The majority of students attempted all questions in Section A, and there was a fairly even split between questions B5, B6, B7 and B8. It appeared that students found this exam difficult and were under some time pressure. Specific questions that caused a surprising amount of difficulty were A2, A4 and B7. Lack of confidence with tensors and algebraic manipulation of geometric objects seemed to be the main problem. In addition, I would say that too many had "memorised" rather than "understood" the techniques which lead to big problems. Some people seemed unaware of the formula sheet on the final page of the exam, which didn't help things. Apologies that it ended up rather hidden.

A1 This question was totally standard bookwork for (i) and was typically well answered although a surprising number of people did not **define** Γ^k_{ij} as asked. Some quoted the definition, whereas it was more efficient to work it naturally into the derivation, but this was not penalised.

A2 Here, (i) was bookwork question, but not recognised as such by many. The answer follows by straight substitution into $\mathsf{R}^T\mathsf{R} = \mathsf{I}$. (ii) This did cause confusion because too many people wanted to use the formula $D/Dt = \partial/\partial t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$ for the material derivative. This is not appropriate or necessary; by the product rule

$$\frac{D}{Dt}\left(\mathsf{R}\mathsf{R}^T\right) = \dot{\mathsf{R}}\mathsf{R}^T + \mathsf{R}\dot{\mathsf{R}}^T = 0.$$

Hence,

$$\dot{\mathsf{R}}\mathsf{R}^T = -\mathsf{R}\dot{\mathsf{R}}^T = -\left(\dot{\mathsf{R}}\mathsf{R}^T\right)^T,$$

which means that $\dot{\mathsf{R}}\mathsf{R}^T = \omega$, say, is antisymmetric. The matrix R is orthogonal so $\mathsf{R}^T = \mathsf{R}^{-1}$, which means that

$$\dot{\mathsf{R}}\mathsf{R}^T = \dot{\mathsf{R}}\mathsf{R}^{-1} = \omega \quad \Rightarrow \quad \dot{\mathsf{R}} = \omega\mathsf{R}.$$

The second result follows by differentiating $\mathsf{F} = \mathsf{R}\mathsf{U}$. In retrospect, this question demanded knowledge of matrix manipulation that may not have been as familiar as it should have been to those taking the course.

A3 Reasonably well answered by most, but too many people wanted to dive in and use Reynold's transport theorem. It is possible that way, but it is cleaner (and quicker) to use the transformation to the reference state, which gives

$$\frac{D\boldsymbol{g}}{Dt} = \frac{1}{m}\frac{D}{Dt}\int_{\Omega_0} \rho\boldsymbol{r}J\,\mathrm{d}\mathcal{V}_0,$$

because mass m is fixed. The partial derivative then passes under the fixed integral and we can use conservation of mass to write

$$\rho J = \rho_0,$$

which is itself fixed. Thus

$$\frac{D\boldsymbol{g}}{Dt} = \frac{1}{m}\int_{\Omega_0} \rho_0 \frac{\partial \boldsymbol{r}}{\partial t}\mathrm{d}\mathcal{V}_0.$$

The required result falls out after transforming back to the deformed domain and differentiating again.

The interpretation was not attempted by all, but it is simply Newton's second law for a particle where all the mass is concentrated at the centre of mass.

A4 I was surprised by the struggles that many had with this question. The undeformed volume $V_0 = 1$ and the deformed volume is simply $V = (1+\lambda_1)(1+\lambda_2)(1+\lambda_3)$ — the block undergoes three complementary stretches. Thus the fractional change in volume is

$$(1+\lambda_1)(1+\lambda_2)(1+\lambda_3) - 1,$$

which gives the sum of the three strain invariants of e_{ij}. The fractional volume change in terms of the invariants of the deformed metric tensor is $\sqrt{I_3}$. Here, most people did correctly compute G_{ij} but not all had worked out the fractional volume change.

B5 Part (i) was generally well answered although quite a few people worked backwards to get the answer. The second part (ii) was also correctly derived by the vast majority. The main problem with part (iii) was characterising pure dilation which is given by $X_I = \alpha^{1/3} x_I$ to achieve a volume change of α. Many people wrote $X_I = \alpha x_I$, but this was not heavily penalised if the rest of the working was carried out correctly. A few people fumbled the differentiation of $\mathcal{W}$.

