MATH45061

Three hours

A formula sheet is provided at the end of the examination

THE UNIVERSITY OF MANCHESTER

CONTINUUM MECHANICS

21 January 2020

09:45 – 12:45

Answer **ALL SIX** questions.

University approved calculators may be used.

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1. Show that the derivative

$$\mathring{\boldsymbol{A}} = \frac{\partial \boldsymbol{A}}{\partial t} - \mathsf{W}\boldsymbol{A}$$

of a vector $\boldsymbol{A}$ is objective, where W is the spin tensor. You may use the fact that $\dot{\mathsf{F}} = \mathsf{L}F$ without proof, where F is the deformation gradient tensor and L is the Eulerian velocity gradient tensor. Both L and F are defined on the formula sheet at the end of the examination.

[5 marks]

2. A mixture of two chemicals is transported within a continuum. The concentrations (masses per unit deformed volume) of the two chemicals a and b are given by $A(\boldsymbol{R}, t)$ and $B(\boldsymbol{R}, t)$, respectively. The chemicals react in such a way that a mass of a is produced at a rate R_A per unit deformed volume and a mass of b is produced at a rate R_B per unit deformed volume.

(i) By starting from conservation of mass equations within a material volume, find two partial differential equations that describe the conservation of mass of each chemical.

[5 marks]

(ii) Assuming that there are no other sources or sinks of mass and that the total mass of both chemicals is conserved, find a relationship between R_A and R_B.

[3 marks]

(iii) Assume further that B evaporates from the surface of the continuum with constant mass-loss rate E per unit deformed area and that R_B is also a constant. Find the relationship between R_B and E which ensures that the mass of A does not change.

[2 marks]

3. A solid material is reinforced by a family of fibres, whose initial direction is described by the vector field $\boldsymbol{a}(\boldsymbol{r})$, where $|\boldsymbol{a}| = 1$.

(i) Show that fibres in a deformed configuration, described by a position vector $\boldsymbol{R}$, are given by $\boldsymbol{A} = \mathsf{F}\boldsymbol{a}$, where $\mathsf{F} = \boldsymbol{\nabla}_{\boldsymbol{r}}\boldsymbol{R}$ is the deformation gradient tensor.

[2 marks]

(ii) The strain energy of the material is given by $\mathcal{W}(\mathsf{c}, \boldsymbol{a} \otimes \boldsymbol{a})$ and must remain unchanged if the material and the fibres are simultaneously rotated about any axis in the undeformed configuration. Explain why this leads to the restriction

$$\mathcal{W}(\mathsf{c}, \boldsymbol{a} \otimes \boldsymbol{a}) = \mathcal{W}(\mathsf{Q}\mathsf{c}\mathsf{Q}^T, \mathsf{Q}\boldsymbol{a} \otimes \boldsymbol{a}\mathsf{Q}^T), \tag{1}$$

for any proper orthogonal tensor Q. Here $\mathsf{c} = \mathsf{F}^T\mathsf{F}$ is the Cauchy-Green deformation tensor.

[3 marks]

(iii) If the strain energy is written as a function of the five invariants

$$I_1 = \mathrm{trace}(\mathsf{c}), \quad I_2 = \frac{1}{2}\left\{[\mathrm{trace}(\mathsf{c})]^2 - \mathrm{trace}(\mathsf{c}^2)\right\}, \quad I_3 = \det(\mathsf{c}), \quad I_4 = \boldsymbol{a}\cdot\mathsf{c}\boldsymbol{a}, \quad I_5 = \boldsymbol{a}\cdot\mathsf{c}^2\boldsymbol{a},$$

show that the constraint (1) would be satisfied.

[6 marks]

4. An incompressible, hyperelastic solid sphere has undeformed radius 1. The sphere undergoes a deformation such that it remains spherical with a twist (rotation about an axis passing through its centre) that varies with distance from the centre of the sphere. The strain energy function of the solid material is given by

$$\mathcal{W} = (I_1 - 3) + (I_2 - 3),$$

where I_1 and I_2 are the first and second strain invariants.

