The majority of students attempted all questions in Section A, and questions B5, B6 and B8. It is clear that this was a difficult exam and people were under time pressure. Overall, section A was well answered, but peopled struggle with the section B questions. The main problems seemed to be lack of confidence with tensors and the severe time pressure for section B. Roughly speaking, people knew what to do, but were necessarily as quick, or accurate, as they needed to be. That said, some people did extremely well.

A1 This question revealed a lack of confidence with coordinate transforms, but most people correctly deduced that $e(i,j)$ is a **not** a tensor because it does not obey the appropriate transformation rule. This could be shown explicitly or argued as follows: the transformation the Cartesian base vectors to the covariant base vectors $\boldsymbol{g}_i$ is given by

$$\begin{pmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_1} \\ \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_2} \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}.$$

The transformation matrix is orthogonal, but has a determinant of -1, so it represents a reflection (and a rotation), which means that the components of $e(i,j)$ must be different in the two coordinate systems if it is to be a tensor. (In fact, $e(i,j)$ is a pseudotensor, because it remains invariant if we restrict the transformations to those that preserve the handedness of the coordinate system.)

A2 Almost everybody knew what to do with this question, but the algebra go a little bit involved if you weren't careful. (In hindsight, I could have made that a bit easier, so sorry.) The quickest way through was to realise that because x_2 remains unchanged, you don't need to invert a 3×3 matrix, but only the 2×2 sub-block corresponding to the x_1 and x_3 coordinates. A few people hadn't learned (and couldn't derive from first principles) the definition of the material derivative, which is inexcusable.

A3 Generally well answered and a small variation on the argument that the stress tensor is symmetric in the absence of surface couples. The most efficient method was to use Cartesian coordinates, but well done to those that used general coordinate systems. The argument is to use the divergence theorem to convert the surface integral into a volume integral, use Cauchy's equation to eliminate as many terms as possible and then convert the resulting integral into a PDE by arguing that it must be valid for any material region.

A4 This was essentially a bookwork question and was well answered by nearly everybody. Once again, the divergence theorem is used to convert the surface integral into a volume integral, Cauchy's equation is used to eliminate a number of terms and the remaining term gives the result. In the lecture notes, the argument was made in general coordinates, but you can also use Cartesians and many people did.

B5 A lot of people got tied up in the tensor calculations here. A problem is that a mistake in the metric tensors at the beginning means that the rest of the question becomes very difficult. For this reason, many of the marks awarded were method marks. I realise that you did also have to know the definition of spherical coordinates, which caused (unintended) problems for some people.

(a) There was some confusion about the definition of the position vector in spherical coordinates. In standard spherical polars, the components in the global Cartesian coordinate system are given by

$$\boldsymbol{r} = \begin{pmatrix} r\sin\theta\sin\phi \\ r\sin\theta\cos\phi \\ r\cos\theta \end{pmatrix}.$$

A mistake that occurred a few times was to assume that

$$\boldsymbol{r} = \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix},$$

which is essentially a relabelling of the Cartesian coordinates.

The deformed position is given by

$$\boldsymbol{R} = \begin{pmatrix} R\sin\Theta\sin\Phi \\ R\sin\Theta\cos\Phi \\ R\cos\Theta \end{pmatrix},$$

and because $R(r)$, but $\Theta = \pi - \theta$ and $\Phi = \phi$, we have

$$\boldsymbol{R} = \begin{pmatrix} R(r)\sin\theta\sin\phi \\ R(r)\sin\theta\cos\phi \\ -R(r)\cos\theta \end{pmatrix}.$$

Given the undeformed and deformed position vectors, calculation of the metric tensors and strain invariants is straightforward. After the appropriate calculations, we find

$$I_3 = (R'R^2/r^2)^2,$$

so if the material is incompressible then

$$I_3 = 1 = (R'R^2/r^2)^2 \quad\Rightarrow\quad R'R^2 = \pm r^2 \quad\Rightarrow\quad \frac{R^3}{3} = \pm\frac{r^3}{3} + \frac{A}{3},$$

and so

$$R = \left(A \pm r^3\right)^{1/3},$$

where the negative sign is chosen because R and r must be in opposite directions after eversion of the sphere.

