Every student attempted all the questions in section A. In section B, the most popular choice was questions B4, B6 and B7. Question B5 was largely avoided, but those few that did attempt it scored well. For the section A questions, A2 was the best answered, followed by A1 and lastly A3.

Students were much better at making use of the formula sheet this year and the answers were generally of good quality containing clear logical sequences of steps. There were still a few students who did not always explain what they were doing or why a particular formula was true: the initial statement of a formula or equation should have some accompanying explanation; e.g. conservation of mass requires ...

A1 This question tested understanding of the concept of velocity and the difference between its Eulerian and Lagrangian representations, as well as testing the ability to work with a given deformation field. A surprisingly large number of students made (i) much harder than it needed to be. The Lagrangian viewpoint of the velocity is the natural one: $\boldsymbol{v}(x_1, x_2, x_3) = \partial \boldsymbol{X}/\partial t$. The answer is just $v_1 = v_2 = 2a^2 t x_2$, $v_3 = 0$. Most students got there ... eventually. For (ii) in order to find the velocity in the Eulerian viewpoint you must invert the deformation to find $\boldsymbol{x}(\boldsymbol{X})$ and then substitute this into the expression for the velocity. Everybody attempted this by inverting the appropriate 3×3 matrix that related $\boldsymbol{X}$ and $\boldsymbol{x}$, but actually we need simply to observe that $x_2 = X_2/(1 + a^2t^2)$ and so $V_1 = V_2 = 2a^2 t X_2/(1 + a^2t^2)$. Part (iii) caused more problems than I expected. The idea was simply to use the representation of the plane as (b, x_2, x_3) in the deformation to obtain

$$X_1 = b + a^2t^2 x_2, \quad X_2 = (1 + a^2t^2)x_2, \quad X_3 = x_3.$$

Expressing $\boldsymbol{x}$ in terms of $\boldsymbol{X}$ leads to the two trivial equations $X_2 = X_2$ and $X_3 = X_3$ and one non-trivial equation

$$X_1 = b + \frac{a^2t^2}{1 + a^2t^2} X_2 \quad \Rightarrow \quad X_1 - \frac{a^2t^2}{1 + a^2t^2} X_2 = b,$$

the equation of a plane in the deformed coordinates with normal $N_1 = 1$, $N_2 = -a^2t^2/(1 + a^2t^2)$, $N_3 = 0$.

A2 This question tested understanding of deriving conservation laws in general and part (i) was answered well. Marks were lost for not explicitly stating that the charge was constant so the conservation of charge was the same as conservation of the number of the particles. In part (ii), as for similar questions in previous years, very few students recognised that if $\mathcal{Q}$ is a source of charge per unit volume then it must be specified whether the volume is the deformed or undeformed one. The other common error in part (ii) was to forget to multiply the conservation equation found in (i) by q to convert it back into an equation for charge.

A3 This question tested understanding of objectivity and objective rates. Many students made heavy weather of part (i), which requires only the observation that for the new observer

$$\frac{D\boldsymbol{B}^*}{Dt^*} = \frac{D}{Dt}\left[Q\boldsymbol{b}\right] = \dot{Q}\boldsymbol{B} + Q\dot{\boldsymbol{B}};$$

the $\dot{Q}\boldsymbol{B}$ term is non-objective. For part (ii) students could either reproduce a derivation of the upper-convected derivative showing that it is objective by construction or use direct verification to show that the derivative satisfies the appropriate objective transformation. Many students forgot that all quantities must be transformed so for the new observer

$$\boldsymbol{B}^{*\nabla} = \frac{D\boldsymbol{B}^*}{Dt^*} - L^*\boldsymbol{B}^* = \dot{Q}\boldsymbol{B} + Q\dot{\boldsymbol{B}} - L^*Q\boldsymbol{B}.$$

L is not an objective tensor and, in fact $L^* = \dot{Q}Q^{-1} + QLQ^{-1}$ (see lecture notes) from which the result follows.

B4 This question tested whether students could determine strain invariants and work with the constitutive laws for general hyperelastic materials (given in the formula sheet). It was answered well by the majority that attempted it, apart from part (i).

(i) The most common mistake here was not actually answering the question and not stating the assumptions clearly. The required assumptions are that there is no twist and that cross-sectional planes do not tilt (or as some said that the material planes remain in the same coordinate planes) and that there is a **linear** stretch in the axial direction.

(ii) Everybody use the correct formulæ to find the strain invariants. The most common mistake was to forget that $R(r)$ which means that $\partial R/\partial r \neq 1$ and must be written as an unknown in the strain invariants.