(i) Explain why the radius of the sphere cannot change under the imposed deformation.

[2 marks]

(ii) Write down the deformed position in components in a global Cartesian coordinate system as a function of the undeformed position described using spherical polar coordinates ($\xi^1 = r, \xi^2 = \theta, \xi^3 = \phi$). The global coordinate system is chosen so that the axis of rotation is the z axis. You may assume that the twist is given by a function of the form $\Phi = \phi + f(r)$, where Φ is the aziumthal angle in the deformed configuration.

[2 marks]

(iii) Hence, compute the three strain invariants corresponding to this deformation.

[10 marks]

(iv) Compute the component of the stress tensor T^{13} and find a condition on f which ensures that $T^{13} = 0$. Give a description of the deformation in this case.

[4 marks]

You may use the fact that in spherical polar coordinates the position is given by

$$\boldsymbol{r} = r \sin\theta \cos\phi \boldsymbol{e}_x + r \sin\theta \sin\phi \boldsymbol{e}_y + r \cos\theta \boldsymbol{e}_z,$$

and the only non-zero Christoffel symbols are

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \frac{\cos\theta}{\sin\theta},$$

$$\Gamma^1_{22} = -r, \quad \Gamma^1_{33} = -r\sin^2\theta, \quad \Gamma^2_{33} = -\cos\theta\sin\theta.$$

5. A one-dimensional bar of thermoelastic material, initially of unit density, is stretched from its undeformed length of 1 to a length λ. The unidirectional Cauchy stress is given by a scalar quantity P. The fact that the problem is entirely one-dimensional reduces all tensors to scalar quantities. The Helmholtz free energy takes the form $\Psi(\lambda, \Theta)$.

(i) Explain why the second Piola—Kirchhoff stress tensor is given by $S = \lambda P$ and find an expression for the Green–Lagrange strain, E.

[3 marks]

(ii) Use the Clausius–Duhem inequality to show that

$$P = \rho \frac{\partial \Psi}{\partial \lambda}, \quad \eta = -\frac{\partial \Psi}{\partial \Theta}.$$

[5 marks]

(iii) Hence show that

$$\left.\frac{\partial (P/\rho)}{\partial \Theta}\right|_{\lambda} = \left.\frac{\partial S}{\partial \Theta}\right|_{\lambda} = -\left.\frac{\partial \eta}{\partial \lambda}\right|_{\Theta}.$$

[5 marks]

(iv) Show that for an isentropic material (total entropy is fixed)

$$\mathrm{d}S = \left[\frac{\partial S}{\partial E} + \frac{1}{\lambda}\frac{\left(\frac{\partial S}{\partial \Theta}\right)^2}{\frac{\partial \eta}{\partial \Theta}}\right] \mathrm{d}E,$$

and interpret the result in terms of the elastic and thermal components.

[5 marks]

Recall that the differential (infinitesimal change) of a function of two variables $A(x, y)$ is such that

$$\mathrm{d}A = \frac{\partial A}{\partial x}\,\mathrm{d}x + \frac{\partial A}{\partial y}\,\mathrm{d}y.$$

6. An incompressible, generalised Oldroyd B fluid has the constitutive relationship

$$\mathsf{T} = -P\mathsf{I} + 2\mu\mathsf{D} + G\mathsf{A},$$

where the tensor A satisfies

$$\mathsf{A}^{\nabla} = -\frac{1}{\tau}\left(\mathsf{A} - \mathsf{I}\right);$$

here, P is the fluid pressure and μ, G and τ are constants. The upper-convected derivative is defined by

$$\mathsf{A}^{\nabla} = \frac{D\mathsf{A}}{Dt} - \mathsf{L}\mathsf{A} - \mathsf{A}\mathsf{L}^{T},$$

where L is the Eulerian velocity gradient tensor and D is the symmetric part of L.

(i) Confirm that the constitutive relationships are objective.

