{21}=\epsilon/2$ and $D_{11}=D_{22}=0$. It follows that

$$\mathsf{T}=\begin{pmatrix}-P+GA_{11} & \mu\epsilon+GA_{12}\\ \mu\epsilon+GA_{21} & -P+GA_{22}\end{pmatrix},$$

and it remains to calculate A from the equation

$$\mathsf{A}^{\nabla}=-\frac{1}{\tau}(\mathsf{A}-\mathsf{I}).$$

We are told that the flow is steady and there is no variation with space so $\frac{D\mathsf{A}}{Dt} = 0$ and hence

$$-\mathsf{LA} - \mathsf{AL}^T = -\frac{1}{\tau}\left(\mathsf{A} - \mathsf{I}\right).$$

Many students struggled to solve this matrix equations with a number commiting all kinds of mathematical sins like dividing by matrices and/or assuming that matrix multiplication commutes. The matrix equation above leads to the four simultaneous equations

$$\begin{aligned} -\epsilon\left(A_{21} + A_{12}\right) &= (1 - A_{11})/\tau, \\ -\epsilon A_{22} &= -A_{12}\tau, \\ -\epsilon A_{22} &= -A_{21}\tau, \\ 0 &= \left(1 - A_{22}\right)/\tau; \end{aligned}$$

from which it follows that $A_{22} = 1$, $A_{12} = A_{21} = \epsilon\tau$ and $A_{11} = 1 + 2\epsilon^2\tau^2$. The final result for the stress is then

$$\mathsf{T} = \begin{pmatrix} -P + G(1 + 2\epsilon^2\tau^2) & \mu\epsilon + G\epsilon\tau \\ \mu\epsilon + G\epsilon\tau & -P + G \end{pmatrix}.$$

All the terms in the stress are constant apart from the pressure P. Thus Cauchy's equations are satisfied provided that P is constant. In (iii) the normal stress difference is $2\epsilon^2\tau^2 G$ and hence the fluid is **non-Newtonian** because the normal stress difference is not zero. A surprising number of students got this the wrong way round and said that a non-zero normal stress meant the fluid was Newtonian. Students that had calculated the wrong expression for T but calculated the correct corresponding normal stress difference and made the correct conclusion about whether the fluid was Newtonian were given full marks for this part.