Three hours

A formula sheet is provided at the end of the examination

THE UNIVERSITY OF MANCHESTER

CONTINUUM MECHANICS

21 January 2020

09:45 – 12:45

Answer ALL SIX questions.

University approved calculators may be used.
1. Show that the derivative
\[ \dot{A} = \frac{\partial A}{\partial t} - WA \]
of a vector \( A \) is objective, where \( W \) is the spin tensor. You may use the fact that \( \dot{F} = LF \) without proof, where \( F \) is the deformation gradient tensor and \( L \) is the Eulerian velocity gradient tensor. Both \( L \) and \( F \) are defined on the formula sheet at the end of the examination.  

[5 marks]

2. A mixture of two chemicals is transported within a continuum. The concentrations (masses per unit deformed volume) of the two chemicals \( a \) and \( b \) are given by \( A(R, t) \) and \( B(R, t) \), respectively. The chemicals react in such a way that a mass of \( a \) is produced at a rate \( R_A \) per unit deformed volume and a mass of \( b \) is produced at a rate \( R_B \) per unit deformed volume.

(i) By starting from conservation of mass equations within a material volume, find two partial differential equations that describe the conservation of mass of each chemical.  

[5 marks]

(ii) Assuming that there are no other sources or sinks of mass and that the total mass of both chemicals is conserved, find a relationship between \( R_A \) and \( R_B \).  

[3 marks]

(iii) Assume further that \( B \) evaporates from the surface of the continuum with constant mass-loss rate \( E \) per unit deformed area and that \( R_B \) is also a constant. Find the relationship between \( R_B \) and \( E \) which ensures that the mass of \( A \) does not change.  

[2 marks]
3. A solid material is reinforced by a family of fibres, whose initial direction is described by the vector field \( a(r) \), where \(|a| = 1\).

(i) Show that fibres in a deformed configuration, described by a position vector \( R \), are given by \( A = F a \), where \( F = \nabla_r R \) is the deformation gradient tensor. [2 marks]

(ii) The strain energy of the material is given by \( W(c, a \otimes a) \) and must remain unchanged if the material and the fibres are simultaneously rotated about any axis in the undeformed configuration. Explain why this leads to the restriction

\[
W(c, a \otimes a) = W(QcQ^T, Qa \otimes aQ^T),
\]

for any proper orthogonal tensor \( Q \). Here \( c = F^T F \) is the Cauchy-Green deformation tensor. [3 marks]

(iii) If the strain energy is written as a function of the five invariants

\[
I_1 = \text{trace}(c), \quad I_2 = \frac{1}{2} \{ [\text{trace}(c)]^2 - \text{trace}(c^2) \}, \quad I_3 = \det(c), \quad I_4 = a \cdot ca, \quad I_5 = a \cdot c^2 a,
\]

show that the constraint (1) would be satisfied. [6 marks]
4. An incompressible, hyperelastic solid sphere has undeformed radius 1. The sphere undergoes a deformation such that it remains spherical with a twist (rotation about an axis passing through its centre) that varies with distance from the centre of the sphere. The strain energy function of the solid material is given by

\[ W = (I_1 - 3) + (I_2 - 3), \]

where \( I_1 \) and \( I_2 \) are the first and second strain invariants.

(i) Explain why the radius of the sphere cannot change under the imposed deformation. [2 marks]

(ii) Write down the deformed position in components in a global Cartesian coordinate system as a function of the undeformed position described using spherical polar coordinates \((\xi^1 = r, \xi^2 = \theta, \xi^3 = \phi)\). The global coordinate system is chosen so that the axis of rotation is the \( z \) axis. You may assume that the twist is given by a function of the form \( \Phi = \phi + f(r) \), where \( \Phi \) is the azimuthal angle in the deformed configuration. [2 marks]

(iii) Hence, compute the three strain invariants corresponding to this deformation. [10 marks]

(iv) Compute the component of the stress tensor \( T^{13} \) and find a condition on \( f \) which ensures that \( T^{13} = 0 \). Give a description of the deformation in this case. [4 marks]

You may use the fact that in spherical polar coordinates the position is given by

\[ r = r \sin \theta \cos \phi e_x + r \sin \theta \sin \phi e_y + r \cos \theta e_z, \]

and the only non-zero Christoffel symbols are

\[ \Gamma_{12}^{2} = \Gamma_{21}^{2} = \Gamma_{13}^{3} = \Gamma_{31}^{3} = \frac{1}{r}, \quad \Gamma_{23}^{3} = \Gamma_{32}^{3} = \frac{\cos \theta}{\sin \theta}, \]

\[ \Gamma_{22}^{1} = -r, \quad \Gamma_{33}^{1} = -r \sin^2 \theta, \quad \Gamma_{33}^{2} = -\cos \theta \sin \theta. \]
5. A one-dimensional bar of thermoelastic material, initially of unit density, is stretched from its undeformed length of 1 to a length $\lambda$. The unidirectional Cauchy stress is given by a scalar quantity $P$. The fact that the problem is entirely one-dimensional reduces all tensors to scalar quantities. The Helmholtz free energy takes the form $\Psi(\lambda, \Theta)$.

