Three hours

THE UNIVERSITY OF MANCHESTER

CONTINUUM MECHANICS

23 January 2015
14:00 – 17:00

Answer ALL FOUR questions in section A (21 marks in total).
Answer THREE of the FOUR questions in section B (54 marks in total). If more than THREE questions from Section B are attempted, then credit will be given for the best THREE answers.

Electronic calculators may be used, provided that they cannot store text.
SECTION A

A1.

(i) If a general coordinate system is orthogonal explain why the covariant metric tensor takes the form

\[ g_{ij} = \begin{pmatrix} (h_1)^2 & 0 & 0 \\ 0 & (h_2)^2 & 0 \\ 0 & 0 & (h_3)^2 \end{pmatrix} \]

and find explicit expressions for the entries \( h_i \) in terms of the covariant base vectors \( g_i \).

(ii) Find an expression for the Laplacian of a scalar field \( \nabla^2 u = \nabla \cdot \nabla u \) in terms of the quantities \( h_i \) and partial derivatives with respect to \( \xi^i \).

[6 marks]

A2. Explain why a constitutive law of the form

\[ \frac{D T}{D t} = A D \]

is not objective. Here, \( T \) is the Cauchy stress tensor; \( D = (L + L^T)/2 \) is the Eulerian rate of deformation tensor, \( L = \nabla R V \); and \( A \) is a constant fourth-rank tensor. Give an example of what could be done to make the law objective.

[5 marks]
A3. Two different miscible fluids have densities $\rho_1$ and $\rho_2$ and respective velocities $\mathbf{V}_1(\mathbf{R}, t)$ and $\mathbf{V}_2(\mathbf{R}, t)$. The respective production rates of each fluid per unit volume are $S_1(\mathbf{R}, t)$ and $S_2(\mathbf{R}, t)$.

(i) By considering the conservation of mass of each fluid within a control volume show that

$$\frac{D\rho_1}{Dt} + \rho_1 \nabla \cdot \mathbf{V}_1 = S_1, \quad \frac{D\rho_2}{Dt} + \rho_2 \nabla \cdot \mathbf{V}_2 = S_2.$$ 

(ii) Hence, show that conservation of total mass is expressed by

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} = S_1 + S_2,$$

where $\rho = \rho_1 + \rho_2$ and $\mathbf{V}$ is to be found.

(iii) If the total mass must be conserved state the constraint on $S_1$ and $S_2$. [6 marks]

A4. A continuum is deformed from an initial configuration with position vector $\mathbf{r}$ to a new configuration described by position vector $\mathbf{R}$. The initial position is described by a (Lagrangian) coordinate system $\xi^i$, the new position by an (Eulerian) coordinate system $\chi^j$.

The force acting on an infinitesimal surface of a continuum is given by

$$\mathbf{F} = T^{ij} G_j \text{d}a_i,$$

where $T$ is the Cauchy stress tensor; $\text{d}a$ is the deformed surface area; and $G_j = \frac{\partial \mathbf{R}}{\partial \chi^j}$. The second Piola–Kirchhoff stress tensor, $\mathbf{s}$, is defined by the relationship

$$\mathbf{F} = s^{ij} G_j \text{d}a_i,$$

where $\text{d}a$ is the undeformed surface area; and $G_j = \frac{\partial \mathbf{R}}{\partial \xi^j}$.

Find the relationship between $\mathbf{s}$ and $T$. [4 marks]
A spherical ball of hyperelastic material is deformed such that material originally at radius $r$ is moved to the radius $R(r)$, but the ball maintains its spherical shape.

(i) By computing the eigenvalues of the mixed metric tensor $g^{ij}G_{jk}$, or otherwise, show that the stretch in the radial direction, $\lambda_r$, is given by $\frac{dR}{dr}$, whereas the stretch in the directions tangential to the sphere’s surface is given by $\lambda_t = R/r$.

(ii) Show that the three strain invariants are given by

$$I_1 = \lambda_r^2 + 2\lambda_t^2, \quad I_2 = \lambda_t^4 + 2\lambda_r^2\lambda_t^2, \quad I_3 = \lambda_r^2\lambda_t^4.$$

(iii) By using the properties of $I_3$ show that $R = r$ for an incompressible material.

You should assume that spherical coordinates are defined by $\xi^1 = r$, $\xi^2 = \theta$, $\xi^3 = \phi$, where the undeformed position is given by $r = r \sin \theta \cos \phi \mathbf{e}_x + r \sin \theta \sin \phi \mathbf{e}_y + r \cos \theta \mathbf{e}_z$, where $\mathbf{e}_I$ are global Cartesian base vectors.

[18 marks]
B6. A hyperelastic material consists of a family of fibres embedded in a solid matrix that undergoes a general deformation from an undeformed configuration described by the position vector \( \mathbf{r} \) to a current position described by \( \mathbf{R} \). The direction of the fibres in the undeformed configuration is represented by the unit direction vectors, \( \mathbf{a}(\mathbf{r}, t) \).

