

Three hours

A formula sheet is provided at the end of the examination

**THE UNIVERSITY OF MANCHESTER**

**CONTINUUM MECHANICS**

**16 January 2019**

**14:00 – 17:00**

Answer **ALL SIX** questions.

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University approved calculators may be used.

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1. You have been given the following constitutive law for the Helmholtz free energy,  $\Phi$ , of an unknown continuum material:

$$\Phi = \Theta + \frac{D}{Dt} (\mathbf{F} : \mathbf{F}),$$

where  $\mathbf{F} = \nabla_{\mathbf{r}} \mathbf{R}$  is the deformation gradient tensor.

- (i) Is this law more suitable for a solid-like material, a fluid-like material or a material with aspects of fluid and solid behaviour? Explain your reasons.
- (ii) Thermal effects are included by adding the term  $\nabla_{\mathbf{R}} \Theta$  to the free energy, where  $\Theta$  is the temperature. Explain whether this would lead to a valid constitutive law, giving reasons for your answer.

[4 marks]

2. A chemical transported within a continuum has concentration (masses per unit deformed volume) given by  $C(\mathbf{R}, t)$  and a net production rate of  $S(C, \mathbf{R}, t)$

- (i) Explain why the equation for conservation of mass can be written in the form

$$\frac{D}{Dt} \int_{\Omega_t} C \, d\mathcal{V}_t = \int_{\Omega_t} S \, d\mathcal{V}_t,$$

and hence define the units of  $S$ .

- (ii) If there is an additional flux of mass across the boundaries of the domain show that the conservation of mass can be written in the form

$$\frac{\partial C}{\partial t} + \nabla_{\mathbf{R}} \cdot (C\mathbf{V} + \mathbf{J}) = S,$$

where  $\mathbf{V}$  is the velocity of the continuum and  $\mathbf{J}$  is the mass flux vector, which should be defined precisely.

[10 marks]

3. In Cartesian coordinates,  $X_I$ , the force per unit deformed area on a surface with outer unit normal  $\mathbf{N}$  is given by a stress vector with components  $T_I = T_{JI}N_J$ , where  $T_{JI}$  are the Cartesian components of the Cauchy stress tensor.

- (i) By considering a transformation to a new set of Cartesian coordinate, or otherwise, show that the Cauchy stress tensor does satisfy tensor transformation properties.
- (ii) Is the Cauchy stress tensor objective?
- (iii) Explain why the material derivative of the Cauchy stress tensor is **not** objective.
- (iv) Show that the Truesdell stress rate

$$\frac{DT}{Dt} + \text{trace}(\mathbf{D})\mathbf{T} - \mathbf{L}\mathbf{T} - \mathbf{T}\mathbf{L}^T,$$

is objective. Here,  $\mathbf{L} = \nabla_{\mathbf{R}}V$  is the velocity gradient tensor and  $\mathbf{D}$  is the symmetric part of  $\mathbf{L}$ . You may use the fact that  $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ , where  $\mathbf{F}$  is the deformation gradient tensor.

[12 marks]

4. An incompressible, hyperelastic solid cylinder has undeformed radius 1 and undeformed length 12. The cylinder undergoes a deformation such that it remains cylindrical with no twist and its radius is doubled uniformly. The strain energy function of the solid material is given by

$$\mathcal{W} = (I_1 - 3) + (I_2 - 3).$$

- (i) Write down the deformed position in components in a global Cartesian coordinate system as a function of the cylindrical polar coordinates  $(r, \theta, z)$ , aligned with the undeformed cylinder. State any additional assumptions that you have made.
- (ii) Write down the deformed position in components in a global Cartesian coordinate system as a function of the undeformed Cartesian coordinates  $(x, y, z)$ .
- (iii) Hence, compute the three strain invariants corresponding to this deformation using both polar and Cartesian undeformed coordinates and confirm that they are indeed invariant.
- (iv) Compute the stress tensor  $T^{ij}$  in the deformed solid in either polar or Cartesian coordinates and find the conditions on the solid pressure,  $P$ , such that Cauchy's equations are satisfied in the absence of body forces and accelerations.

