Three hours

THE UNIVERSITY OF MANCHESTER

CONTINUUM MECHANICS

21 January 2016

14:00 – 17:00

Answer **ALL THREE** questions in section A (21 marks in total).
Answer **THREE** of the **FOUR** questions in section B (54 marks in total). If more than THREE questions from Section B are attempted, then credit will be given for the best THREE answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

A1.

(i) Briefly explain the fundamental differences in constitutive behaviour for fluids and solids.

(ii) Given the following constitutive assumptions for the Cauchy stress, T, state whether the resulting behaviour will be fluid-like, solid-like or a mixture of the two:

$$(a)\mathsf{T}(\rho, \mathsf{D}), \qquad (b)\mathsf{T}(\mathsf{E}, \Theta), \quad (c)\mathsf{T}(\mathsf{F}, \mathsf{D}, \Theta),$$

where ρ is the density; Θ is the temperature; E is the Almansi strain tensor; F is deformation gradient tensor; and D is the rate of deformation tensor.

[5 marks]

A2. A substance is transported within a moving continuum. Its concentration (mass per unit volume) is denoted by $C(\boldsymbol{R}, t)$, where $\boldsymbol{R}$ are the Eulerian coordinates and t is time.

(i) Explain why conservation of mass of the substance within a material region Ω_t is given by the expression

$$\frac{D}{Dt}\int_{\Omega_t} C \mathrm{d}\mathcal{V}_t = 0.$$

(ii) Show that an equivalent equation for conservation of mass is

$$\frac{\partial C}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} (C\boldsymbol{V}) = 0,$$

where $\boldsymbol{V}$ is the velocity of the continuum.

[6 marks]

A3. A continuum is deformed from an initial quarter cylinder to a new configuration. The position vector to the original configuration in a global Cartesian basis is

$$\boldsymbol{r} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix},$$

where $r \in [0, a]$, $\theta \in [0, \pi/2]$, $z \in [0, h]$ are cylindrical polar coordinates.

The deformed configuration is given by

$$\boldsymbol{R} = \begin{pmatrix} R(r,\theta)\cos\theta \\ R(r,\theta)\sin\theta \\ Z(z) \end{pmatrix}, \quad \text{where} \quad R(r,\theta) = \begin{cases} \frac{rL}{a\cos\theta} & \theta \in [0, \pi/4] \\ \frac{rL}{a\sin\theta} & \theta \in [\pi/4, \pi/2] \end{cases}, \quad Z(z) = \frac{Hz}{h}.$$

(i) Describe the deformed configuration.

(ii) Find the undeformed and deformed metric tensors g_{ij} and G_{ij}. Is the deformation is physically admissible?

[10 marks]

SECTION B

B4. A length of hyperelastic material with uniform cross-section is subjected to a uniform tension in the axial direction, but remains traction-free otherwise. The undeformed geometry is defined by a Cartesian coordinate system (x_1, x_2, x_3) where x_1 is aligned with the axial direction and (x_2, x_3) spans the cross-section.

(i) Assuming that the uniform tension leads to uniform extension in the axial direction explain under what circumstances the deformed position is given by $X_1 = \lambda x_1$, $X_2 = \mu x_2$, $X_3 = \mu x_3$. Be explicit about any additional assumptions that have been made.

(ii) For the given deformation, compute the three strain invariants I_1, I_2 and I_3.

(iii) By using the boundary conditions on the stress, find an explicit expression for μ and hence the applied tension T as functions of the axial stretch λ and the derivatives $\frac{\partial \mathcal{W}}{\partial I_1}$, $\frac{\partial \mathcal{W}}{\partial I_2}$ and $\frac{\partial \mathcal{W}}{\partial I_3}$, where $\mathcal{W}$ is the strain energy function of the material.

(iv) Find the equivalent relationship when the material is incompressible.

