In a change to previous years there was no choice in the exam and so all students attempted all questions. Overall the students answered the questions well and presented clear and logical answers, explaining their steps.

1 This question tested understanding of the constitutive modelling and the general difference between fluid and solid behaviour. There were two possible answers to part (i) depending on how one chose to interpret the $\frac{D}{Dt}(\mathsf{F} : \mathsf{F})$ term. One argument is that this is the rate of change of a deformation measure so the behaviour should be fluid-like. Alternatively, some students used the product rule to write the derivative as $\dot{\mathsf{F}}{:}\mathsf{F} + \mathsf{F}{:}\dot{\mathsf{F}}$ and then argued that the material had aspects of fluid and solid behaviour because it depended on F and $\dot{\mathsf{F}}$. Either answer, if properly justified, was given full marks.

For (ii), the simplest answer is that it doesn't make mathematical sense to add a vector (gradient) to a scalar. Alternative answers showing lack of objectivity (which is a consequence of illegally adding a vector to a scalar); or that the energy can't depend on the temperature gradient from thermodynamics were awarded full marks if properly justified, and partial marks if not.

2 This question tested understanding of deriving conservation laws and was answered well. For (i) a number of students did not define the units of S sufficiently carefully: S is the production rate per unit **deformed** volume of the continuum. In part (ii), most people arrived at the correct equation, but the justification wasn't always perfect and the flux $\boldsymbol{J}$ was sometimes not defined. $\boldsymbol{J}$ is the mass-flux into the domain per unit deformed boundary area. The correct starting point is that the additional flux is $\boldsymbol{J} \cdot \boldsymbol{N}$ integrated over the surface, which is then converted in the term shown in the equation via the divergence theorem. The derivative is brought under the integral via the Reynolds Transport Theorem and then we make the usual arguments about the integral being zero for any (suitable smooth) volume and hence the integrand must be zero.

3 This question tested understanding of the definition of a tensor, objectivity and the connections between the two. Part (i) was missed out or poorly answered by a surprisingly large number of students. The question uses essentially the same methods as question 4 on example sheet III and was intended to be straightforward. The rest of the question was generally answered well. For (ii) the answer is that the Cauchy stress **is** objective (precisely because it satisfies the appropriate tensor transformation properties). For (iii) many students said that material derivatives of objective quantities are not objective in general, which is essentially true, but you did need to show that was the case for this particular quantity. For example, the material derivative of a scalar quantity remains objective because the scalar does not change. For full marks, students needed to show that the expansion of $\frac{D}{Dt}\left(\mathsf{QTQ}^T\right)$ from the product rule leads to non-objective terms if Q is not constant. Part (iv) was generally answered correctly and was very similar to question 13 on example sheet IV.

4 This question tested whether students could determine strain invariants and work with the constitutive laws for general hyperelastic materials (given in the formula sheet) in two different coordinate systems. It was answered well.

(i) The most common mistake here was not taking into account that expanding the radius must cause a change in length of the cylinder. In fact, you can work out exactly what the length change must be from the incompressibility constraint.

(ii) The same common mistake was carried over from question (i), but most people wrote down consistent displacements and were, therefore, awarded full marks.

(iii) Everybody use the correct formulæ to find the strain invariants. The most common mistakes were to forget that $R(r)$ and $Z(z)$ were both linear in the respective coordinates, which means that the derivative terms in the strain invariants are not zero. Most people correctly deduced that I_1, I_2 and I_3 have to be the same in both coordinate systems if they are invariant.

(iv) There was some confusion in this part about whether P was an independent variable or not. It **is independent** because the material is incompressible. It is easiest to work in Cartesian (and most people did). Some people didn't realise that A and B should have been calculated from the given strain energy function: they were both 2. Cauchy's equations reduce to

$$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0,$$

which means that the solid pressure must be a constant.

5 This question tested thermodynamic modelling and use of the Clausius–Duhem inequality in an unseen context. It was answered reasonably well, despite the fact that there were two typographical errors, for which I apologise.

(i) Almost everybody got the correct interpretation that when $\zeta = 0$ there is no damage and when $\zeta = 1$ the material is fully damaged: cannot build-up strain energy.

(ii) This part followed from the standard methods applied to the Clausius–Duhem inequality in the Lagrangian formulation. This was the site of the first typographical error, a missing ρ_0. The correct answer should be

$$s^{ij} = (1-\zeta)\rho_0 \frac{\partial \Psi_0}{\partial \gamma_{ij}};$$

Most people wrote either the correct answer, or assumed that $\rho_0 = 1$, all of which were given full marks. The function to be found is $f = \rho_0 \Psi_0$.