(i) The second Piola–Kirchhoff stress tensor can be derived from a strain energy \( W \) by the expression

\[
\mathbf{s}^{ij} = \frac{\partial W}{\partial \gamma_{ij}},
\]

where \( \gamma_{ij} \) is the Green–Lagrange strain tensor. By converting to Cartesian coordinates (or otherwise) show that

\[
\mathbf{s} = 2 \frac{\partial W}{\partial \mathbf{c}},
\]

where \( \mathbf{c} = \mathbf{F}^T \mathbf{F} \) and \( \mathbf{F} = \nabla \mathbf{r} \mathbf{R} \) is the deformation gradient tensor.

(ii) Show that the unit direction vectors become \( \mathbf{A} = \mathbf{F} \mathbf{a} \), in the deformed configuration.

(iii) The strain energy can be written in the form \( W(\mathbf{c}, \mathbf{a} \otimes \mathbf{a}) \) and there are now five possible invariants

\[
I_1 = \text{trace}(\mathbf{c}), \quad I_2 = \frac{1}{2} \left\{ \left[\text{trace}(\mathbf{c})\right]^2 - \text{trace}(\mathbf{c}^2) \right\}, \quad I_3 = \det(\mathbf{c}), \quad I_4 = \mathbf{a} \cdot \mathbf{c} \mathbf{a}, \quad I_5 = \mathbf{a} \cdot \mathbf{c}^2 \mathbf{a}.
\]

Hence, find an expression for the second Piola–Kirchhoff stress tensor of the form

\[
\mathbf{s} = \sum_{\alpha=1}^{5} D_\alpha \frac{\partial W}{\partial I_\alpha},
\]

where \( D_\alpha \) are all second-rank tensors that are to be found. You may assume the result \( \frac{\partial I_3}{\partial \mathbf{c}} = I_3 \mathbf{c}^{-1} \).

(iv) If the strain energy function is given by

\[
W = \frac{1}{2} (I_1 - 3) - \frac{1}{2} p (I_3 - 1) - \frac{1}{2} q (I_4 - 1),
\]

where \( p \) and \( q \) are constants, find the form of the corresponding Cauchy stress, \( \mathbf{T} \). You may use the relationship \( \mathbf{s} = J \mathbf{F}^{-T} \mathbf{T} \mathbf{F}^{-T} \) without proof.
B7. The Rivlin–Ericksen tensor of order $m$, $A^{(m)}$, is defined by
\[ \frac{D^m}{Dt^m}[d\mathbf{R}]^2 = A^{(m)}_{ij} d\chi^i d\chi^j, \]
where $\chi^i$ are general Eulerian coordinates.

(i) Show that $A^{(1)} = L + L^T$, where $L = \nabla_R V$ is the velocity gradient tensor.

(ii) Prove the recurrence relationship
\[ A^{(m+1)} = \frac{D A^{(m)}}{Dt} + L^T A^{(m)} + A^{(m)} L. \]

(iii) A third-order incompressible fluid has Cauchy stress given by
\[ \mathbf{T} = \left\{ a_1 + a_2 \text{trace} \left[ \left( A^{(1)} \right)^2 \right] \right\} A^{(1)} + a_3 A^{(2)} + a_4 \left( A^{(1)} \right)^2 + a_5 A^{(3)} + a_6 \left( A^{(1)} A^{(2)} + A^{(2)} A^{(1)} \right), \]
where the quantities $a_i$ are constants.

(a) A simple shear flow in Cartesian coordinates is given by $V_1 = \gamma X_2$, $V_2 = V_3 = 0$, where $\gamma$ is a constant shear rate. Find an explicit expression for the shear stress, $T_{12}$, of the third-order fluid and hence find an expression for the effective viscosity $\mu(\gamma)$ such that
\[ T_{12} = \mu(\gamma) \gamma. \]

(b) If the fluid is shear-thinning ($\mu$ decreases with increasing $\gamma$) show that if the shear stress is to increase monotonically with shear rate that there is a maximum allowable shear rate. Hence comment on the validity of the model.

[18 marks]
B8.

(i) A perfect thermoelastic material is modelled using the constitutive assumptions (in the usual notation)

$$\Psi = \Psi(\mathbf{F}, \Theta, \nabla_r \Theta), \quad T = T(\mathbf{F}, \Theta, \nabla_r \Theta), \quad \eta = \eta(\mathbf{F}, \Theta, \nabla_r \Theta), \quad Q = Q(\mathbf{F}, \Theta, \nabla_r \Theta).$$

Use the Clausius–Duhem inequality to show that the free energy is a function only of deformation and temperature,

$$\Psi = \Psi(\mathbf{F}, \Theta),$$

and that

$$T = \rho \frac{\partial \Psi}{\partial \mathbf{F}} T, \quad \eta = -\frac{\partial \Psi}{\partial \Theta}.$$

(ii) The specific heat capacity is defined to be

$$c(\mathbf{F}, \Theta) = -\Theta \frac{\partial^2 \Psi}{\partial \Theta^2} > 0.$$

Show that

$$c = \Theta \frac{\partial \eta}{\partial \Theta} = \frac{\partial \Phi}{\partial \Theta},$$

where $\Phi = \Psi + \eta \Theta$ is the internal energy.