**Hint:** In cylindrical polar coordinates  $(\xi^1, \xi^2, \xi^3) = (r, \theta, z)$ , where the Cartesian coordinates are given by  $x = r \cos \theta$ ,  $y = r \sin \theta$ , and the only non-zero Christoffel symbols are given by

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r} \quad \text{and} \quad \Gamma_{22}^1 = -r.$$

[18 marks]

5. Damage to the internal structure of rubber-like materials can be modelled using a Helmholtz free energy function,  $\Psi$ , of the form

$$\Psi(\gamma_{ij}, \zeta) = (1 - \zeta)\Psi_0(\gamma_{ij}),$$

where  $\Psi_0$  is the strain energy function of an undamaged material,  $\gamma_{ij}$  is the Green–Lagrange strain tensor, and  $\zeta \in [0, 1]$  is a scalar known as the damage variable.

(i) Interpret the above equation in the limits  $\zeta = 0$  and  $\zeta = 1$ ?

(ii) By using the Clausius–Duhem inequality and assuming isothermal conditions show that

$$s^{ij} = (1 - \zeta)\rho_0 \frac{\partial \Psi_0}{\partial \gamma_{ij}}$$

and that

$$\mathcal{D}_{\text{int}} \equiv f\dot{\zeta} \geq 0,$$

where  $f$  is to be found.

(iii) Show that  $f$  represents an effective stress power per unit mass.

(iv) Hence, show the evolution equation for the stress tensor is

$$\dot{s}^{ij} = (1 - \zeta)\dot{s}_0^{ij} - \dot{\zeta}s_0^{ij},$$

where  $s_0^{ij}$  is to be defined.

(v) If  $\Psi_0 = I_1 - 3$  and  $\zeta = 1 - e^{-t}$  for  $t \geq 0$ , find an explicit expression for  $s^{ij}$  as a function of  $t$ , assuming that  $s^{ij} = 0$  at  $t = 0$ .

[18 marks]

6. An incompressible, generalised Oldroyd B fluid has the constitutive relationship

$$\mathbf{T} = -P\mathbf{I} + \tilde{\mathbf{T}},$$

where  $P$  is the fluid pressure and the extra-stress,  $\tilde{\mathbf{T}}$ , satisfies the relationship

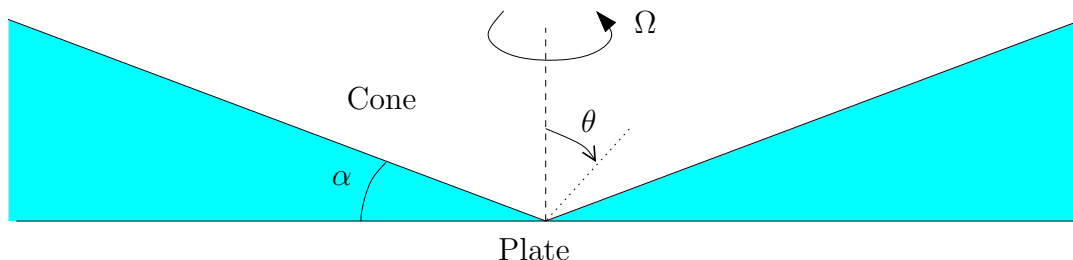
$$\tilde{\mathbf{T}} + \lambda_1 \tilde{\mathbf{T}}^\nabla = \eta_0 \mathbf{D} + \lambda_2 \mathbf{D}^\nabla,$$

where  $\eta_0$ ,  $\lambda_1$  and  $\lambda_2$  are constants. The upper-convected derivative is defined by

$$\mathbf{A}^\nabla = \frac{D\mathbf{A}}{Dt} - \mathbf{L}\mathbf{A} - \mathbf{A}\mathbf{L}^T,$$

where  $\mathbf{L} = \nabla_{\mathbf{R}} \otimes \mathbf{V}$  is the Eulerian velocity gradient tensor and  $\mathbf{D}$  is the symmetric part of  $\mathbf{L}$ .