(iii) The incompressibility constraint is that $I_3 = 1$ and starting from the correct equation for $I_3 = (R')^2 R^2 \lambda^2 / r^2$, solving the resulting ODE gives $R = r/\sqrt{\lambda}$. Thus $C = c/\sqrt{\lambda}$ and $D = d/\sqrt{\lambda}$. If people correctly solved the wrong equation (because they had made a mistake in deriving I_3) full marks were awarded.

(iv) This was straightforward algebra and although a few people got lost, most got close to the correct answer. Note that a few people forgot to write explicitly that $T^{ij} = 0$ when $i \neq j$.

B5 This question was very unpopular, with only a few students attempting it. It may have been the objectivity part that put people off. Those that did attempt it did well. Anyway, part (i) is similar to question B7 from the 2014/15 paper. The easiest method is to work out

$$\frac{D}{Dt}|d\boldsymbol{R}|^2 = \frac{D}{Dt}\left(dX_I dX_I\right) = 2dX_I \frac{DdX_I}{Dt} = 2dX_I V_{I,K} dX_K,$$

after using the formula from the formula sheet. Thus,

$$A^{(1)}_{IJ} dX_I dX_J = 2L_{IJ} dX_I dX_J = L_{IJ} dX_I dX_J + L_{JI} dX_J dX_I = (L_{IJ} + L^T_{IJ}) dX_I dX_J,$$

which gives the result.

For (ii) the description of the problem means that if we use cylindrical polar coordinates aligned with the pipe then only V_3 is non-zero and it is a function only of ξ^1. Thus, the only non-zero entries of $\mathsf{A}^{(1)}$ are $A^{(1)}_{13} = A^{(1)}_{31} = V^3_{,1}$ and so the stress tensor has the form

$$T^{ij} = \begin{pmatrix} a_3\left(V^3_{,1}\right)^2 & 0 & a_1 V^3_{,1} + 2a_2\left(V^3_{,1}\right)^3 \\ 0 & 0 & 0 \\ a_1 V^3_{,1} + 2a_2\left(V^3_{,1}\right)^3 & 0 & a_3\left(V^3_{,1}\right)^2 \end{pmatrix}.$$

From the axial component of Cauchy's equation we have that

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rT^{13}\right) = -G \quad \Rightarrow \quad T^{13} = -\frac{1}{2}Gr + \frac{C}{r}.$$

We expect the stress to be finite at the centre of the cylinder so $C = 0$. It follows that

$$a_1 V^3_{,1} + 2a_2\left(V^3_{,1}\right)^3 = -\frac{1}{2}Gr,$$

which is the desired expression. It can be exactly integrated if $a_2 = 0$.

(iii) The radial component of Cauchy's equation gives

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rT^{11}\right) = -\rho F^1.$$

and from (ii) $T^{11} = a_3 V^3_{,1}$ is non-zero and its derivative when multiplied by r is not zero either, which means that there must be a non-zero body force in the radial direction, $F^1 \neq 0$. (The value can be found explicitly if we assume $a_2 = 0$ or $a_1 = 0$.)

B6 This question tested working with the first and second laws of thermodynamics for a previously unstudied system. Although the system was unseen the methods used in chapter 4 of the lecture notes can be directly applied. Most students obtained near full marks for parts (i) and (iii), but part (ii) proved to be more challenging.

For part (i) the method is to use the modified conservation of energy (first law of thermodyanmics) given on the formula sheet with $W_e = -\boldsymbol{M}\boldsymbol{\cdot}\dot{\boldsymbol{B}}$

$$\rho \frac{D\Phi}{Dt} = \mathsf{T} : \mathsf{D} + \rho B - \boldsymbol{\nabla_R}\boldsymbol{\cdot}\boldsymbol{Q} - \boldsymbol{M}\boldsymbol{\cdot}\dot{\boldsymbol{B}},$$

to replace the heat flux and body heating terms in the second law of thermodynamics after expansion of the gradient of heat flux term and multiplication by Θ

$$\rho\Theta\dot{\eta} + \boldsymbol{\nabla_R}\boldsymbol{\cdot}\boldsymbol{Q} - \frac{\boldsymbol{Q}}{\Theta}\boldsymbol{\cdot}\boldsymbol{\nabla_R}\Theta - \rho B \geq 0.$$

After using the fact that $\Phi = \Psi + \eta\Theta$, we have

$$-\rho B + \boldsymbol{\nabla_R}\boldsymbol{\cdot}\boldsymbol{Q} = -\rho\left(\dot{\Psi} + \dot{\eta}\Theta + \eta\dot{\Theta}\right) + \mathsf{T} : \mathsf{D} - \boldsymbol{M}\boldsymbol{\cdot}\dot{\boldsymbol{B}},$$

which gives the desired result when substituted into the second law. The most common mistake was failing to expand the heat flux term correctly in the second law.

