The majority of the students attempted all questions in section A and B5, B6 and B8. Many students did very well on this exam and the proportion of first-class marks was approximately 60%. Index notation still caused a few problems as well as manipulation of tensors in matrix representation.

A1 This question tested understanding of tensor calculus and the relationship between metric tensors and the properties of the coordinate system.

(i) This was generally well answered, but a few people didn't explicitly write the condition for orthogonality that $\boldsymbol{g}_i\cdot\boldsymbol{g}_j = 0$.

(ii) In contrast, this was not well answered. The solution is simply to combine the formulæ given in the back of the paper for the gradient and divergence to obtain

$$\boldsymbol{\nabla}^2 u = \frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^k}\left(\sqrt{g}g^{ik}\frac{\partial u}{\partial \xi^i}\right),$$

which can then be simplified for the specific metric tensor computed in part (i).

A2 This question tested general knowledge of what is means to be objective. After noting that T, D and $\boldsymbol{\mathsf{A}}$ are objective tensors, the key observation is that the material derivative of an objective quantity is not necessarily objective and, indeed, $D\mathsf{T}/Dt$ is not objective. The law could be made objective by using an objective derivative such as the upper-convected derivative (or any of the others discussed in the course). A common mistake was to claim that the fourth-rank tensor $\boldsymbol{\mathsf{A}}$ was not objective because it didn't obey the transformation $\boldsymbol{\mathsf{A}}^* = \mathsf{Q}\boldsymbol{\mathsf{A}}\mathsf{Q}^{-1}$, but this is the transformation for a second-rank tensor.

A3 This question tested derivation of governing equations in differential form from integral formulations.

(i) The majority correctly used Reynolds transport theorem to derive the required equations, but a few didn't distinguish between the velocities of the two fluids.

(ii) The easiest way to get to the final equation is simply to add the two equations found in part (i) and then rearrange. The required velocity $\boldsymbol{V}$ is the density-weighted average

$$\boldsymbol{V} = \frac{\rho_1\boldsymbol{V}_1 + \rho_2\boldsymbol{V}_2}{\rho}.$$

(iii) Almost everybody correctly stated that mass is conserved when $S_1 + S_2 = 0$.

A4 This question was in the lecture notes and tested index manipulations. The correct solution is actually given in B5, which a number of students spotted. The idea is to transform from Eulerian to Lagrangian coordinates and use Nanson's relation to convert between deformed and undeformed area elements.

B5 This was perhaps the most consistently well-answered question on the paper.

(i) Almost everybody was able to compute the eigenvalues of $g^{ij}G_{jk}$ which are $(\mathrm{d}R/\mathrm{d}r)^2$ and $(R/r)^2$ (twice) and correctly stated that the eigenvalues are the squares of the stretches. The only mistakes were forgetting that R is a function of r and that you need to differentiate with respect to r when computing the deformed metric tensor.

(ii) Everybody that attempted this question correctly computed the invariants.

(iii) If the material is incompressible I_3 which leads to an ODE that can be solved to show that $R = r$. Errors here failing to put in full working to show the solution of the ODE (or not realising that an ODE needed to be solved).

B6 This question was attempted by a number of students with varying degrees of success.

(i) For this part, one needs to transform from general coordinates to Cartesians and then note that

$$s_{IJ} = \frac{\partial \mathcal{W}}{\partial \gamma_{IJ}} = \frac{\partial \mathcal{W}}{\partial c_{KL}}\frac{\partial c_{KL}}{\partial \gamma_{IJ}},$$

and use the definition of γ in terms of c to compute the derivative.

(ii) This result follows from the arguments about deformation of a line element given in the lecture notes.

(iii) This is a generalisation of the argument given for the derivation of the expression of T in a hyperelastic material given in the formula sheet. Using the fact that $\mathsf{c}(I_1, I_2, I_3, I_4, I_5)$, we apply the chain rule to write

$$\frac{\partial \mathcal{W}}{\partial \mathsf{c}} = \sum_{\alpha=1}^{5} \frac{\partial \mathcal{W}}{\partial I_\alpha} \frac{\partial I_\alpha}{\partial \mathsf{c}},$$

and the result follows. It remains to calculate the partial derivatives of the invariants with respect to c, which is essentially an exercise in index notation. The results for the first three invariants were done in the lecture notes, but

$$\frac{\partial I_4}{\partial \mathsf{c}} = \boldsymbol{a} \otimes \boldsymbol{a}, \quad \frac{\partial I_4}{\partial \mathsf{c}} = \boldsymbol{a} \otimes \mathsf{c}\boldsymbol{a} + \boldsymbol{a}\mathsf{c} \otimes \boldsymbol{a}.$$

Relatively few people got both these terms correct.

(iv) Most people correctly found the derivatives of the strain energy function with respect to the invariants, but errors in (iii) meant that it was difficult to get to the final form for T. If the correct transformation was applied to the wrong s then full marks were given for this section.

B7 The least popular question in section B, probably because it looked unfamiliar. The question was intended to test understanding of rates of change of line elements and constitutive modelling in fluids.

(i) This is essentially the derivation of D in the lecture notes, section 2.5. It is quickest in Cartesians using the expression for the derivative of a line element given in the formula sheet.

(ii) The easiest way to prove the recursion is to note that

$$A_{IJ}^{(m+1)} \mathrm{d}X_I \mathrm{d}X_J = \frac{D}{Dt}\left(A_{IJ}^{(m)} \mathrm{d}X_I \mathrm{d}X_J\right),$$

and then use the product rule and the expression for the derivative of a line element.

(iii) Those that attempted this part generally did well. Problems in matrix multiplication were the only mistakes. Many of the terms in the expression are zero for this flow and you should find that

$$\mu(\gamma) = a_1 + 2(a_2 + a_6)\gamma^2.$$

(iv) I think that people were running out of steam at this point. If the shear stress is increasing monotonically with shear rate than $\mathrm{d}T_{12}/\mathrm{d}\gamma > 0$, which means that

$$a_1 + 6(a_2 + a_6)\gamma^2 > 0. \tag{1}$$

We are told that μ decreases with increasing γ, which means that $a_2 + a_6 < 0$. Thus, equation (1) becomes

$$a_1 - 6|a_2 + a_6|\gamma^2 > 0 \quad \Rightarrow \quad \gamma^2 < \frac{a_1}{6|a_2 + a_6|}.$$

The model is therefore only valid over a limited range of shear rates.

B8 This was generally well answered and the majority of students who attempted it obtained almost all marks. This was designed to test understanding of the thermodynamic concepts that we covered in the course, but was largely an exercise in straightforward calculus, which is probably why people did so well!

(i) This is standard bookwork seen in the lecture notes.

(ii) Use of the thermodynamic relationships found in part (i) can straightforwardly be used to establish the required result.

(iii) Integrating the relationship $c(\Theta) = \partial\Phi/\partial\Theta$ gives the first result. The second follows from

$$\frac{\partial \eta}{\partial \Theta} = \frac{c(\Theta)}{\Theta} \quad \Rightarrow \quad \eta(\mathsf{F}, \Theta) - \eta(\mathsf{F}, \Theta_0) = \int_{s=\Theta_0}^{s=\Theta} c(s)/s \mathrm{d}s.$$

(iv) Substitution of the results found in (iii) gives the desired answer.