[18 marks]

B5. Consider a non-reacting continuum mixture of m constituents in the absence of body forces and external sources of heat or mass. We assume that every constituent is present at every point in space, and that each constituent of the mixture obeys its own internal balance and conservation equations

$$\text{conservation of mass} \qquad \frac{\partial \rho_a}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} (\rho_a \boldsymbol{V}_a) = 0,$$

$$\text{balance of linear momentum} \qquad \rho_a \frac{D\boldsymbol{V}_a}{Dt} = \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \mathsf{T}_a,$$

where the standard notation has been used, and the subscript $_a$ indicates the quantity associated with the a-th constituent. Note that the summation convention is **not** used in this question.

(i) Show that the corresponding laws for the entire mixture

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} (\rho \boldsymbol{V}) = 0, \quad \text{and} \quad \rho \frac{D\boldsymbol{V}}{Dt} = \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \mathsf{T},$$

are satisfied if

$$\rho = \sum_{a=1}^{m} \rho_a, \quad \boldsymbol{V} = \frac{1}{\rho} \sum_{a=1}^{m} \rho_a \boldsymbol{V}_a, \quad \mathsf{T} = \sum_{a=1}^{m} \left[\mathsf{T}_a - \rho_a (\boldsymbol{V}_a - \boldsymbol{V}) \otimes (\boldsymbol{V}_a - \boldsymbol{V}) \right].$$

Hint: You will find it useful to establish the relationship

$$\rho_a \frac{D\boldsymbol{V}_a}{Dt} = \frac{\partial}{\partial t} (\rho_a \boldsymbol{V}_a) + \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} (\rho_a \boldsymbol{V}_a \otimes \boldsymbol{V}_a).$$

(ii) The energy equation for the mixture is given by

$$\rho \frac{D\Phi}{Dt} = \mathsf{T} : \mathsf{D} - \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} (\boldsymbol{Q} + \boldsymbol{J}),$$

where $\boldsymbol{J}$ is the additional energy flux. Show that the Clausius–Duhem inequality for the entire mixture,

$$\rho \frac{D\eta}{Dt} \geq -\boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \left(\frac{\boldsymbol{Q}}{\Theta} \right),$$

can be written in the form

$$-\rho \dot{\Psi} - \rho \eta \dot{\Theta} - \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{J} - \frac{1}{\Theta} \boldsymbol{Q} \boldsymbol{\cdot} \boldsymbol{\nabla}_{\boldsymbol{R}} \Theta + \mathsf{T} : \mathsf{D} \geq 0,$$

where Ψ is to be defined.

(iii) Assuming that $\Psi(\rho, \Theta, C_a)$, where $C_a = \rho_a / \rho$ is the concentration of the a-th constituent show that

$$\eta = -\frac{\partial \Psi}{\partial \Theta}, \quad \mathsf{T} = -\rho^2 \frac{\partial \Psi}{\partial \rho} \mathsf{I},$$

and deduce that

$$\boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \left(\boldsymbol{J} - \sum_a \mu_a \boldsymbol{F}_a \right) + \sum_a \boldsymbol{F}_a \boldsymbol{\cdot} \boldsymbol{\nabla}_{\boldsymbol{R}} \mu_a + \frac{1}{\Theta} \boldsymbol{Q} \boldsymbol{\cdot} \boldsymbol{\nabla}_{\boldsymbol{R}} \Theta \leq 0,$$

where $\boldsymbol{F}_a$ is the flux associated with the a-th constituent such that $DC_a/Dt = \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{F}_a$ and μ_a is to be defined.

[18 marks]

B6. A particular strain energy function is given by

$$\mathcal{W} = \frac{C_1}{2}(I_1 - 3) + \frac{C_2}{2}\log^2\sqrt{I_3} - C_1 \log\sqrt{I_3},$$

where C_1 and C_2 are constants.