In part (ii), many people forgot that incompressibility means that the volume doesn't change which means that $J = 1$ and $\dot{\rho} = 0$. In addition, constant uniform temperature means that all gradients of Θ are zero. Hence, the expression found in part (i) becomes

$$-\rho\dot{\Psi} + \mathsf{T} : \mathsf{D} - \boldsymbol{M}\boldsymbol{\cdot}\dot{\boldsymbol{B}} \geq 0,$$

and the using the facts that

$$\dot{\omega} = \rho\dot{\Psi} + \frac{1}{\mu_0}\dot{\boldsymbol{B}}\boldsymbol{\cdot}\boldsymbol{B} \quad \text{and} \quad \frac{1}{\mu_0}\boldsymbol{B} = \boldsymbol{H} + \boldsymbol{M},$$

means that

$$-\rho\dot{\Psi} = -\dot{\omega} + (\boldsymbol{H} + \boldsymbol{B})\boldsymbol{\cdot}\dot{\boldsymbol{B}},$$

and

$$-\dot{\omega} + \mathsf{T} : \mathsf{D} + \boldsymbol{H}\boldsymbol{\cdot}\dot{\boldsymbol{B}} \geq 0.$$

The symmetry of T means that $\mathsf{T} : \mathsf{D} = \mathsf{T} : \mathsf{L}$.

Many students got this far, but the final part (converting to the Lagrangian representation) is where the most common mistakes were made. Using the given transformation rules for $\boldsymbol{H}$ and $\boldsymbol{B}$ leads to the expression

$$H_I\dot{B}_I = H_I B_J L_{IJ} + h_I\dot{b}_I,$$

which means that the inequality can be written as

$$-\dot{\omega} + (T_{IJ} + H_I B_J)\, L_{IJ} + h_I \cdot b_I \geq 0.$$

Expressing this in general coordinates gives

$$-\frac{D\omega}{Dt} + \left(T^{ij} + H^i B^j\right) V_i||_j + \boldsymbol{h}\boldsymbol{\cdot}\dot{\boldsymbol{b}} \geq 0,$$

which can be expressed in the form given in the question if

$$\hat{s}^{ij} = s^{ij} + \frac{1}{2}\left(H^i B^j + H^j B^i\right),$$

the mechanical stress with additional magnetic terms.

For part (iii) the key argument is that the strain and magnetic effects are independent and so using the chain rule and considering the appropriate thermodynamical processes leads to the desired result.

B7 This question tested working with the second law of thermodynamics to determine constraints on physical parameters in the system. It also tested understanding of the connection between thermodynamic and mechanical pressure. Part (i) is standard bookwork and was covered in the lecture notes in section 7.1.2. A common mistake here was for students to show equivalence of the two pressures for the cases given, but this doesn't actually answer the question. You needed to show that it is **only** possible for the two pressures to be equivalent in the cases given.

Parts (ii) and (iii) are very similar to questions 1 and 2 on Example Sheet 6, so should have been straightfoward. Many students did well on part (ii), but part (iii) was less well answered. Using the fact that $\Psi(\rho, \Theta)$ in Clausius–Duhem inequality together with conservation of mass leads to the expression

$$-\frac{1}{\Theta}\boldsymbol{Q}\cdot\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta + \lambda\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V}\mathsf{I} : \mathsf{D} + 2\mu\mathsf{D} : \mathsf{D} \geq 0.$$

Considering only variations in temperature and the form given for the flux implies that

$$\frac{\kappa}{\Theta}\left(\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta\right)^2 \geq 0,$$

and because Θ and its gradient squared are both positive it follows that $\kappa \geq 0$.

Considering only mechanical variations we have

$$\lambda\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V}\mathsf{I} : \mathsf{D} + 2\mu\mathsf{D} : \mathsf{D} \geq 0,$$

and the problem is that these two terms are not independent. We can use the decomposition from (ii)

$$\mathsf{D} = \widetilde{\mathsf{D}} + \frac{1}{3}\left(\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V}\right)\mathsf{I},$$

to derive the expression

$$\left(\lambda + \frac{2}{3}\mu\right)\left(\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V}\right)^2 + 2\mu\widetilde{\mathsf{D}} : \widetilde{\mathsf{D}} \geq 0,$$

where the two terms are independent. This leads directly to the result.