Three hours

THE UNIVERSITY OF MANCHESTER

CONTINUUM MECHANICS

15 January, 2013

09:45 – 12:45

Answer **ALL FOUR** questions in section A (21 marks in total).
Answer **THREE** of the **FOUR** questions in section B (54 marks in total). If more than THREE questions from Section B are attempted, then credit will be given for the best THREE answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

A1. A two-dimensional Euclidean space is represented by two different coordinate systems: a Cartesian system (x_1, x_2) and an alternative system (ξ^1, ξ^2), where

$$x_1 = \frac{1}{\sqrt{2}}\left(\xi^1 + \xi^2\right) \quad \text{and} \quad x_2 = \frac{1}{\sqrt{2}}\left(\xi^1 - \xi^2\right).$$

A quantity $e(i, j)$ is defined such that in **both** coordinate systems

$$e(1,1) = e(2,2) = 0, \quad e(1,2) = 1 \quad \text{and} \quad e(2,1) = -1.$$

Do the quantities $e(i, j)$ represent the components of a tensor?

[5 marks]

A2. A region in three-dimensional Euclidean space is deformed such that the positions of its material points in Cartesian coordinates X_I at time t are given by

$$X_1 = \mathrm{e}^t x_1 + t\, x_3, \quad X_2 = x_2, \quad X_3 = x_3 - t\, x_1,$$

where the Lagrangian coordinates x_I give the position of the material points in the same Cartesian coordinate system at $t = 0$.

(a) For the Lagrangian field $a(x_I, t) = x_1 + t$ find the corresponding Eulerian field $A(X_I, t)$, such that $A(X_I, t) = a(x_J(X_I), t)$.

(b) Find the material derivatives of the field a in both the Lagrangian and Eulerian representations and confirm that they are the same.

[6 marks]

A3. A surface couple $\boldsymbol{M}$ and surface traction $\boldsymbol{T}$ act on a material region, Ω_t, of a continuous body so that the balance of angular momentum when the region is in equilibrium is given by

$$\int_{\partial\Omega_t} \boldsymbol{M} + \boldsymbol{R} \times \boldsymbol{T}\, \mathrm{d}\mathcal{S}_t = \boldsymbol{0},$$

where $\boldsymbol{R}$ is the position vector to the material points of the body. No body forces or body couples are acting on the body.

Show that if the entire body is in equilibrium and $\boldsymbol{M} \neq \boldsymbol{0}$, the Cauchy stress tensor T in the body is not necessarily symmetric. In addition, find the relationship between the Cauchy stress tensor and the couple stress tensor, M. The tensor M is defined so that in a Cartesian coordinate system the components of the surface couple are $M_I = M_{IJ} N_J$, where $\boldsymbol{N}$ is the outer unit normal to the surface.

You may use Cauchy's equation of equilibrium

$$\boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \mathsf{T} + \rho \boldsymbol{F} = \boldsymbol{0},$$

without proof.

[5 marks]

A4. Show that the Cauchy stress tensor, T, and the Eulerian velocity gradient tensor, $\mathsf{L} = \boldsymbol{\nabla}_{\boldsymbol{R}} \otimes \boldsymbol{V}$, are a work conjugate pair and determine whether the rate of work given by $\mathsf{T}\boldsymbol{:}\mathsf{L}$ is per unit volume of the deformed or undeformed body.

You may use Cauchy's equation

$$\rho\dot{\boldsymbol{V}} = \boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\mathsf{T} + \rho\boldsymbol{F},$$

and the fact that the total rate of work on the deformed body Ω_t is given by

$$\dot{\mathcal{W}} = \int_{\partial\Omega_t} \boldsymbol{T}\boldsymbol{\cdot}\boldsymbol{V}\,\mathrm{d}\mathcal{S}_t + \int_{\Omega_t} \rho\left(\boldsymbol{F} - \dot{\boldsymbol{V}}\right)\boldsymbol{\cdot}\boldsymbol{V}\,\mathrm{d}\mathcal{V}_t,$$

without proof.