(i) Explain why the second Piola—Kirchhoff stress tensor is given by $S = \lambda P$ and find an expression for the Green–Lagrange strain, $E$.

(ii) Use the Clausius–Duhem inequality to show that

$$P = \rho \frac{\partial \Psi}{\partial \lambda}, \quad \eta = -\frac{\partial \Psi}{\partial \Theta}. $$

(iii) Hence show that

$$\frac{\partial (P/\rho)}{\partial \Theta} \bigg|_\lambda = \frac{\partial S}{\partial \Theta} \bigg|_\lambda = -\frac{\partial \eta}{\partial \lambda} \bigg|_\Theta. $$

(iv) Show that for an isentropic material (total entropy is fixed)

$$dS = \left[ \frac{\partial S}{\partial E} + \frac{1}{\lambda} \left( \frac{\partial S}{\partial \Theta} \right)^2 \right] \, dE, $$

and interpret the result in terms of the elastic and thermal components.

Recall that the differential (infinitesimal change) of a function of two variables $A(x, y)$ is such that

$$dA = \frac{\partial A}{\partial x} \, dx + \frac{\partial A}{\partial y} \, dy.$$
6. An incompressible, generalised Oldroyd B fluid has the constitutive relationship

\[ T = -Pl + 2\mu D + GA, \]

where the tensor \( A \) satisfies

\[ A^\nabla = -\frac{1}{\tau} (A - I); \]

here, \( P \) is the fluid pressure and \( \mu, G \) and \( \tau \) are constants. The upper-convected derivative is defined by

\[ A^\nabla = \frac{DA}{Dt} - LA - AL^T, \]

where \( L \) is the Eulerian velocity gradient tensor and \( D \) is the symmetric part of \( L \).

(i) Confirm that the constitutive relationships are objective.

You may assume that the Cauchy stress \( T \) is objective; that the deformation gradient tensor, \( F \), transforms as \( F^* = QF \) and \( \dot{F} = LF \), where \( Q \) is an orthogonal matrix that expresses the relative rotation between observers.

A steady two-dimensional shear flow in Cartesian coordinates is given by \( V = (\epsilon Y, 0) \), where \( \epsilon \) is the constant shear rate and \( X \) and \( Y \) are the Cartesian coordinates.

(ii) Find the Cauchy stress components \( T_{XX}, T_{YY}, T_{XY} \) of an Oldroyd B fluid subject to this flow and find a condition on the fluid pressure which ensures that Cauchy’s equation is satisfied.

(iii) Calculate the normal stress difference \( T_{XX} - T_{YY} \) for the flow and hence determine whether the fluid is Newtonian.
FORMULA SHEET

- For a general (Lagrangian) coordinate system $\xi^i$:

$$
g_i = \frac{\partial r}{\partial \xi^i}, \quad g_i \cdot g^j = \delta^j_i, \quad g_{ij} = g_i \cdot g_j, \quad g = \det(g_{ij}).
$$

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$$

- For a scalar field $f(x)$ and vector field $u(x)$

$$
\nabla f = g^i \frac{\partial f}{\partial \xi^i}, \quad \text{div} \, u = \frac{1}{\sqrt{g}} \frac{\partial (u^i \sqrt{g})}{\partial \xi^i}, \quad \text{curl} \, u = \epsilon^{ijk} u_j |_{i} g_k.
$$

- The material derivative in general coordinates is

$$
\frac{DU^i}{Dt} = \frac{\partial U^i}{\partial t} + V^j U^i ||_j,
$$

where $V$ is the velocity of the continuum and

$$
U^i ||_j = U^i + \Gamma^i_{jk} U^k,
$$

where $\Gamma^i_{jk}$ are the Christoffel symbols for the chosen coordinate system in the deformed configuration.

- The deformation gradient tensor $\mathbf{F} = \nabla_r \mathbf{R}$ has components Cartesian coordinates given by

$$
F_{IJ} = \frac{\partial X_I}{\partial x_J}.
$$

The determinant of $\mathbf{F}$ is denoted by $J$.