(b) Again, assuming that you have the correct metric tensors, the stress tensor follows by straightforward calculation. The most difficult quantity to compute is B^{ij}, and the second term $g^{ir}g^{js}G_{rs}$ is most easily computed by the matrix computation $\mathsf{g}^{-1}\mathsf{G}\mathsf{g}^{-T}$, where g and G are the undeformed and deformed covariant metric tensors, which means that g^{-1} is the contravariant metric tensor.

The only non-zero components of the tensor are T^{11}, T^{22} and T^{33}, which many people correctly deduced and the deduction about the pressure follows directly from Cauchy's equations.

B6 **(a)** Most people correctly deduced that you only need to show that D is objective in order to show objectivity of the constitutive law. We showed that D is objective in the lecture notes and the majority correctly reproduced the argument. Note that L is **not** objective, but the symmetric part, $\mathsf{D} = (\mathsf{L} + \mathsf{L}^T)/2$ is.

(b) Nearly everybody correctly found that the only non-zero term in L is $\overline{L}_1^3 = W'(R)$, which means that

$$\overline{D}_1^3 = \overline{D}_3^1 = \frac{1}{2}W'(R).$$

Sometimes factors of 1/2 went missing and a couple of people forgot that D is supposed to be symmetric. The correct form for the mixed stress tensor is given by

$$\overline{T}_j^i = \begin{pmatrix} -\pi + \alpha_2(W')^2/4 & 0 & \alpha_1 W'/2 \\ 0 & -\pi & 0 \\ \alpha_1 W'/2 & 0 & -\pi + \alpha_2(W')^2/4 \end{pmatrix},$$

and most people were close... It's then a simple transform to get to the contravariant form.

(c) Many had lost heart by this point, but once you have the stress tensor, then forming the equations follows from plugging in values and solving the resulting simplified PDEs by separation of variables.

B7 Very few people attempted this question, perhaps because it looked unfamiliar, but it was very close to an example sheet question.

(a) The way into the question is to assume that if the undeformed rope is given by (x_1, x_2, x_3), the deformed position is $(\lambda_1 x_1, \lambda_2 x_2, \lambda_2 x_3)$, where the main extension is along the x_1 axis. The load on the rope is then given by $T_{11} = T$, $T_{22} = T_{33} = 0$. These give enough conditions to work out λ_2

and T, assuming that $\lambda_1 = 2$. The solution in the linear case is $T = \frac{3\lambda+2\mu}{\lambda+\mu}\mu$ and in the nonlinear case, $T = 5\frac{1}{4}\mu$.

(b) Taking the limit $\lambda \to \infty$ in the linear case gives $T \to 3\mu$, so the linear theory underpredicts the load by about 43%. Most that got this far correctly deduced that the linear theory is an underprediction.

B8 **(a)** This section was bookwork from the lecture notes and was correctly answered by the majority. A few people had the arguments in the wrong order, so the chain of inference fell apart.

(b) Again, generally well answered by the majority. Applying the conditions for objectivity, using the fact that Θ is a scalar and standard transformation rules for the derivatives, it follows that $K_{IJ}(\theta)$ must be invariant under all rotations, which means that it can only be a multiple of the identity.

(c) Nobody got to the end of this question, which may well have been due to the time pressure. Most people correctly argued that

$$\frac{DE}{Dt} = \int_{\Omega_t} \rho\dot{\boldsymbol{V}}\boldsymbol{\cdot}\boldsymbol{V} + \rho\dot{\Phi}\,\mathrm{d}\mathcal{V}_t.$$

The procedure is then to use Cauchy's equation to replace $\dot{\boldsymbol{V}}$ and the energy equation to replace $\dot{\Phi}$. It's easiest to work in Cartesian components when doing this. After a little rearrangement, it follows that

$$\frac{DE}{Dt} = \int_{\Omega_t} \left[-PV_I + \lambda V_{J,J}V_I + \mu(V_{J,J}V_I + V_{J,I}V_J) - Q_I\right]_{,I}\,\mathrm{d}\mathcal{V}_t,$$

and using the divergence theorem gives

$$\frac{DE}{Dt} = \int_{\partial\Omega_t} \left[-PV_I + \lambda V_{J,J}V_I + \mu(V_{J,J}V_I + V_{J,I}V_J) - Q_I\right] N_I\,\mathrm{d}\mathcal{S}_t.$$

Hence, if $V_I = 0$ and $Q_I N_I = 0$ on the boundary, then the energy is conserved, as required.