[7 marks]

You may assume that the Cauchy stress T is objective; that the deformation gradient tensor, F, transforms as $\mathsf{F}^* = \mathsf{Q}\mathsf{F}$ and $\dot{\mathsf{F}} = \mathsf{L}\mathsf{F}$, where Q is an orthogonal matrix that expresses the relative rotation between observers.

A steady two-dimensional shear flow in Cartesian coordinates is given by $\boldsymbol{V} = (\epsilon Y, 0)$, where ϵ is the constant shear rate and X and Y are the Cartesian coordinates.

(ii) Find the Cauchy stress components T_{XX}, T_{YY}, T_{XY} of an Oldroyd B fluid subject to this flow and find a condition on the fluid pressure which ensures that Cauchy's equation is satisfied.

[9 marks]

(iii) Calculate the normal stress difference $T_{XX} - T_{YY}$ for the flow and hence determine whether the fluid is Newtonian.

[2 marks]

FORMULA SHEET

- For a general (Lagrangian) coordinate system ξ^i:

$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{r}}{\partial \xi^i}, \quad \boldsymbol{g}_i \cdot \boldsymbol{g}^j = \delta_i^j, \quad g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j, \quad g = \det(g_{ij}).$$

$$\boldsymbol{G}_i = \frac{\partial \boldsymbol{R}}{\partial \xi^i}, \quad \boldsymbol{G}_i \cdot \boldsymbol{G}^j = \delta_i^j, \quad G_{ij} = \boldsymbol{G}_i \cdot \boldsymbol{G}_j, \quad G = \det(G_{ij}).$$

- For a scalar field $f(\boldsymbol{x})$ and vector field $\boldsymbol{u}(\boldsymbol{x})$

$$\boldsymbol{\nabla} f = \boldsymbol{g}^i \frac{\partial f}{\partial \xi^i}, \quad \operatorname{div} \boldsymbol{u} = \frac{1}{\sqrt{g}} \frac{\partial \left(u^i \sqrt{g}\right)}{\partial \xi^i}, \quad \operatorname{curl} \boldsymbol{u} = \epsilon^{ijk} u_j|_i \boldsymbol{g}_k.$$

- The material derivative in general coordinates is

$$\frac{DU^i}{Dt} = \frac{\partial U^i}{\partial t} + V^j U^i||_j,$$

where $\boldsymbol{V}$ is the velocity of the continuum and

$$U^i||_j = U^{i,j} + \Gamma^i_{jk} U^k,$$

where Γ^i_{jk} are the Christoffel symbols for the chosen coordinate system in the deformed configuration.

- The deformation gradient tensor $\mathsf{F} = \boldsymbol{\nabla}_{\boldsymbol{r}} \boldsymbol{R}$ has components Cartesian coordinates given by

$$F_{IJ} = \frac{\partial X_I}{\partial x_j}.$$

The determinant of F is denoted by J.

- The Eulerian velocity gradient tensor, L, has components in Cartesian coordinates given by

$$L_{IJ} = \frac{\partial V_I}{\partial X_J}.$$

- The deformation rate tensor, D and spin tensor, W are defined by

$$\mathsf{D} = \frac{1}{2}\left(\mathsf{L} + \mathsf{L}^T\right), \quad \mathsf{W} = \frac{1}{2}\left(\mathsf{L} - \mathsf{L}^T\right).$$

- Cauchy's equation in the usual notation in components in general coordinates ξ^i is

$$T^{ji}||_j + \rho F^i = \rho \ddot{U}^i = \rho \frac{DV^i}{Dt}, \quad \text{where} \quad T^{ji}||_j = T^{ji}_{,j} + \Gamma^j_{jr} T^{ri} + \Gamma^i_{jr} T^{jr}.$$

- The material derivative of the determinant of the deformation gradient tensor is

$$\frac{DJ}{Dt} = J \boldsymbol{\nabla}_{\boldsymbol{R}} \cdot \boldsymbol{V}.$$

- The Reynolds Transport theorem states that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\phi\,\mathrm{d}\mathcal{V}_t = \int_{\Omega_t}\left(\frac{D\phi}{Dt} + \phi\boldsymbol{\nabla_R}\boldsymbol{\cdot}\boldsymbol{V}\right)\,\mathrm{d}\mathcal{V}_t,$$

where Ω_t is a material volume, ϕ is a scalar field and $\boldsymbol{V}$ is the velocity of the continuum.