(i) Show that the corresponding second Piola–Kirchhoff stress tensor is

$$s^{ij} = C_1 g^{ij} + \left(C_2 \log\sqrt{I_3} - C_1\right) G^{ij}.$$

(ii) Find the Cauchy stress τ^{ij} in the limit of infinitesimal deformations. **Hint:** You will need to establish the results that

$$I_3 = 1 + 2e^k_k,$$

and

$$G^{ij} = [G_{ij}]^{-1} \approx g^{ij} - 2e^{ij},$$

where e_{ij} is the infinitesimal strain tensor.

(iii) Hence find the values of C_1 and C_2 to ensure that the strain energy function is consistent with the linear, isotropic constitutive law in the absence of heating and pre-stress:

$$\tau^{ij} = \lambda e^k_k g^{ij} + 2\mu e^{ij},$$

where λ and μ are constants.

[18 marks]

B7.

A particular fluid has the constitutive relationship

$$\mathsf{T} = -P\mathsf{I} + \mathsf{T}_1 + \mathsf{T}_2,$$

where P is the fluid pressure and the stresses T_1 and T_2 satisfy the relationships

$$\mathsf{T}_1 = \mu_1 \mathsf{D}, \quad \mathsf{T}_2 + \lambda_2 \mathsf{T}_2{}^{\nabla} + \alpha\frac{\lambda_2}{\mu_2}\mathsf{T}_2\mathsf{T}_2 = \mu_2 \mathsf{D},$$

where μ_1, μ_2, α and λ_2 are constants. The upper-convected derivative is defined by

$$\mathsf{A}^{\nabla} = \frac{D\mathsf{A}}{Dt} - \mathsf{L}\mathsf{A} - \mathsf{A}\mathsf{L}^T,$$

where $\mathsf{L} = \boldsymbol{\nabla}_{\boldsymbol{R}} \otimes \boldsymbol{V}$ is the Eulerian velocity gradient tensor and D is the symmetric part of L.

(i) Confirm that the constitutive relationships are objective.

You may assume that the Cauchy stress T is objective; that upper-convected derivatives of objective quantities remain objective; and that the deformation gradient tensor, F, transforms as $\mathsf{F}^* = \mathsf{Q}\mathsf{F}$ and $\dot{\mathsf{F}} = \mathsf{L}\mathsf{F}$, where Q is an orthogonal matrix that expresses the relative rotation between observers.

A steady two-dimensional extensional flow is given by $\boldsymbol{V} = (\epsilon X, -\epsilon Y)$, where ϵ is a constant rate of extension and X and Y are Cartesian coordinates.

(ii) Assuming that the stress T_2 is constant in time and space, find the Cauchy stress components T_{XX}, T_{YY}, T_{XY} of the fluid in this flow. You may leave P as an unknown variable. Explain your choice of any solution branches.

[18 marks]

FORMULA SHEET

- For a general (Lagrangian) coordinate system ξ^i:

$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{r}}{\partial \xi^i}, \quad \boldsymbol{g}_i \boldsymbol{\cdot} \boldsymbol{g}^j = \delta_i^j, \quad g_{ij} = \boldsymbol{g}_i \boldsymbol{\cdot} \boldsymbol{g}_j, \quad g = \det(g_{ij}).$$

$$\boldsymbol{G}_i = \frac{\partial \boldsymbol{R}}{\partial \xi^i}, \quad \boldsymbol{G}_i \boldsymbol{\cdot} \boldsymbol{G}^j = \delta_i^j, \quad G_{ij} = \boldsymbol{G}_i \boldsymbol{\cdot} \boldsymbol{G}_j, \quad G = \det(G_{ij}).$$

- For a scalar field $f(\boldsymbol{x})$ and vector field $\boldsymbol{u}(\boldsymbol{x})$

$$\boldsymbol{\nabla} f = \boldsymbol{g}^i \frac{\partial f}{\partial \xi^i}, \quad \operatorname{div} \boldsymbol{u} = \frac{1}{\sqrt{g}} \frac{\partial \left(u^i \sqrt{g}\right)}{\partial \xi^i}, \quad \operatorname{curl} \boldsymbol{u} = \epsilon^{ijk} u_j|_i \boldsymbol{g}_k.$$

- The material derivative in general coordinates is

$$\frac{DU^i}{Dt} = \frac{\partial U^i}{\partial t} + V^j U^i||_j,$$

where $\boldsymbol{V}$ is the velocity of the continuum.