[5 marks]

SECTION B

For the next set of questions you may assume Cauchy's equation in the usual notation in components in general coordinates ξ^i

$$T^{ji}||_j + \rho F^i = \rho \ddot{U}^i, \quad \text{where} \quad T^{ji}||_j = T^{ji}_{,j} + \Gamma^j_{jr} T^{ri} + \Gamma^i_{jr} T^{jr},$$

and Γ^i_{jk} are the Christoffel symbols for the chosen coordinate system in the deformed configuration.

B5. A spherical shell of hyperelastic material is everted (turned inside out) so that the deformed position in spherical polar coordinates (R, Θ, Φ) is given by

$$R(r), \quad \Theta = \pi - \theta, \quad \Phi = \phi,$$

where (r, θ, ϕ) represents the undeformed position in spherical polar coordinates. Both sets of coordinates are taken relative to the same origin at the centre of the sphere.

(a) Find general expressions for the three strain invariants

$$I_1 = g^{ij} G_{ji} = G^i_i, \quad I_2 = \frac{1}{2}\left(G^i_i G^j_j - G^i_j G^j_i\right) \quad \text{and} \quad I_3 = G/g;$$

and show that if the deformation is incompressible

$$R = (A - r^3)^{\frac{1}{3}}, \quad \text{where } A \text{ is a constant.}$$

(b) The incompressible material is described by a strain energy function $\mathcal{W}(I_1, I_2)$ such that

$$T^{ij} = PG^{ij} + Ag^{ij} + BB^{ij},$$

where

$$A = 2\frac{\partial \mathcal{W}}{\partial I_1}, \quad B = 2\frac{\partial \mathcal{W}}{\partial I_2} \quad \text{and} \quad B^{ij} = \left[I_1 g^{ij} - g^{ir} g^{js} G_{rs}\right].$$

Find explicit expressions for the non-zero stress components and show that the pressure is a function only of r. (You do not need to find an explicit expression for p).

You may use the fact that in spherical polar coordinates $(\xi^1, \xi^2, \xi^3) = (r, \theta, \phi)$, the only non-zero Christoffel symbols are

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \frac{\cos\theta}{\sin\theta},$$
$$\Gamma^1_{22} = -r, \quad \Gamma^1_{33} = -r\sin^2\theta, \quad \Gamma^2_{33} = -\cos\theta\sin\theta.$$

[18 marks]

B6. A particular Reiner–Rivlin fluid has the constitutive relation

$$T^i_j = -\pi\delta^i_j + \alpha_1 D^i_j + \alpha_2 D^i_k D^k_j,$$

where T^i_j are the components of the Cauchy stress tensor in general Eulerian coordinates; δ^i_j is the Kronecker delta; and D^i_j are components of the Eulerian rate of deformation tensor. The quantity π is the thermodynamic pressure and α_1 and α_2 are constants.

(a) Confirm that the constitutive law is invariant under a distance-preserving change in Eulerian observer.

You may assume that the Cauchy stress T is objective; that the deformation gradient tensor, F, transforms as $\mathsf{F}^* = \mathsf{QF}$; and that $\dot{\mathsf{F}} = \mathsf{LF}$, where L is the Eulerian velocity gradient tensor $\mathsf{D} = (\mathsf{L} + \mathsf{L}^T)/2$ and Q is an orthogonal matrix that expresses the relative rotation between observers.

(b) A quantity of this fluid flows through a cylindrical pipe of radius $R = a$ and we assume that the velocity field has the form

$$\boldsymbol{V} = W(R)\boldsymbol{e}_Z,$$

where $\boldsymbol{e}_Z$ is a unit vector directed along the axis of the pipe. There are no body forces acting.

Find the general form of the contravariant components of the Cauchy stress tensor T^{ij} in cylindrical polar coordinates for this flow.

(c) Hence, use Cauchy's equations to find explicit expressions for π and W. Note that these expressions should still contain two arbitrary constants representing the axial pressure gradient and a reference pressure. Indicate which is which in your solution.