- For a Cartesian line element $\mathrm{d}X_I$ in the deformed configuration

$$\frac{D\mathrm{d}X_I}{Dt} = V_{I,K}\mathrm{d}X_K,$$

where V_I is the I-th Cartesian component of the velocity.

- Nanson's relation states that

$$\mathrm{d}A_{\bar{i}} = J\frac{\partial \xi^j}{\partial \chi^{\bar{i}}}\mathrm{d}a_j,$$

where ξ^j are the Lagrangian coordinates, $\chi^{\bar{i}}$ are the Eulerian coordinates, J is the determinant of the deformation gradient tensor, $\mathrm{d}\boldsymbol{A}$ is an area element in the deformed configuration and $\mathrm{d}\boldsymbol{a}$ is an area element in the undeformed configuration.

- The Green–Lagrange strain tensor is defined by

$$\gamma_{ij} = \frac{1}{2}\left(G_{ij} - g_{ij}\right).$$

- The strain invariants are defined by

$$I_1 = g^{ij}G_{ji}, \quad I_2 = \frac{1}{2}\left(I_1^2 - g^{ir}g^{js}G_{ij}G_{rs}\right), \quad I_3 = G/g,$$

where $g = det(g_{ij})$ and $G = det(G_{ij})$

- A hyperelastic material is described by a strain energy function $\mathcal{W}(I_1, I_2, I_3)$ such that

$$T^{ij} = PG^{ij} + Ag^{ij} + BB^{ij},$$

where

$$A = \frac{2}{\sqrt{I_3}}\frac{\partial \mathcal{W}}{\partial I_1}, \quad B = \frac{2}{\sqrt{I_3}}\frac{\partial \mathcal{W}}{\partial I_2}, \quad P = 2\sqrt{I_3}\frac{\partial \mathcal{W}}{\partial I_3},$$
$$\text{and} \quad B^{ij} = \left[I_1 g^{ij} - g^{ir}g^{js}G_{rs}\right].$$

- The physical components of the stress tensor are given by $\sigma_{(ij)} = T^{ij}\sqrt{G_{jj}/G^{ii}}$ (no summation).

- The body stress tensor T^{ij} and second Piola–Kirchhoff stress tensor s^{ij} are related by the expression $JT^{ij} = s^{ij}$.

- The first law of thermodynamics can be written as

$$\rho\frac{D\Phi}{Dt} = \mathsf{T} : \mathsf{D} + \rho B - \boldsymbol{\nabla_R}\boldsymbol{\cdot}\boldsymbol{Q} + \mathcal{W}_e,$$

where $\mathcal{W}_e$ is any additional non-thermomechanical rates of work.

- The second law of thermodynamics for continuum mechanics can be written as

$$\rho\dot{\eta} \geq -\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\left(\frac{\boldsymbol{Q}}{\Theta}\right) + \rho\frac{B}{\Theta}.$$

- The Clausius–Duhem inequality is

$$-\rho\dot{\Psi} - \rho\eta\dot{\Theta} - \frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta + \mathsf{T} : \mathsf{D} \geq 0,$$

  where $\Psi = \Phi - \eta\Theta$; or (in the Lagrangian viewpoint)

$$-\rho_0\dot{\psi} - \rho_0\eta_0\dot{\theta} - \frac{1}{\theta}\boldsymbol{q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{r}}\theta + s^{ij} : \dot{\gamma}_{ij} \geq 0,$$

  where $\psi = \Psi$.

- The most general transformation of position and time between observers in Euclidean space is

$$\boldsymbol{R}^*(t^*) = \mathsf{Q}(t)\boldsymbol{R}(t) + \boldsymbol{C}(t), \quad t^* = t - a,$$

  where Q is an orthogonal matrix, $\boldsymbol{C}$ is a translation vector and a is a constant time shift.

END OF EXAMINATION PAPER