(iii) For an entropic material $c$ is assumed to be a function of $\Theta$ only. Use the results from (ii) to show that the change in internal energy for an entropic material caused by an increase in temperature from $\Theta_0$ to $\Theta$ is given by

$$\Phi(\mathbf{F}, \Theta) - \Phi(\mathbf{F}, \Theta_0) = \int_{s=\Theta_0}^{s=\Theta} c(s)ds,$$

and find a similar expression for the change in entropy, $\eta$.

(iv) Hence show that if $c(\Theta) = C$, a constant,

$$\Psi(\mathbf{F}, \Theta) = \Phi(\mathbf{F}, \Theta_0) - \Theta \eta(\mathbf{F}, \Theta_0) + F(\Theta),$$

where

$$F(\Theta) = C \left[ \Theta - \Theta_0 - \Theta \ln \left( \frac{\Theta}{\Theta_0} \right) \right].$$

[18 marks]
FORMULA SHEET

- For a general (Lagrangian) coordinate system $\xi^i$:
  \[ g_i = \frac{\partial r}{\partial \xi^i}, \quad g_i g^j = \delta_i^j, \quad g_{ij} = g_i g_j, \quad g = \det(g_{ij}). \]
  \[ G_i = \frac{\partial R}{\partial \xi^i}, \quad G_i G^j = \delta_i^j, \quad G_{ij} = G_i G_j, \quad G = \det(G_{ij}). \]
- For a scalar field $f(x)$ and vector field $u(x)$:
  \[ \nabla f = g^i \frac{\partial f}{\partial \xi^i}, \quad \text{div} u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^i} \left( u^i \sqrt{g} \right), \quad \text{curl} u = \epsilon^{ijk} u_j |_{\xi^i} g^k. \]
- The material derivative in general coordinates is
  \[ \frac{DU^i}{Dt} = \frac{\partial U^i}{\partial t} + V^j U^i |_{||j}, \]
  where $V$ is the velocity of the continuum.
- Cauchy’s equation in the usual notation in components in general coordinates $\xi^i$ is
  \[ T^{ji} |_{||j} + \rho F^i = \rho \ddot{U}^i = \rho \left( \frac{DV^i}{Dt} \right), \]
  where $T^{ji} |_{||j} = T^{ji} + \Gamma^i_{jr} T^{jr} + \Gamma^i_{jr} T^{jr}$,
  and $\Gamma^i_{jk}$ are the Christoffel symbols for the chosen coordinate system in the deformed configuration.
- The Reynolds Transport theorem states that
  \[ \frac{d}{dt} \int_{\Omega_t} \phi \, dV_t = \int_{\Omega_t} \left( \frac{D\phi}{Dt} + \phi \nabla \cdot V \right) \, dV_t, \]
  where $\Omega_t$ is a material volume, $\phi$ is a scalar field and $V$ is the velocity of the continuum.
- For a Cartesian line element $dX_I$ in the deformed configuration
  \[ \frac{DDX_I}{Dt} = V_{I,K} dX_K, \]
  where $V_I$ is the $I$-th Cartesian component of the velocity.
- Nanson’s relation states that
  \[ dA_i = J \frac{\partial \xi^j}{\partial \chi^i} \, da_j, \]
  where $\xi^j$ are the Lagrangian coordinates, $\chi^i$ are the Eulerian coordinates, $J$ is the determinant of the deformation gradient tensor, $dA$ is an area element in the deformed configuration and $da$ is an area element in the undeformed configuration.
• The Green–Lagrange strain tensor is defined by
  \[ \gamma_{ij} = \frac{1}{2} (G_{ij} - g_{ij}) . \]

• The strain invariants are defined by
  \[ I_1 = g^{ij} G_{ji}, \quad I_2 = \frac{1}{2} \left( I_1^2 - g^{ir} g^{js} G_{ij} G_{rs} \right), \quad I_3 = G/g, \]
  where \( g = det(g_{ij}) \) and \( G = det(G_{ij}) \)

• An incompressible hyperelastic material is described by a strain energy function \( W(I_1, I_2) \) such that
  \[ T^{ij} = P G^{ij} + A g^{ij} + B B^{ij}, \]
  where
  \[ A = 2 \frac{\partial W}{\partial I_1}, \quad B = 2 \frac{\partial W}{\partial I_2} \quad \text{and} \quad B^{ij} = \left[ I_1 g^{ij} - g^{ir} g^{js} G_{rs} \right]. \]

• The Clausius–Duhem inequality is
  \[ -\rho \dot{\Psi} - \rho \eta \dot{\Theta} - \frac{1}{\Theta} Q \cdot \nabla \Theta + T : D \geq 0. \]