You may use the fact that in cylindrical polar coordinates $(\xi^1, \xi^2, \xi^3) = (r, \theta, z)$ the only non-zero Christoffel symbols are given by

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \quad \text{and} \quad \Gamma^1_{22} = -r.$$

[18 marks]

B7. A cable is made of an elastic material and a bungee jumper calculates safety factors by determining the load required to cause the rope to double in length. The jumper calculates the required load by using linear elasticity, assuming that

$$\mathsf{T} = \lambda\,\mathrm{tr}(\mathsf{E})\mathsf{I} + 2\mu\mathsf{E},$$

where T is the Cauchy stress tensor; E is the displacement gradient tensor; I is the identity and the material constants λ and μ are the Lamé coefficients. Unfortunately, the material is incompressible and hyperelastic with strain energy function

$$\mathcal{W} = \frac{\mu}{2}\left[(I_1 - 3) + (I_2 - 3)\right],$$

where I_1 and I_2 are the first and second strain invariants.

(a) Calculate the tension required to double the length of the rope using the linear and nonlinear theories, assuming that the rope is subject only to a constant axial tension T. You may assume that any cross-sectional deformation is uniform.

(b) The jumper realises that the material is incompressible, but simply adjusts his linear theory by taking the limit $\lambda \to \infty$. Find the relative error in applied load between the modified linear and nonlinear theories. Does the linear theory under-predict or over-predict the required load?

You may also assume that for an incompressible hyperelastic material

$$T^{ij} = PG^{ij} + Ag^{ij} + BB^{ij},$$

where

$$A = 2\frac{\partial \mathcal{W}}{\partial I_1}, \quad B = 2\frac{\partial \mathcal{W}}{\partial I_2} \quad \text{and} \quad B^{ij} = \left[I_1 g^{ij} - g^{ir} g^{js} G_{rs}\right],$$

and the physical components of the stress are given by $\sigma_{(ij)} = \sqrt{(G_{jj})/(G^{ii})}T^{ij}$, where the repeated indices are not summed.

[18 marks]

B8. Consider a Newtonian fluid for which the Cauchy stress is of the form

$$\mathsf{T} = -P\mathsf{I} + \lambda(\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V})\,\mathsf{I} + 2\mu\mathsf{D},$$

in the usual notation. The thermodynamic pressure P and internal energy Φ are functions only of the density ρ and temperature Θ.

(a) Show that the thermodynamic pressure P is not equal to the mechanical pressure $P_m = -\mathrm{tr}(\mathsf{T})/3$ unless either (i) the fluid is incompressible or (ii) $\lambda + 2\mu/3 = 0$.

(b) If the heat flux is given by components in a Cartesian basis by

$$Q_I = -K_{IJ}(\boldsymbol{R}, \Theta)\,\partial\Theta/\partial X_J,$$

show that the axiom of objectivity is only satisfied if $K_{IJ}(\boldsymbol{R}, \Theta) = \kappa(\Theta)\delta_{IJ}$ for some scalar function $\kappa(\Theta)$.

(c) If the total energy of the fluid is given by

$$E(t) = \int_{\Omega_t} \left(\frac{1}{2}\rho|\boldsymbol{V}|^2 + \rho\Phi\right)\,\mathrm{d}\mathcal{V}_t,$$

show in the absence of any body forces or heat sources that the energy is conserved under any motion such that there is no slip and no heat flux on the boundaries of a fixed domain.

You may use without proof the facts that mass is conserved

$$\frac{D\rho}{Dt} + \rho\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V} = 0;$$

the energy equation in the absence of heat sources is given by

$$\rho\frac{D\Phi}{Dt} = -P\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V} - \boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{Q} + \lambda\,(\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V})^2 + 2\mu\mathsf{D} : \mathsf{D};$$

and the material derivative of the Jacobian of the mapping from Lagrangian to Eulerian coordinates is

$$\frac{DJ}{Dt} = J\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V}.$$

[18 marks]

END OF EXAMINATION PAPER