The fluid is driven within a cone-and-plate device in which the cone rotates at a constant angular velocity  $\Omega$  and the plate is fixed. The fluid occupies the region  $\pi/2 - \alpha \leq \theta \leq \pi/2$ , in a spherical polar coordinate system  $(\xi^1, \xi^2, \xi^3) = (r, \theta, \phi)$ , with origin at the point where the cone meets the plate.



- (i) Write down no-slip boundary conditions on the cone and the plate. Why is  $\mathbf{V} = r \sin \theta F(\theta) \mathbf{e}_\phi$  a plausible form for the velocity field? Here  $\mathbf{e}_\phi$  is a unit vector in the direction of rotation and  $F(\theta)$  is an arbitrary function of  $\theta$ .
- (ii) Let the velocity  $\mathbf{V} = V^i \mathbf{g}_i$ , where  $\mathbf{g}_i$  are the covariant base vectors of the spherical polar coordinate system. Assuming that  $\mathbf{V}$  has the form given in (i) find  $V^i$  in terms of  $F(\theta)$ .
- (iii) In the case when  $\lambda_1 = 0$  find expressions for the non-zero components of stress tensor  $T^{ij}$  in terms of  $F$  and its derivatives, as well as  $\theta$ ,  $r$ ,  $P$  and the constants arising in the constitutive law.
- (iv) Find the stress in the case when  $F$  is a constant and explain the physical circumstances under which this can occur.

**You may use** the fact that in spherical polar coordinates the position is given by

$$\mathbf{r} = r \sin \theta \cos \phi \mathbf{e}_x + r \sin \theta \sin \phi \mathbf{e}_y + r \cos \theta \mathbf{e}_z,$$

and the only non-zero Christoffel symbols are

$$\begin{aligned} \Gamma_{12}^2 = \Gamma_{21}^2 = \Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{r}, \quad \Gamma_{23}^3 = \Gamma_{32}^3 = \frac{\cos \theta}{\sin \theta}, \\ \Gamma_{22}^1 = -r, \quad \Gamma_{33}^1 = -r \sin^2 \theta, \quad \Gamma_{33}^2 = -\cos \theta \sin \theta. \end{aligned}$$

[18 marks]

## FORMULA SHEET

- For a general (Lagrangian) coordinate system  $\xi^i$ :

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial \xi^i}, \quad \mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j, \quad g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j, \quad g = \det(g_{ij}).$$

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- For a scalar field  $f(\mathbf{x})$  and vector field  $\mathbf{u}(\mathbf{x})$

$$\nabla f = \mathbf{g}^i \frac{\partial f}{\partial \xi^i}, \quad \operatorname{div} \mathbf{u} = \frac{1}{\sqrt{g}} \frac{\partial (u^i \sqrt{g})}{\partial \xi^i}, \quad \operatorname{curl} \mathbf{u} = \epsilon^{ijk} u_j |_{,i} \mathbf{g}_k.$$

- The material derivative in general coordinates is

$$\frac{DU^i}{Dt} = \frac{\partial U^i}{\partial t} + V^j U^i |_{,j},$$

where  $\mathbf{V}$  is the velocity of the continuum and

$$U^i |_{,j} = U^{i,j} + \Gamma_{jk}^i U^k,$$

where  $\Gamma_{jk}^i$  are the Christoffel symbols for the chosen coordinate system in the deformed configuration.

- Cauchy's equation in the usual notation in components in general coordinates  $\xi^i$  is

$$T^{ji} |_{,j} + \rho F^i = \rho \ddot{U}^i = \rho \frac{DV^i}{Dt}, \quad \text{where} \quad T^{ji} |_{,j} = T_{,j}^{ji} + \Gamma_{jr}^j T^{ri} + \Gamma_{jr}^i T^{jr}.$$

- The material derivative of the determinant of the deformation gradient tensor is

$$\frac{DJ}{Dt} = J \nabla_{\mathbf{R}} \cdot \mathbf{V}.$$

- The Reynolds Transport theorem states that

$$\frac{d}{dt} \int_{\Omega_t} \phi \, d\mathcal{V}_t = \int_{\Omega_t} \left( \frac{D\phi}{Dt} + \phi \nabla_{\mathbf{R}} \cdot \mathbf{V} \right) d\mathcal{V}_t,$$

where  $\Omega_t$  is a material volume,  $\phi$  is a scalar field and  $\mathbf{V}$  is the velocity of the continuum.