- The Eulerian velocity gradient tensor, $\mathbf{L}$, has components in Cartesian coordinates given by

$$
L_{IJ} = \frac{\partial V_I}{\partial X_J}.
$$

- The deformation rate tensor, $\mathbf{D}$ and spin tensor, $\mathbf{W}$ are defined by

$$
\mathbf{D} = \frac{1}{2} (\mathbf{L} + \mathbf{L}^T), \quad \mathbf{W} = \frac{1}{2} (\mathbf{L} - \mathbf{L}^T).
$$

- Cauchy’s equation in the usual notation in components in general coordinates $\xi^i$ is

$$
T^{ji} ||_j + \rho F^i = \rho \ddot{U}^i = \rho \frac{DV^i}{Dt}, \quad \text{where} \quad T^{ji} ||_j = T^{ji} + \Gamma^j_{jr} T^{ri} + \Gamma^i_{jr} T^{jr}.
$$

- The material derivative of the determinant of the deformation gradient tensor is

$$
\frac{DJ}{Dt} = J \nabla_r \cdot \mathbf{V}.
$$
• The Reynolds Transport theorem states that
\[ \frac{d}{dt} \int_{\Omega_t} \phi \, dV_t = \int_{\Omega_t} \left( \frac{D\phi}{Dt} + \phi \nabla_R \cdot \mathbf{V} \right) \, dV_t, \]
where \( \Omega_t \) is a material volume, \( \phi \) is a scalar field and \( \mathbf{V} \) is the velocity of the continuum.

• For a Cartesian line element \( dX_I \) in the deformed configuration
\[ \frac{DdX_I}{Dt} = V_{I,K} dX_K, \]
where \( V_I \) is the \( I \)-th Cartesian component of the velocity.

• Nanson’s relation states that
\[ dA_I = J \frac{\partial \xi^j}{\partial \chi^i} da_j, \]
where \( \xi^j \) are the Lagrangian coordinates, \( \chi^i \) are the Eulerian coordinates, \( J \) is the determinant of the deformation gradient tensor, \( dA \) is an area element in the deformed configuration and \( da \) is an area element in the undeformed configuration.

• The Green–Lagrange strain tensor is defined by
\[ \gamma_{ij} = \frac{1}{2} (G_{ij} - g_{ij}), \]

• The strain invariants are defined by
\[ I_1 = g^{ij}G_{ji}, \quad I_2 = \frac{1}{2} \left( I_1^2 - g^{ir}g^{js}G_{ij}G_{rs} \right), \quad I_3 = G/g, \]
where \( g = \det(g_{ij}) \) and \( G = \det(G_{ij}) \)

• A hyperelastic material is described by a strain energy function \( W(I_1, I_2, I_3) \) such that
\[ T^{ij} = PG^{ij} + Ag^{ij} + BB^{ij}, \]
where
\[ A = \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_1}, \quad B = \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_2}, \quad P = 2\sqrt{I_3} \frac{\partial W}{\partial I_3}, \]
and \( B^{ij} = \left[ I_1 g^{ij} - g^{ir}g^{js}G_{rs} \right] \).

• The physical components of the stress tensor are given by \( \sigma_{(ij)} = T^{ij} \sqrt{G_{jj}}/G^{ii} \) (no summation).

• The body stress tensor \( T^{ij} \) and second Piola–Kirchhoff stress tensor \( s^{ij} \) are related by the expression \( JT^{ij} = s^{ij} \).

• The first law of thermodynamics can be written as
\[ \rho \frac{D\Phi}{Dt} = \mathbf{T} : \mathbf{D} + \rho B - \nabla_R \cdot \mathbf{Q} + \mathcal{W}_e, \]
where \( \mathcal{W}_e \) is any additional non-thermomechanical rates of work.
• The second law of thermodynamics for continuum mechanics can be written as

\[ \rho \dot{\eta} \geq -\nabla_r \cdot \left( \frac{Q}{\Theta} \right) + \frac{B}{\Theta}. \]

• The Clausius–Duhem inequality is

\[ -\rho \dot{\Psi} - \rho \eta \dot{\Theta} - \frac{1}{\Theta} Q \cdot \nabla_r \Theta + T : D \geq 0, \]

where \( \Psi = \Phi - \eta \Theta \); or (in the Lagrangian viewpoint)

\[ -\rho \dot{\psi} - \rho \eta \dot{\theta} - \frac{1}{\theta} q \cdot \nabla_r \theta + s^{ij} : \dot{\gamma}^{ij} \geq 0, \]

where \( \psi = \Psi \).

• The most general transformation of position and time between observers in Euclidean space is

\[ R^* (t^*) = Q(t) R(t) + C(t), \quad t^* = t - a, \]

where \( Q \) is an orthogonal matrix, \( C \) is a translation vector and \( a \) is a constant time shift.