B6 (i) was straightforward and well answered. Under the assumptions given you simply have to show that $\widetilde{\mathsf{T}}$ is objective. You did need to state the P was a scalar. Those that claimed that D and the upper-convected derivatives were objective because we said so in the notes got a few marks, but the full marks were available for showing that was the case, which most managed.

Part (ii) was more difficult. The majority correctly derived that

$$\mathsf{L} = \mathsf{D} = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix}.$$

The correct technique to proceed is then to substitute these into the equation for $\widetilde{\mathsf{T}}$ and rearrange to find

$$\widetilde{T}_{XX} = \frac{\eta_0 \epsilon (1 - 2\lambda_2 \epsilon)}{1 - 2\lambda_1 \epsilon}, \quad \widetilde{T}_{XY} = 0, \quad \widetilde{T}_{YY} = -\frac{\eta_0 \epsilon (1 + 2\lambda_2 \epsilon)}{1 + 2\lambda_1 \epsilon}.$$

You then need to use Cauchy's equation to determine the pressure so that you can calculate T.

For part (iii) you should find that the stress difference diverges as $\epsilon \to 1/2\lambda_1$, which means that the model is not valid in this limit. If people correctly took the limit and interpreted the results consistently for wrong expressions for the stress full marks were still given.

B7 This question was attempted by nearly everybody and nobody managed to complete it, which was disappointing. The question was possibly too long, but there was a consistent conceptual error about unit vectors and covariant base vectors that made things much harder than they needed to be. The covariant base vectors are orthogonal, but they are not all unit vectors. Almost everybody correctly derived that

$$\boldsymbol{g}_3 = \begin{pmatrix} -r \sin\theta \sin\phi \\ r \sin\theta \cos\phi \\ 0 \end{pmatrix},$$

but then did not normalise properly. In fact, $|\boldsymbol{g}_3| = r \sin\theta$, which means that

$$\boldsymbol{V} = r \sin\theta F(\theta) \boldsymbol{e}_\phi = F(\theta) \boldsymbol{g}_3,$$

so $V^1 = V^2 = 0$ and $V^3 = F(\theta)$. This makes the subsequent algebra much easier.

A problem here was that people did not seem to be comfortable with the required manipulations to compute the components of the stress. They are all given in examples in the course, but perhaps there was not enough practice at this type of question. That said, the wrong expression for V^i just complicates everything. When $i = 3$, Cauchy's equation for this flow becomes

$$0 = T^{j3}_{,j} + \Gamma^j_{jr} T^{r3} + \Gamma^3_{jr} T^{jr}.$$

Using the constitutive law the components $T^{12} = T^{21} = T^{13} = T^{31} = 0$ and then the only non-zero components of the Christoffel symbols leads to

$$0 = T^{32}_{,2} + \Gamma^3_{32} T^{23} + \Gamma^3_{32} T^{32} + \Gamma^3_{23} T^{23}.$$

Using the symmetry of the stress tensor gives

$$\frac{\partial T^{32}}{\partial \theta} + 3 \frac{\cos\theta}{\sin\theta} T^{32} = 0,$$

which can be integrated by separation of variables to give the result.

B8 Part (i) was correctly answered by everybody that attempted it and is essentially a standard technique from the notes. The biggest problem with the rest of the question was with people being unable to define the trace of a tensor/matrix, but again this was defined in the notes. For a tensor object A the trace is A^i_i. Here we have the trace of e^2. In Cartesian components $e^2_{IJ} = e_{IK}e_{KJ}$, so its trace is $e_{IK}e_{KI} = (e_{IK})^2$, after using symmetry properties. The differentiation is then straightforward:

$$\psi^\infty = \mu_\infty (e_{IJ})^2 \quad \Rightarrow \quad \frac{\partial \psi^\infty}{\partial e_{IJ}} = 2\mu_\infty e_{IJ} \quad \Rightarrow \quad \frac{\partial \psi^\infty}{\partial \mathsf{e}} = 2\mu_\infty \mathsf{e}.$$

Once you have this methodology, the rest of part (iii) is accessible.