- For a Cartesian line element  $dX_I$  in the deformed configuration

$$\frac{DdX_I}{Dt} = V_{I,K} dX_K,$$

where  $V_I$  is the  $I$ -th Cartesian component of the velocity.

- Nanson's relation states that

$$dA_{\bar{i}} = J \frac{\partial \xi^j}{\partial \chi^{\bar{i}}} da_j,$$

where  $\xi^j$  are the Lagrangian coordinates,  $\chi^{\bar{i}}$  are the Eulerian coordinates,  $J$  is the determinant of the deformation gradient tensor,  $d\mathbf{A}$  is an area element in the deformed configuration and  $d\mathbf{a}$  is an area element in the undeformed configuration.

- The Green–Lagrange strain tensor is defined by

$$\gamma_{ij} = \frac{1}{2} (G_{ij} - g_{ij}).$$

- The strain invariants are defined by

$$I_1 = g^{ij} G_{ji}, \quad I_2 = \frac{1}{2} (I_1^2 - g^{ir} g^{js} G_{ij} G_{rs}), \quad I_3 = G/g,$$

where  $g = \det(g_{ij})$  and  $G = \det(G_{ij})$

- A hyperelastic material is described by a strain energy function  $\mathcal{W}(I_1, I_2, I_3)$  such that

$$T^{ij} = P G^{ij} + A g^{ij} + B B^{ij},$$

where

$$A = \frac{2}{\sqrt{I_3}} \frac{\partial \mathcal{W}}{\partial I_1}, \quad B = \frac{2}{\sqrt{I_3}} \frac{\partial \mathcal{W}}{\partial I_2}, \quad P = 2\sqrt{I_3} \frac{\partial \mathcal{W}}{\partial I_3},$$

and  $B^{ij} = [I_1 g^{ij} - g^{ir} g^{js} G_{rs}]$ .

- The physical components of the stress tensor are given by  $\sigma_{(ij)} = T^{ij} \sqrt{G_{jj}/G^{ii}}$  (no summation).
- The body stress tensor  $T^{ij}$  and second Piola–Kirchhoff stress tensor  $s^{ij}$  are related by the expression  $JT^{ij} = s^{ij}$ .
- The first law of thermodynamics can be written as

$$\rho \frac{D\Phi}{Dt} = \mathbb{T} : \mathbb{D} + \rho B - \nabla_{\mathbf{R}} \cdot \mathbf{Q} + \mathcal{W}_e,$$

where  $\mathcal{W}_e$  is any additional non-thermomechanical rates of work.

- The second law of thermodynamics for continuum mechanics can be written as

$$\rho \dot{\eta} \geq -\nabla_{\mathbf{R}} \cdot \left( \frac{\mathbf{Q}}{\Theta} \right) + \rho \frac{B}{\Theta}.$$

- The Clausius–Duhem inequality is

$$-\rho \dot{\Psi} - \rho \eta \dot{\Theta} - \frac{1}{\Theta} \mathbf{Q} \cdot \nabla_{\mathbf{R}} \Theta + \mathbb{T} : \mathbb{D} \geq 0,$$

where  $\Psi = \Phi - \eta\Theta$ ; or (in the Lagrangian viewpoint)

$$-\rho_0 \dot{\psi} - \rho_0 \eta_0 \dot{\theta} - \frac{1}{\theta} \mathbf{q} \cdot \nabla_{\mathbf{r}} \theta + s^{ij} : \dot{\gamma}_{ij} \geq 0,$$

where  $\psi = \Psi$ .

- The most general transformation of position and time between observers in Euclidean space is

$$\mathbf{R}^*(t^*) = \mathbf{Q}(t)\mathbf{R}(t) + \mathbf{C}(t), \quad t^* = t - a,$$

where  $\mathbf{Q}$  is an orthogonal matrix,  $\mathbf{C}$  is a translation vector and  $a$  is a constant time shift.

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**END OF EXAMINATION PAPER**