- Cauchy's equation in the usual notation in components in general coordinates ξ^i is

$$T^{ji}||_j + \rho F^i = \rho \ddot{U}^i = \rho \frac{DV^i}{Dt}, \quad \text{where} \quad T^{ji}||_j = T^{ji}_{,j} + \Gamma^j_{jr} T^{ri} + \Gamma^i_{jr} T^{jr},$$

and Γ^i_{jk} are the Christoffel symbols for the chosen coordinate system in the deformed configuration.

- The Reynolds Transport theorem states that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \phi \, \mathrm{d}\mathcal{V}_t = \int_{\Omega_t} \left(\frac{D\phi}{Dt} + \phi \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V} \right) \mathrm{d}\mathcal{V}_t,$$

where Ω_t is a material volume, ϕ is a scalar field and $\boldsymbol{V}$ is the velocity of the continuum.

- For a Cartesian line element $\mathrm{d}X_I$ in the deformed configuration

$$\frac{D\mathrm{d}X_I}{Dt} = V_{I,K} \mathrm{d}X_K,$$

where V_I is the I-th Cartesian component of the velocity.

- Nanson's relation states that

$$\mathrm{d}A_{\bar{i}} = J \frac{\partial \xi^j}{\partial \chi^{\bar{i}}} \mathrm{d}a_j,$$

where ξ^j are the Lagrangian coordinates, $\chi^{\bar{i}}$ are the Eulerian coordinates, J is the determinant of the deformation gradient tensor, $\mathrm{d}\boldsymbol{A}$ is an area element in the deformed configuration and $\mathrm{d}\boldsymbol{a}$ is an area element in the undeformed configuration.

- The Green–Lagrange strain tensor is defined by

$$\gamma_{ij} = \frac{1}{2}\left(G_{ij} - g_{ij}\right).$$

- The strain invariants are defined by

$$I_1 = g^{ij}G_{ji}, \quad I_2 = \frac{1}{2}\left(I_1^2 - g^{ir}g^{js}G_{ij}G_{rs}\right), \quad I_3 = G/g,$$

where $g = det(g_{ij})$ and $G = det(G_{ij})$

- A hyperelastic material is described by a strain energy function $\mathcal{W}(I_1, I_2, I_3)$ such that

$$T^{ij} = PG^{ij} + Ag^{ij} + BB^{ij},$$

where

$$A = \frac{2}{\sqrt{I_3}}\frac{\partial \mathcal{W}}{\partial I_1}, \quad B = \frac{2}{\sqrt{I_3}}\frac{\partial \mathcal{W}}{\partial I_2}, \quad P = 2\sqrt{I_3}\frac{\partial \mathcal{W}}{\partial I_3},$$

$$\text{and} \quad B^{ij} = \left[I_1 g^{ij} - g^{ir}g^{js}G_{rs}\right].$$

- The physical components of the stress tensor are given by $\sigma_{(ij)} = T^{ij}\sqrt{G_{jj}/G^{ii}}$ (no summation).
- The body stress tensor T^{ij} and second Piola–Kirchhoff stress tensor s^{ij} are related by the expression $JT^{ij} = s^{ij}$.
- The Clausius–Duhem inequality is

$$-\rho\dot{\Psi} - \rho\eta\dot{\Theta} - \frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta + \mathsf{T} : \mathsf{D} \geq 0.$$

END OF EXAMINATION PAPER