(iii) This part was answered reasonably well and I was generous with the marks, given the typo in the question above. The full argument is that $\dot{f} = \rho_0 \dot{\Psi}_0$, because ρ_0 is constant. This can be re-written in the form

$$\dot{f} = \rho_0 \frac{\partial \Psi_0}{\partial \gamma_{ij}} \dot{\gamma}_{ij} \equiv \rho_0\, \hat{s}_0^{ij}\, \dot{\gamma}_{ij},$$

where $\hat{s}_0$ is the second Piola–Kirchhoff stress tensor associated with the undamaged material. Thus, $\dot{f}$ is an effective stress power per unit mass.

(iv) This followed from differentiating the the expression from part (ii) via the product rule:

$$\dot{s}_{ij} = (1-\zeta)\rho_0 \frac{\partial \dot{\Psi}_0}{\partial \gamma_{ij}} - \dot{\zeta}\rho_0 \frac{\partial \Psi_0}{\partial \gamma_{ij}},$$

which is equal to the given result if we define $s_0^{ij} = \rho_0 \frac{\partial \Psi_0}{\partial \gamma_{ij}}$. Given the missing factor of ρ_0 in (ii), if it was missed out here full marks were still given.

(v) This question was not answered so well, perhaps because it was one of the hardest parts of the exam. Also, it was getting towards the end of the question and there was a more significant typo: the γ should have been a ζ. Again, I'm very sorry. I compensated, by awarding full marks to those that had made any sensible attempts at this question. The idea is that we need to use the given information to solve the equation determined in part (iv). Thus, we need to find s_0^{ij} by differentiating Ψ_0. Using the definition of I_1 and γ_{ij}, we find that

$$\Psi_0 = 2g^{ij}\gamma_{ji} \quad \Rightarrow \quad s_0^{ij} = \rho_0 \frac{\partial \Psi_0}{\partial \gamma_{ij}} = 2\rho_0 g^{ij},$$

after using the symmetry properties of γ_{ij} and hence $\dot{s}_0^{ij} = 0$, because all the terms of the undeformed metric tensor are fixed in time. Thus we have

$$\dot{s}^{ij} = -2\rho_0 \dot{\zeta} g^{ij},$$

and integrating gives

$$s^{ij} = K^{ij} - 2\rho_0 \zeta g^{ij},$$

where K^{ij} is a constant tensor. Now we need to know $\zeta = 1 - \mathrm{e}^{-t}$ and we can use the initial conditions to find that

$$s^{ij} = -2\rho_0(1 - \mathrm{e}^{-t})g^{ij}.$$

6 This question tested working with non-Newtonian fluids and calculations with tensors in non-Cartesian coordinate systems. It was surprisingly poorly answered given that similar questions have been asked in previous exams, but the algebra does get involved. The main problem was incorrect calculations of the velocity components and then the entries of D in spherical polar coordinates.

(i) Most people wrote down the correct no-slip condition on the plate $\boldsymbol{V} = \mathbf{0}$, but the condition on the cone was more challenging $\boldsymbol{V} = R\Omega\boldsymbol{e}_\phi$, where R is the distance from the axis of rotation. Hence in spherical polar coordinates $\boldsymbol{V} = r\sin\theta\Omega\boldsymbol{e}_\phi$. The argument is then that we expect the velocity to vary with θ, but nothing else (in the simplest case), so we can pose the given form with $F(\theta) = 0$ on the plate and $F(\theta) = \Omega$ on the cone.

(ii) Here the answer is simply that $V^3 = F(\theta)$, because $\boldsymbol{g}_3 = r\sin\theta\boldsymbol{e}_\phi$. A surprisingly small number of people got this correct, suggesting that the distinction between components and coordinates in non-Cartesian coordinate systems is not fully understood by the majority of students.

(iii) In order to work out the stress we need to know L and D in spherical coordinates. In general coordinates

$$L^i_j = V^i||_j = V^i_{,j} + \Gamma^i_{jk}V^k,$$

and then

$$D^{ij} = \frac{1}{2}\left(g^{jk}V^i||_k + g^{ik}V^j||_k\right),$$

and the only non-zero component is $D^{23} = D^{32} = \frac{1}{2}F'/r^2$, and one can then calculate the entries in the stress tensor.

$$T^{11} = -P, \quad T^{12} = T^{21} = -\lambda_2\sin^2\theta FF'/2r, \quad T^{13} = T^{31} = 0,$$

$$T^{22} = -P/r^2 + \lambda_2 FF'\cos\theta\sin\theta/r^2, \quad T^{23} = T^{32} = \eta_0 F'/2r^2,$$

$$T^{33} = -\lambda_2 F'/r^2\left(F' + \frac{\cos\theta}{\sin\theta}F\right) - P/(r^2\sin^2\theta).$$

(iv) If F is a constant then $F' = 0$ and all terms apart from the (constant) pressure in the stress tensor are zero. If F is constant, the velocity on both cone and plate must be the same and this is only possible if $\Omega = 0$: the cone is not rotating.